03/30/21

What, who, where, when and how
(why –) 5Ws

What is "calculus of variations" ?

Who are main players ? – many prominent figures
you & me

Where to find the minimizer of an extremal ? ← problem dictates
weak solutions

when to expect a minimizer ? ← sufficent conditions

how to find it ? ← Find & solve ODE
PDE

Problems : • Kepler planimetric problem
Wine barrel – maximize volume

• Steiner problem :

Villiage

$\cdot V_1$ – Town 1 {Tesla Charging Station ?

V_2 · $*$

Trainstation

· V_3

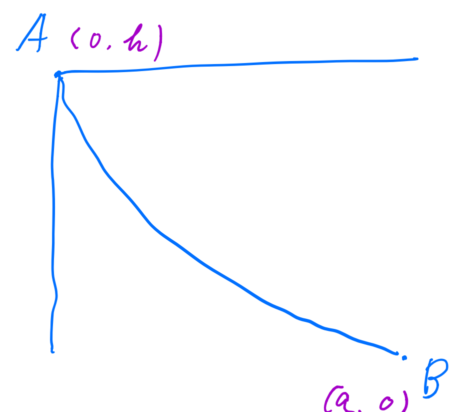
Geometric nature

(A) Calculus of variation concerns functionals

involves derivatives & integrals of functions

Brachistochrone:

1696 Johann Bernoulli



Find the path for the particle to travel in the shortest time under the gravity.

$$\begin{cases} f(0) = h \\ f(a) = 0 \end{cases}$$

Solving this is a presentation ~ HW

Grades: Do a presentation on one of projects (5 total

or do a presentation on HW

during the office hour $\longrightarrow$ A

I select the one with excellent solution

to present in class $\longrightarrow$ A^+

$\underline{Ex}$: $D_f(\varphi) := \int_\Omega |\underset{=}{\nabla}\varphi|^2$, $\varphi|_{\partial\Omega} = f$

(∇ = D)

a functional on φ

$\varphi \in \underline{\underline{W^{1,2}}}(\Omega)$ makes sense $\varphi|_{\partial\Omega} = f$ also makes sense if Ω is smooth

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Calculus of variation concerns how to find the minimizer

Assume u is a minimizer

$\forall\ \eta \in W_0^{1,2}(\Omega)$ $\quad \begin{cases} u+\eta \in W^{1,2}(\Omega) \\ u+\eta|_{\partial\Omega} = u|_{\partial\Omega} = f \end{cases}$

$u \rightsquigarrow u+\varepsilon\eta$

$$D(\varepsilon) = \int_\Omega |\nabla u + \varepsilon\nabla\eta|^2$$

$$0 = D'(0) = \boxed{2\int_\Omega \langle \nabla u\ \ \nabla\eta \rangle} = -2\int \Delta u \cdot \eta$$

$\forall \eta \in W_0^{1,2}(\Omega)$

$D'(0) \sim \delta D$ called the 1st variation

Hence u satisfies $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \underline{u|_{\partial\Omega} = f} \end{cases}$

PDE people use this to study the PDE – Euler–Lagrange equation

Lagrangian U

$L : \mathbb{R}^n \times \mathbb{R} \times \Omega$

$L(p, z, x)$

$$I(z) = \int_U \underbrace{L(Dz, z, x)}_{z \rightsquigarrow z+\varepsilon\eta}$$

$z : U \to \mathbb{R}$ a function (or vector valued function $(z^1, \dots z^m)$)

$U \subset \mathbb{R}^n$

e.g. $L(p) = |p|^2$

δI – called the 1st variation

$z + \varepsilon\eta$

$\eta \in W_0^{1,2}(U)$ $\quad z \in W^{1,2}$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(z+\varepsilon\eta) = \int_U L_{p_j} D_j\eta + L_z\,\eta$$

$$= \int_U \Big[\underbrace{-D_j(L_{p_j}) + L_z}\Big]\cdot\eta$$

If η v-valued $\qquad L = L(p, \underline{z}, x) \quad \underline{\mathbb{R}^{m\times n}} \times \mathbb{R}^m \times \underline{U}$

m total

$$= \int L_{p^i_j}\, D_j\eta^i + L_{z^k}\,\underline{\eta}^k$$

n

$$= \int -D_j(L_{p^k_j})\,\eta^k + L_{z^k}\,\underline{\eta}^k$$

$z \to z + \varepsilon\eta$

$\begin{pmatrix} z^1 + \varepsilon\eta^1 \\ \vdots \\ z^m + \varepsilon\eta^m \end{pmatrix}$

$$\int \sum_{k=1}^{3} \varphi^k \cdot \eta^k = 0 \quad \forall\, \eta$$

$\Rightarrow \varphi^k = 0 \quad \forall\ 1 \le k \le m$

$$\Rightarrow \quad \boxed{-D_j\left(L_{p_j^k}\right) + L_{z^k} = 0} \quad \forall\, 1 \le k \le m$$

- 2nd-variation & ellipticity

Hadamard–Legendre condition

Say u is a minimizer

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(u+\varepsilon\eta) = 0 \quad \& \quad \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} I(u+\varepsilon\eta) \geq 0$$

$$\frac{d}{d\varepsilon} I(u+\varepsilon\eta) = \int L_{p_j} D_j\eta + L_z \eta$$

$$L = L(p, z\ x)$$

$$\Rightarrow \quad L(u+\varepsilon\eta) = L\left(D_j u + \varepsilon D_j\eta,\ u+\varepsilon\eta,\ x\right)$$

$$\frac{d^2}{d\varepsilon^2} I(u+\varepsilon\eta) = \boxed{\int \boxed{L_{p_j p_k}} D_j\eta\, D_k\eta + \boxed{2 L_{p_j z}} D_j\eta\, \eta + \boxed{L_{zz}}\,\eta^2 \geq 0}$$

We have a lot of freedom in chose η

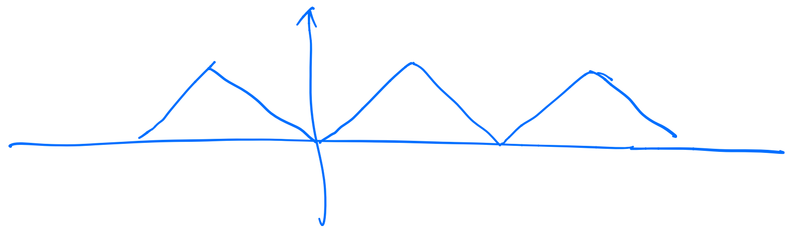
Now we let

$\xi \in \mathbb{R}^n$

$$\eta_\xi(x) = \delta \rho\left(\frac{x\cdot\xi}{\delta}\right)\theta(x)$$

$\delta > 0$

ρ :



$$\rho = \begin{cases} t & 0 \le t \le \frac{1}{2} \\ 1-t & \frac{1}{2} \le t \le 0 \end{cases}$$

$\rho(t+1) = \rho(t)$

$|\rho'|^2 = 1$ a.e

$$D_j \eta = \rho' \xi_j \theta + \delta \rho\left(\frac{x\cdot\xi}{\delta}\right) D_j\theta$$

$$\Rightarrow I''(0) = \boxed{\int L_{p_j p_k} (\rho')^2 \xi_j \xi_k \theta^2} + O(\delta)$$

$\delta \to 0$

e.g $L = |p|^2$

$$\Rightarrow \boxed{L_{p_j p_k} \xi_j \xi_k \ge 0 \quad \text{in } U}$$

$\ge a|\xi|^2$ uniform ellipticity

necessary condition for the minimizer.

In our discussion of

$$-D_j\left(L_{p_j}(Du; u, x)\right) + L_z = 0$$

(

$\downarrow$ Euler-Lagrange equation

We pose the restriction $\quad L_{p_i p_j} \xi_i \xi_j \geqslant \lambda |\xi|^2$

So-called the ellipticity assumption

The similar result holds for systems $\left(\begin{array}{l} u \text{ is vector} \\ \text{valued case} \end{array} \right)$

(B) Null Lagrangians & Applications

$L(p, u, x)$ is called Null-L if

E-L equation holds $\forall\, u$ $\quad \left(-\frac{\partial}{\partial x_j} (L_{p_j}) + L_z = 0 \right)$

(E.g.) ex1: $\underline{\Phi(u, x)}$ any function of $\mathbb{R}^n \times \mathbb{R}$, $n = 1$

$$\Psi \doteq \frac{d\Phi}{dx} = \Phi_{u^i} u^i_x + \Phi_x$$

$$= \underbrace{\Phi_{u^i} p^i + \Phi_x}$$

$$L(p\, u.\, x) \doteq \underbrace{\Phi_{u_j}(u) p^j + \Phi_x(u,x)}$$

Now $\quad -\boxed{(\Psi_{p_i})_x} + \Psi_{u^i}$

$$= -(\Phi_{u^i})_x + \underline{\Phi_{u^i u^j} p^j + \Phi_{x u^i}}$$

$$= -\left(\Phi_{u^i u^j} \frac{du^j}{dx} + \Phi_{u^i x} \right) + \qquad = 0$$

$\frac{du^j}{dx} = p^j$

Ex 2. $\underline{L(p, u, x)} = \sum a^i p_i \qquad m = 1$

$$\underline{-(L_{p_i})_x + \underset{0}{\underset{\parallel}{L_u}} = 0}$$

Ex 3. $L(p) = \det(p) \qquad P = (p_i^j)_{n\times n}$

$u: \mathbb{R}^n \to \mathbb{R}^n \qquad p = \left(\frac{\partial u^i}{\partial x^j}\right)$ — Jacobian matrix.

This requires some work:

$$\boxed{-\frac{\partial}{\partial x^j}\left(L_{p_j^k}\right) = 0 \quad \text{— need to prove}}$$

$A = (a_j^i) \qquad A_j^i$ — cofactor $\qquad u: \mathbb{R}^n \to \mathbb{R}^n$

$$\boxed{\sum_{l=1}^n a_i^l A_j^l = \delta_{ij} \det(A)} \leftarrow \text{Cramer's rule}$$

$\Rightarrow \quad L_{p_j^k} = A_j^k \qquad A = \left(\frac{\partial u^i}{\partial x_j}\right) \rightsquigarrow a_j^i = \frac{\partial u^i}{\partial x^j}$

$$\boxed{-\frac{\partial}{\partial x^j}\left(A_j^k\right)} \qquad \frac{\partial}{\partial x^j} \qquad \boxed{\det(A) = a_k^l A_k^l}$$

taking $\frac{\partial}{\partial x^j}$ & sum j

$$\frac{\partial}{\partial x^j}\left(\sum a_i^l A_j^l\right) = \frac{\partial}{\partial x^j}\left(\sum \delta_{ij} \boxed{\det(A)}\right)$$

$$\Rightarrow \quad \underbrace{u_{ij}^l A_j^l}_{\mathrm{I}} + \boxed{a_i^l (A_j^l)_j} = \delta_{ij} \boxed{A_k^l} \cdot \boxed{a_{kj}^l}$$

$$\boxed{a_i^l (A_j^l)_j} = 0 \qquad (A_j^l)_j = 0$$

$$\delta_{ij} A_k^l a_{kj}^l = \underbrace{A_k^l u_{x_k x_i}^l}_{\mathrm{I}}, \qquad a_{kj}^l = u^l_{x_k x_j}$$

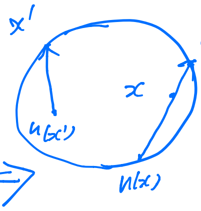
$$\boxed{\frac{d}{dt} f = \frac{\partial f}{\partial x^i} \frac{dx^i}{dt}}$$

04/01/2021

Application — Brouwer Fix Point Theorem:

— observation:

$\left\{ \begin{array}{l} \text{If } \forall x \in B \\ u(x) \neq x \end{array} \right. \Longrightarrow$



$\rightarrow w(x)$: $\quad w\big|_{\partial B} = \text{identity}$

$\rightarrow w: B \to \partial B$

— Assume w.L.G w is smooth — by approximation

w & id $\qquad w\big|_{\partial B} = id\big|_{\partial B}$

$$0 = \int_B \det(Dw)\, dx = \int_B \det(D(id)) = |B|$$

$\uparrow$ follows from the Lemma below.

$w: B \to \partial B$

$|w|^2 = 1$

$\Rightarrow D(w^T w) = 2w^T \cdot Dw = 0$

$w^T \cdot \left(\frac{\partial w^i}{\partial x_j}\right) = 0$

Lemma: If $u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega}$

$$\Rightarrow I(u_1) = \int_\Omega L(Du_1, u_1, x) = \int_\Omega L(Du_2, u_2, x) = I(u_2)$$

$u_t = t u_1 + (1-t) u_2 \qquad u_t\big|_{\partial\Omega} = u_1 = u_2$

$\quad = u_2 + (u_1 - u_2) t \qquad \eta = u_1 - u_2$

$$\Rightarrow I'(t) = \frac{d}{dt} \int_\Omega L(D(u_2 + t\eta), u_2 + t\eta, x)$$

$$= \int_\Omega L_{p_k^i} \eta^i_{x_k} + L_{u^k} \eta^k$$

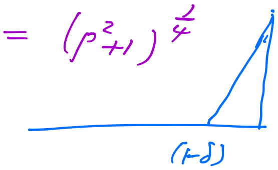
$$= \int_\Omega \Big[\underbrace{-\frac{\partial}{\partial x^k}\big(L_{p_k^i}\big)}_{u_t} + \underbrace{L_{u^i}}_{=0} \Big] \eta^i \equiv 0$$

$\Rightarrow I(0) = I(1)$

When & where? — Existence is not automatic

Ex 1: $$I(z) = \int_0^1 \underline{(1+z_x^2)}^{\frac14}\,dx \qquad \begin{cases} z(0)=0 \\ z(1)=1 \end{cases}$$

$$L(p) = (p^2+1)^{\frac14}$$



$z \in$ piecewise C' or $C' = \mathcal{A}$

$$z_\delta(x) = \begin{cases} 0 & 0 \le x \le 1-\delta \\ \frac{1}{\delta}\left(x-(1-\delta)\right), & 1-\delta \le x \le 1 \end{cases}$$

$$I(z_\delta) = 1-\delta + \int_{1-\delta}^{1} \left(1+\frac{1}{\delta^2}\right)^{\frac14}$$

$$= \underline{1-\delta + \delta^{\frac12}(1+\delta^2)^{\frac14}} \le \underline{1+\delta^{\frac12}}$$

$$\Longleftarrow \quad \underline{(1+\delta^2)^{\frac14} \le 1+\delta^{\frac12}}$$

$\Rightarrow \quad I(z_\delta) \longrightarrow 1 \quad$ but $\quad I(z) \ge 1 \quad \forall z \in \mathcal{A}$

One can write down $\tilde{z}_\delta \in C'[0,1]$ by modification.

Example shows $\boxed{\text{not always one can find a minimizer}}$

The sequence z_δ with $I(z_\delta) \longrightarrow \inf I(z)$ is called a "minimizing sequence"

Ex 2: $D(\varphi) := \int_\Omega |\nabla\varphi|^2$ $\qquad \varphi|_{\partial\Omega} = \psi$

Stands for Dirichlet

$\partial\Omega$ smooth. ψ smooth.

ψ extends to some smooth function on Ω

$W_0^{1,2}(\Omega)$ is the $\mathcal{A}$ we consider, namely

$$\varphi = \psi + \eta \qquad \eta \in W_0^{1,2}(\Omega)$$

The minimizing sequence will be φ_n satisfying

$$\varphi_n \in W^{1,2}(\Omega) \qquad \varphi_n|_{\partial\Omega} = \psi$$

$$\int |\nabla\varphi_n|^2 = D(\varphi_n) \longrightarrow \inf_{\varphi\in\mathcal{A}} D(\varphi)$$

Poincaré inequality

$$\varphi_n - \psi \in W_0^{1,2}(\Omega) \Rightarrow \int_\Omega |\varphi_n - \psi|^2 \leq C \int_\Omega |\nabla\varphi_n - \nabla\psi|^2$$

$\Rightarrow$ φ_n is bounded in $W^{1,2}(\Omega)$

$\Rightarrow \exists$ subsequence & $\varphi_\infty \in W^{1,2}(\Omega)$ $\quad \varphi_n \xrightarrow{\text{weakly}} \varphi_\infty$

$$\varphi_n - \psi \longrightarrow \varphi_\infty - \psi$$

In particular, since $\varphi_n - \psi \in W_0^{1,2}$ $\quad \varphi_n - \psi \xrightarrow{\text{by}} \varphi_\infty - \psi \Rightarrow \varphi_\infty|_{\partial\Omega} = \psi$

(Mazur theorem)

We need to show

(*) $\boxed{\liminf_{n\to\infty} D(\varphi_n) \geq D(\varphi_\infty)}$ to verify that φ_∞ is a minimizer found. (*) is the so-called Semi-lower continuity of the functional.

$\boxed{\text{This is relatively easy!}}$ for $D(\varphi)$ — read Jost's PDE (which you may download for free from Springer's website. Jost also has a good "Riem Geom & Geom. Analysis")

(We usually assume $L : \mathbb{R}^n \times \mathbb{R} \times \Omega$ smooth)

Theorem (existence) $U = \Omega$ - bounded, $\partial\Omega$ nice

Assumptions: $1 < q < \infty$

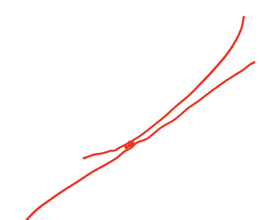
(1) $L(p, z, x) \geq \alpha |p|^q - \beta$ $\quad \alpha, \beta \in \mathbb{R}^+$

(2) $\underline{L(p,z,x) \text{ is convex in}}$ p — L dictates where

$$A = \{ u \in W^{1,q}(\Omega) \mid u|_{\partial\Omega} = f \}$$

Then $\exists\, u \in A$

$$I(u) = \inf_{w \in A} I(w)$$



$$L(p, z, x) \geq L(p_0, z, x) + L_p\big|_{p_0} \cdot (p - p_0)$$

Step 1: Reduction to the weak lower-semicontinuity.

Take $u_k \in A$ $\quad I(u_k) \longrightarrow \inf_{w\in A} I(w) \doteq m$

W.L.G $\quad \underline{L(p, z, x) \geqslant \alpha |p|^q}$ by adding a constant

$$\Downarrow$$

$$\alpha \int |Du_k|^q \leqslant I(u_k) \leqslant m + A.$$

$Du_k = \nabla u_k$

Pick $\underline{w_0 \in A}$ $\Rightarrow$ $\boxed{u_k - w_0 \in W_0^{1,q}(\Omega)}$

$$\Rightarrow \quad \int |u_k - w_0|^q \leqslant C_q \int |D(u_k - w_0)|^q$$

By the Poincaré inequality

$$\leqslant C\left[\frac{(m+A)}{\alpha} + B\right]$$

$$\Rightarrow \quad \|u_k\|_{L^q} \leqslant \|u_k - w_0\|_{L^q} + \|w_0\|_{L^q} \leqslant C$$

So $\{u_k\}$ forms a bounded sequence in $W^{1,q}(\Omega)$

$\Rightarrow$ By passing subsequence $\Rightarrow$ $\{u_k\} \longrightarrow u$ weakly in $W^{1,q}(\Omega)$

$\int u_k \varphi \longrightarrow \int u\varphi$ $\quad \varphi \in L^{q'}$ $\qquad \int \nabla u_k \cdot \psi \longrightarrow \int \nabla u \cdot \psi$ $\quad \psi$ v.c. $\in L^{q'}$

$\Rightarrow$ Lower semi-continuity asserts

$$\left\{ \underline{I(u) \leqslant \varliminf_{k\to\infty} I(u_k)} \right. \qquad (*)$$

$\Rightarrow$ $u - w_0 \in W_0^{1,q}(\Omega)$ By Mazur's theorem

$\Rightarrow \quad I(u) = m$

Hence the key is to prove $(*)$.

This needs the convexity!

Read Jost for the special case $L(p, z, x) = |p|^2$.

"(1) $L \geq \alpha |p|^q \geq 0$

(2) $u_k \rightharpoonup u$ weakly

(3) L convex in p

$\Rightarrow \quad \underline{\lim}_{k \to \infty} I(u_k) \geq I(u)$."

This is slightly more than what is needed.

$$\begin{cases} u_k \rightharpoonup u \text{ weakly} \iff \forall \varphi \in L^{q'}(U), \ \psi \in L^{q'}(U, \mathbb{R}^n) \\ \int (u_k - u)\varphi \to 0 \ \& \ \int (Du_k - Du)\cdot \psi \to 0 \end{cases}$$

$q' = \frac{q}{q-1} \iff \frac{1}{q'} + \frac{1}{q} = 1$

Uniform-boundedness-principle

$$\left.\begin{aligned} &\Rightarrow \sup_k \int |u_k|^q + |Du_k|^q \leq A \\ &\& \quad \int |u|^q + |Du|^q \leq A \end{aligned}\right\}$$

Shall show $\forall \varepsilon > 0 \ \exists G_\varepsilon \quad |U \setminus G_\varepsilon| < \gamma(\varepsilon) \to 0$

& G_ε increasing

$$\int_{G_\varepsilon} \underbrace{L(Du, u, x)}_{\chi_{G_\varepsilon}} \leq \underbrace{l := \underline{\lim}_{k \to \infty} I(u_k)}$$

(passing to a subsequence)

$u_k \longrightarrow u$ in L^q & a.e
$\forall \varepsilon \ \exists \ E_\varepsilon$ such that $|U \setminus E_\varepsilon| < \varepsilon$
& $u_k \longrightarrow u$ uniformly. 240

$$F_\varepsilon := \left\{ x \in U \;\middle|\; (|u| + |Du|)^{(x)} < \frac{1}{\varepsilon} \right\}$$

$$\Rightarrow \quad |F_\varepsilon^c| \cdot \left(\frac{1}{\varepsilon}\right)^q \leq A \quad \Rightarrow \quad \underline{|F_\varepsilon^c| \leq A\varepsilon^q} \longrightarrow 0$$

$$\left(\frac{1}{\varepsilon}\right)^q |F_\varepsilon^c| \leq \int_{F_\varepsilon^c} |Du|^q + |u|^q \leq A$$

$$\Rightarrow \quad \underline{G_\varepsilon = E_\varepsilon \cap F_\varepsilon} \qquad |G_\varepsilon^c| \leq A\varepsilon^q + \varepsilon$$

But

$$I(u_k) \geq \underline{\int_{G_\varepsilon} L(Du_k, u_k, x)}$$

$$\geq \int_{G_\varepsilon} L(\underline{Du}, u_k, x) + (D_p L)(Du, u_k, x) \cdot (\underline{Du_k - Du})$$

$$\underline{\text{1st}} \quad \lim_{k \to \infty} \int_{G_\varepsilon} \underbrace{L(Du, u_k, x)} = \underline{\int_{G_\varepsilon} L(Du, u, x)}$$

$\underline{u_k \to u}$ uniformly on G_ε $\qquad |\underline{L(Du, u_\varepsilon, x)}| \leq \underline{C(\varepsilon)}$

$$\Rightarrow \quad |u_\varepsilon| + |u| \leq \frac{3}{\varepsilon} \text{ — bounded}$$

$$|Du| \leq \frac{1}{\varepsilon}$$

DCT $\Rightarrow$ the claim

2nd $\displaystyle\int_{G_\varepsilon} \underbrace{(D_pL)(Du, u_k, x)}_{\psi} \cdot \underline{(Du_k - Du)} \longrightarrow 0$ as $k \to \infty$

$(D_pL)(Du, u, x) \in L^{q'}$

Since $|Du| + |u| \leq \frac{1}{\varepsilon}$ on G_ε

$\Rightarrow \displaystyle\lim_{k\to\infty} \int_{G_\varepsilon} \underbrace{\overset{\psi}{\overset{\parallel}{(D_pL)(Du, u, x)}}}_{\text{on } G_\varepsilon \text{ does not cause problem}} \cdot (Du_k - Du) = 0$

we can multiply by χ_{G_ε} to ψ.

On the other hand $\underset{\psi_k}{(D_pL)(Du_k, u_k, x)} \to \underset{\psi}{(D_pL)(Du, u, x)}$ uniformly

& bounded.

$\forall \delta > 0 \ \exists K$

$\Rightarrow k > K \qquad |\psi_k - \psi| < \delta$

$\Rightarrow \displaystyle\int_{G_\varepsilon} (\psi_k - \psi) \cdot (Du_k - Du)$

$\displaystyle\leq \delta \int_{G_\varepsilon} |Du_k - Du| \leq A'\delta$

Note $\displaystyle\int_U |Du_k|^q + |Du|^q < A$

Hence $\varliminf_{k\to\infty} I(u_k) \geq \int_{G_\varepsilon} L(Du, u, x)$

By MCT, $\Rightarrow I(u) = \int_U L(Du, u, x) \leq \varliminf_{k\to\infty} I(u_k)$