Alternate approach to local L^2-estimate of the pressure

Stokes Problem: Let $\Omega \subset \mathbb{R}^3$ be a open bounded domain with smooth boundary. Additionally we assume that $\pi_1(\Omega) = \pi_1(\Omega_\epsilon) = 0$ for $\epsilon > 0$ sufficient small.

Let $I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \langle f, u\rangle\, dx$ be the functional defined for $u \in H_0^1(\Omega, \mathbb{R}^3)$ and $f \in L^2(\Omega, \mathbb{R}^3)$.

The technique via a minimizing sequence gives a minimizer $u \in H_0^1(\Omega, \mathbb{R}^3)$ within the admissible set $\mathcal{A}$ which is defined by

$$\mathcal{A} = \{v \in H_0^1(\Omega, \mathbb{R}^3) \mid \operatorname{div}(v) = 0\},$$

since $\mathcal{A}$ is a linear subspace (hence a convex subset) of $H_0^1(\Omega, \mathbb{R}^3)$. In particular, the minimizer u satisfies $\operatorname{div} u = 0$. It is also easy to see, via the convexity of $L(p, z) = \frac{1}{2}|p|^2 - \langle f, z\rangle$ on p, the minimizer is unique.

Theorem 0.1. *There exists a (pressure) function $p(x) \in L^2_{\text{loc}}(\Omega, \mathbb{R})$ such that*

$$\int_M \langle \nabla u, \nabla \eta\rangle = \int_M p \operatorname{div}(\eta) + \langle f, \eta\rangle \tag{1}$$

for all $\eta \in H^1(\Omega, \mathbb{R}^3)$ and η has compact support in Ω. Namely u and p satisfy weakly that

$$-\Delta u = -\nabla p + f, \qquad \textit{and pointwisely } \operatorname{div} u = 0. \tag{2}$$

Proof. The proof is built upon that u satisfies the Euler-Lagrange equation

$$\int_\Omega \langle \nabla u, \nabla \eta\rangle - \langle f, \eta\rangle = 0, \quad \forall \eta \in H_0^1(\Omega, \mathbb{R}^3), \operatorname{div} \eta = 0. \tag{3}$$

The first half of the proof is the same as Ch 8.4 of Evans' book in terms of using the mollifiers. Namely we let η_ϵ, u_ϵ, f_ϵ be the mollified approximation of η, u and f. They are defined in Ω_δ if $\epsilon < \delta$. It is easy to check that $\operatorname{curl}(\eta_\epsilon) = (\operatorname{curl}(\eta))_\epsilon$. And they are controlled by the corresponding norms of η, u, f (see Theorem 1 of Ch5.3 of Evans).

Moreover since

$$\int_\Omega \langle F, G_\epsilon\rangle = \int_\Omega F_\epsilon, G\rangle, \quad \frac{\partial}{\partial x_j} F_\epsilon = \left(\frac{\partial}{\partial x_j} F\right)_\epsilon$$

we have that for any $v \in H_0^1(\Omega, \mathbb{R}^3)$

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla u, \nabla \operatorname{curl}(v_\epsilon)\rangle - \langle f, \operatorname{curl}(v_\epsilon)\rangle \\ &= \int_\Omega \langle -\Delta u_\epsilon - f_\epsilon, \operatorname{curl}(v)\rangle \\ &= \int_\Omega \langle \operatorname{curl}(-\Delta u_\epsilon - f_\epsilon), v\rangle. \end{aligned}$$

Here in the last line we have used the fact that, after identifying a vector with a 1-form, $\operatorname{curl} \alpha = *d\alpha$ and hence

$$(\alpha, *d\beta) = (\alpha, d^* *^{-1} \beta) = (d\alpha, *^{-1}\beta) = (*d\alpha, \beta).$$

It can also be checked via a direct calculation. From the above we infer that

$$\operatorname{curl}(-\Delta u_\epsilon - f_\epsilon) = 0, \text{ over } \Omega_\delta. \tag{4}$$

The difference of the alternate proof is on the construction of p_ϵ and the way of proving a uniform upper bound for the L^2-norm of p_ϵ over a relative compact domain Ω_δ for $\delta > 0$ sufficiently small.

Let $\Delta u_\epsilon + f_\epsilon = (\Delta u^1_\epsilon + f^1_\epsilon, \Delta u^2_\epsilon + f^2_\epsilon, \Delta u^3_\epsilon + f^3_\epsilon)$. Here we construct $p_\epsilon(X)$ by

$$p_\epsilon(X) = \int_{X_0}^{X} (\Delta u^1_\epsilon + f^1_\epsilon)dx + (\Delta u^2_\epsilon + f^2_\epsilon)dy + (\Delta u^3_\epsilon + f^3_\epsilon)dz$$

along any smooth path from a fixed point $X_0 \in \Omega_\delta$ to X. Namely we use $\Delta u_\epsilon + f_\epsilon$ as a 1-form. The function is well defined, namely independent of the choice of the paths, since $\pi_1(\Omega_\delta) = \{0\}$ and $d\omega_\epsilon = 0$, where $\omega_\epsilon = (\Delta u^1_\epsilon + f^1_\epsilon)dx + (\Delta u^2_\epsilon + f^2_\epsilon)dy + (\Delta u^3_\epsilon + f^3_\epsilon)dz$. Here (x, y, z) are the coordinate functions of $\mathbb{R}^3$. The claim $d\omega_\epsilon = 0$ is equivalent to (4).

In order to establish a uniform L^2-bound for p_ϵ it suffices to estimate the L^2-norm of Δu_ϵ since we may choose the connecting path so that the length of the path is uniformly bounded from above.

To estimate the L^2-norm of Δu_ϵ we apply (3) by letting

$$\eta = \left[\operatorname{curl}(\xi^2 \operatorname{curl}(u_\epsilon))\right]_\epsilon.$$

Here ξ is a cut-off function with support in $\Omega_{\delta'}$ with $\delta' < \delta$ and being 1 in Ω_δ. Clearly $\operatorname{div} \eta = 0$. Now $\operatorname{div} u_\epsilon = 0$ implies the equation

$$\operatorname{curl}(\xi^2 \operatorname{curl}(u_\epsilon)) = -\xi^2 \Delta u_\epsilon + \nabla(\xi^2) \wedge \operatorname{curl}(u_\epsilon).$$

Here $\wedge$ means the cross product of the two vectors. Integration by parts yields

$$\begin{aligned} \int_\Omega \xi^2 |\Delta u_\epsilon|^2 &= -\int_\Omega \langle \Delta u_\epsilon, \nabla(\xi^2) \wedge \operatorname{curl}(u_\epsilon)\rangle + \langle f_\epsilon, -\xi^2 \Delta u_\epsilon + \nabla(\xi^2) \wedge \operatorname{curl}(u_\epsilon) \\ &\le \frac{1}{2} \int_M \xi^2 |\Delta u_\epsilon|^2 + A\left(\|f\|^2_{L^2} + \|u\|^2_{H^1_0}\right). \end{aligned}$$

Here A is a constant independent of the sufficiently small ϵ. Hence we have the needed L^2-estimate:

$$\int_{\Omega_\delta} |\Delta u_\epsilon|^2 \le A. \tag{5}$$

Once a uniform L^2-bound on p_ϵ over the domain Ω_δ is obtained, taking limit $\epsilon \to 0$ we have the p as the weak limit, and the equation (1). The existence of p also follows from (3) together with the regularity $\Delta u \in L^2$ (following from 5), via Weyl's (Friedrichs) orthogonal decomposition (cf. Theorem 7.7.7 of Morrey's book for an analogous one). □

The method above is more direct than the exposition of Evans' PDE (pages498-499) where he solved another equation $\operatorname{div} v_\epsilon = p_\epsilon$ with Dirichlet boundary condition and appealed to a result in the paper of Dacorogna-Moser (Ann. Inst. Poincaré 7(1990), 1-26) to get the estimate of $\|p_\epsilon\|_{L^2(\Omega_\delta)}$.