I shall give a proof of Evans' result for $F(\nabla^2 u) = 0$

case assuming ① F is uniformly elliptic. [Pogorelov Estimate ensures this!]

$\Leftrightarrow$ $\boxed{(F_{ij}) = \left(\frac{\partial F}{\partial a_{ij}}\right) \text{ is between } \lambda\, id \ \& \ \Lambda\, id}$

② F is concave. E.g. $\log\det(\nabla^2 u) = 0$ $\quad F_{ij} = (u^{ij})$

by Pogorelov's estimate: $\Rightarrow$ if $\lambda\, id \le \nabla^2 u \le \Lambda\, id$.

The proof can be found on page 454 – 456 of Gilbarg–Trudinger.

[Calabi's local C^3-estimate is equally good for the proof of JCP. Check Pogorelov's 1978 book. P. 37–41. Or CNS P379–382]

Evans' proof is conceptually involved. But essentially no computation. Calabi's on the other hand, is very computational.

The key is the Weak Harnack Estimate for sup-solution

Krylov–Safanov Estimate: $\quad L = a^{ij}\partial^2_{ij}$ $\quad$ [For the application of JCP. But $u \in C^2$. It can be weakened]

On $B_{2R}(0)$, $u \ge 0$ $\quad Lu \le 0$. $\quad \exists\, p > 0$ & C $\quad \lambda\, id \le (a^{ij}) \le \Lambda\, id$

$$\left(\fint_{B_R} u^p\right)^{\frac{1}{p}} \le C \left[\inf_{B_R} u\right]. \qquad \text{(KSE)}$$

Very small. [Can NOT use Minkowski inequ]

Plan: (1) Derive Evans result using the above

(2) Reduce the above to another estimate via Calderon–Zygmund decomposition

(3) Proof the reduced estimates

Ⅰ Evans result: $\quad F(\nabla^2 u) = 0$ on $\Omega \supset B_{2R_0}$

$$\operatorname*{osc}_{B_R} \nabla^2 u \le C\left(\frac{R}{R_0}\right)^{\alpha} \operatorname*{osc}_{B_{R_0}} \nabla^2 u \qquad R < R_0$$

This implies C^α-estimate easily

We only show $\quad \omega(R) \le \alpha\, \omega(2R)$. $\quad$ Lemma 8.23 of [GT]

Algebraic Lemma: [Lemma 17.13, Page 462]

$A = (a_{ij})$ Symmetric $\in S(\lambda, \Lambda)$ $\Leftrightarrow \lambda|\xi|^2 \leq a_{ij}\xi^i\xi^j \leq \Lambda|\xi|^2$

positive symmetric $(a_{ij}) \longrightarrow$

N large compare to n

$A = \sum_{k=1}^{N} \beta_k \gamma_k \otimes \gamma_k$ $\quad \gamma_k \in \mathbb{R}^n$ unit length $\quad N = N(\lambda, \Lambda)$

$\lambda^* \leq \beta_k \leq \Lambda^*$ with $\Lambda^* = \Lambda$ $\quad \lambda^* = \frac{\lambda}{2N}$ $\quad (\gamma_k \otimes \gamma_k)(x) = \langle x, \gamma_k \rangle \gamma_k$

$\Rightarrow \operatorname{tr}\left[\left(\beta_k \underbrace{\gamma \otimes \gamma}_{\parallel}\right) \cdot \nabla^2 u\right]$ (each one γ) $\quad \mathcal{L}u = a^{ij}u_{ij}$

$= \operatorname{tr}(\beta\, \gamma \cdot \gamma^{tr} \cdot \nabla^2 u) = \beta\, \gamma_i \gamma_j \nabla^2_{ij} u$

$= \beta \nabla^2 u(\gamma, \gamma)$

Hence $\underbrace{\operatorname{tr}(A \cdot \nabla^2 u)}_{Lu} = \sum_{k=1}^{N} \beta_k(y) \nabla^2 u(\gamma_k, \gamma_k)$ $\quad$ Motzkin-Wasow 1952

The point is to write Lu in terms of the sum of directional derivative. $\quad F(A) \leq F(A_0) + \langle \nabla F|_{A_0}, A - A_0 \rangle$

$$\underbrace{F(\underbrace{\nabla^2 u(y)}_{A}) \leq F(\underbrace{\nabla^2 u(x)}_{A_0}) + \langle \nabla F|_{A_{ij}, x}, \nabla^2 u(y) - \nabla^2 u(x) \rangle}$$

$\parallel$

a_{ij} – (For our case A^{ij}, the inverse)

If $F(\nabla^2 u) = 0 \Rightarrow$

$\langle \frac{\partial F}{\partial A_{ij}}(x), \nabla^2 u(y) - \nabla^2 u(x) \rangle \geq 0$ $\quad x \leftrightarrow y$

Or $\langle \frac{\partial F}{\partial A_{ij}}(y), \nabla^2 u(y) - \nabla^2 u(x) \rangle \leq 0$

$\Rightarrow \sum_{k=1}^{N} \beta_k(y) \left(\underbrace{\nabla^2_{\gamma_k\gamma_k} u(y)}_{W_k(y)} - \underbrace{\nabla^2_{\gamma_k\gamma_k} u(x)}_{W_k(x)}\right) \leq 0$ $\quad$ ①

This the step ①

$$M_{sk} = \sup_{B_{sR}} w_k \qquad m_{sk} = \inf_{B_{sR}} w_k$$

$$A^{ij} u_{ij,\gamma_k} = 0 \qquad L = A^{ij}\partial_{ij}$$

$$\Rightarrow A^{ij} u_{ij,\gamma_k\gamma_k} - A^{is} u_{st\gamma_k} A^{tj} u_{ij\gamma_k} = 0$$

$$L w_k \geq 0 \qquad \log\det(u_{ij}) = 0$$

$M_{2k} - w_k$ in B_{2R}. By KSE
$M_{2k} - w_k \geq 0$

$L(M_{2k} - w_k) \leq 0$

(2) $$\left[\fint_{B_R} (M_{2k} - w_k)^p \right]^{\frac{1}{p}} \leq C \inf_{B_R} (M_{2k} - w_k)$$

$$= C(M_{2k} - M_k) \leq C\left[(M_{2k} - m_{2k}) - (M_k - m_k) \right]$$

$$\omega_k(2R) \qquad \omega_k(R)$$

$$M_{2k} - m_{2k} - M_k + m_k + \underbrace{m_{2k} - m_k}_{\leq 0}$$

Let $$\omega(sR) = \sum_{k=1}^{N} (M_{sk} - m_{sk})$$

Now apply (1) $\Rightarrow$

$$\beta_k (w_k(y) - w_k(x)) \leq \sum_{\ell \neq k} -\beta_\ell (w_\ell(y) - w_\ell(x))$$

$$= \sum \beta_\ell (w_\ell(x) - w_\ell(y))$$

Taking $\sup_{x \in B_{2R}}$. Fix $y \in B_R$

$$\Rightarrow \lambda^* (w_k(y) - m_{2k}) \leq \Lambda^* \left[\sum_{\ell \neq k} M_{2\ell} - w_\ell(y) \right] \quad (\star)$$

$$\left(\fint_{B_R} (w_k(y) - m_{2k})^p dy \right)^{\frac{1}{p}} \leq \left(\frac{\Lambda^*}{\lambda^*}\right) \left[\fint_{B_R} \left(\sum_{\ell \neq k} (M_{2\ell} - w_\ell(y)) \right)^p \right]^{\frac{1}{p}}$$

$$\left(\sum f_\ell(x)\right)^p \leq N \left[\max_\ell f_\ell\right]^p \leq N\left(\sum f_\ell^p\right)$$

$p < 1$. We cannot use Minkowski ineq

But $$\left(\sum f_\ell\right)^p \leq N \sum f_\ell^p \qquad \leq \left(\frac{\Lambda^*}{\lambda^*}\right) N^{\frac{1}{p}} \left[\fint_{B_R} \sum (M_{2\ell} - w_\ell(y))^p \right]^{\frac{1}{p}}$$

$$\left(\sum_{\ell\neq k} a_\ell\right)^{\frac{1}{p}} \leq \left[\left(\sum_{\ell\neq k} a_\ell^{\frac{1}{p}}\right)^{p}\left(\sum 1^{q'}\right)^{\frac{1}{q'}}\right]^{\frac{1}{p}} \qquad a_\ell = \fint_{B_R}(M_{2\ell}-w_\ell(y))^p$$

$q = \frac{1}{p}$

$q' = \dfrac{\frac{1}{p}}{\frac{1}{p}-1} = \dfrac{1}{1-p}$

$$\leq \left(\sum a_\ell^{\frac{1}{p}}\right) N^{\frac{1-p}{p}}$$

$$\Rightarrow \left[\fint_{B_R}\underbrace{(w_k(y)-m_{2k})^p}\right]^{\frac{1}{p}} \leq \left(\frac{\Lambda^*}{\lambda^*}\right) \underbrace{N^{\frac{1}{p}} N^{\frac{1}{p}-1}}_{N^{\frac{2}{p}-1}} \left[\sum_{\ell\neq k}\underbrace{\left[\fint_{B_R}(M_{2\ell}-w_\ell(y))^p\right]^{\frac{1}{p}}}\right]$$

$\omega_{sk} := \operatorname{osc}_{B_{sR}} w_k$

Now use (2)

$$\leq C\underbrace{\left[\sum_{\ell\neq k}\omega_{2\ell}-\omega_\ell\right]}$$

$$\leq C\left[\omega(2R)-\omega(R)\right] \qquad (3)$$

On the other hand   By KSE & (2)

$$\left[\fint_{B_R}(M_{2k}-w_k)^p\right]^{\frac{1}{p}} \leq C\left[\omega_k(2R)-\omega_k(R)\right]$$

$$\left[\fint\left[(M_{2k}-w_k)+(w_k-m_{2k})\right]^p\right]^{\frac{1}{p}} \leq \left[\fint_{B_R}2\left[(M_{2k}-w_k)^p+(w_k-m_{2k})^p\right]\right]^{\frac{1}{p}}$$

$$\Rightarrow \boxed{\omega(2R)} = \left[\fint 1\right]^{\frac{1}{p}} \leq 2^{\frac{1}{p}}\left[\fint(M_{2k}-w_k)^p+\fint(w_k-m_{2k})^p\right]^{\frac{1}{p}}$$

$= M_{2k}-m_{2k}$

$= \omega_{2k} := \operatorname{osc}_{B_{2R}} w_k$

$$\leq 2^{\frac{1}{p}}\cdot 2^{\frac{1-p}{p}}\left(\left[\fint_{B_R}(M_{2k}-w_k)^p\right]^{\frac{1}{p}}+\left[\fint_{B_R}(w_k-m_{2k})^p\right]^{\frac{1}{p}}\right)$$

(2)   (3)

$$\leq \boxed{C\left[\omega(2R)-\omega(R)\right]}$$

$$\Rightarrow \boxed{C\omega(R) \leq (C-1)\,\omega(2R)} \Rightarrow \boxed{\omega(R) \leq \alpha\,\omega(2R)} \qquad (4)$$

$\alpha < 1$

Evans result follows by iteration. [Lemma 8.23 of [GT]]

Ⅱ KSE - proof via Calderon-Zygmund

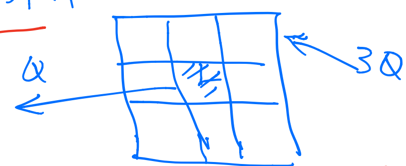
[My argument is different from [GT] §9.8]

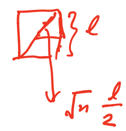
L - uniform elliptic

Theorem 1: $3Q \subset B_{\frac{3\sqrt{n}\ell}{2}}$ $Lu \le 0$, $u \ge 0$. $\exists M' > 1, 1 > \varepsilon' > 0$

If $\inf_{3Q} u \le 1$ then $|\{x \in Q \mid u(x) < M'\}| \ge \varepsilon' |Q|$

or equivalently $|\{x \in Q \mid u(x) \ge M'\}| \le (1-\varepsilon')|Q|$





Theorem 2: Q_0 $3Q_0 \subset \Omega$

$Lu \le 0$ & $u \ge 0$. $\exists M > 1, 1 > \varepsilon > 0$

$\inf_{3Q} u = 1$ $|\{x \in Q_0 \mid u(x) \ge M^k\}| \le (1-\varepsilon)^k |Q_0|$ $\forall k = 1, \dots$

First we show Theorem 2 $\Rightarrow$ KSE

[Folland p. 198]

$$\int |f|^p = p\int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha \qquad \forall\ 0 < p < \infty$$

Pick k, $\boxed{M^k < \alpha \le M^{k+1}}$ $\iff$ $k \log M < \log \alpha \le (k+1)\log M$

$\Rightarrow \boxed{k \ge \frac{\log\alpha - \log M}{\log M}}$

Theorem 2

$$\boxed{\lambda_u(\alpha)} \le P_{Q_0, M^k} |Q_0| \le (1-\varepsilon)^k |Q_0|$$

$\lambda_u(\alpha) = |\{u > \alpha\}|$, $\{u > \alpha\} \subset \{u > M^k\}$

$$= e^{k \log(1-\varepsilon)} |Q_0| \le e^{\frac{\log(1-\varepsilon)}{\log M}[\log\alpha - \log M]} |Q_0|$$

$$P^u_{\Omega, a} := \frac{|\{u \ge a\}|}{|\Omega|}$$

$$= C(M) |Q_0| \, \alpha^{\frac{\log(1-\varepsilon)}{\log M}} = C(M)|Q_0| \alpha^{-p_0}$$

for some $\infty > p_0 > 0$

$$\int_{Q_0} u^p \le p \int_0^1 \alpha^{p-1} |Q_0| + C \int_1^\infty \alpha^{p-1} |Q_0| \alpha^{-p_0}\, d\alpha$$

$$\le \left(1 + C\frac{1}{p_0 - p}\right)|Q_0| \qquad \text{for } p < p_0$$

$\Rightarrow$ $\fint_{Q_0} u^p \le C \cdot 1$ [Here we may assume $\inf_{3Q_0} u = 1$]

$1 = \inf_{3Q_0} u \le C \inf_{Q_0} u$

It is easy to replace Q_0 with B_R

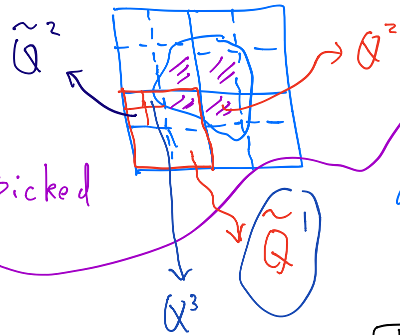
Theorem 1 $\xRightarrow{\text{C-Z-Decomposition}}$ Theorem 2

$\forall\, 0<\delta<1$

$A \subset Q$

$\boxed{|A| \leq \delta |Q|}$

Q^k — k level picked if



$A \subset \bigcup_{k=1}^{\infty} Q_k$ a.e

$|A \cap Q^k| > \delta |Q^k|$

and $|A \cap \tilde{Q}| \leq \delta |\tilde{Q}|$

$\tilde{Q}$ is the parent one

$\lim_{|Q^j| \to 0} \frac{1}{|Q^j|} \int_{Q^j} f(y) = f(x)$ a.e

Q^j shrinks to x $\quad f = \chi_A$

Lemma $\quad A \subset B \subset Q \qquad 0<\delta<1$

(i) $|A| \leq \delta |Q|$, (ii) if Q^i is a dyadic cube in the above with $|A \cap Q^i| > \delta |Q^i| \Rightarrow \tilde{Q}^i \subset B$, $\forall\, \tilde{Q}^i$, parent

Then $|A| \leq \delta |B|$.

Pf: Pick disjoint $\tilde{Q}^{i_k}$ such that $\bigcup \tilde{Q}^i = \bigcup \tilde{Q}^{i_k}$

$|A \cap \tilde{Q}^{i_k}| \leq \delta |\tilde{Q}^{i_k}|$

$A \subset \bigcup \tilde{Q}^i$ a.e.

$$|A| \leq \sum_{k=1}^{\infty} |A \cap \tilde{Q}^{i_k}| \leq \delta \sum_{k=1}^{\infty} |\tilde{Q}^{i_k}| \leq \delta |B| \qquad \square$$

Theorem 2. $k=1$ It is just Thm 1

Assume holds up to k. We show it also holds for $k+1$.

$A = \{x \in Q_0,\ u \geq M^{k+1}\}$ $\qquad A \subseteq B$

$B = \{x \in Q_0,\ u \geq M^k\}$

$|B| \leq (1-\varepsilon)^k |Q_0|$, want $|A| \leq (1-\varepsilon)^{k+1} |Q_0|$

Claim: $|A| \leq (1-\varepsilon)|B|$. by the above Lemma with $\delta = 1-\varepsilon$

Clearly (i) $|A| \leq (1-\varepsilon)|Q_0|$

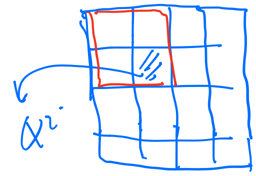
Only need to show (ii) $\tilde{Q}^i \subset B$ if $|Q^i \cap A| > (1-\varepsilon)|Q^i|$

If NOT $\boxed{\tilde{Q}^i \not\subset B} \Rightarrow \exists\ x \in \tilde{Q}^i \quad u(x) < M^k$

$\Rightarrow \boxed{\inf_{\tilde{Q}^i} \frac{u}{M^k} < 1} \Rightarrow$

$\left|\left\{x \in Q^i ; \frac{u(x)}{M^k} > M' \right\}\right| \leq (1-\varepsilon')|Q^i|$

$M' = M \qquad < (1-\varepsilon)|Q^i|$



$\boxed{\text{if } \varepsilon < \varepsilon'}$ $\quad\Updownarrow$

$\boxed{|A \cap Q^i| < (1-\varepsilon)|Q^i|}$

But $|A \cap Q^i| > (1-\varepsilon)|Q^i|$ by the way Q^i was picked.

Applying Lemma $\Rightarrow |A| \leq \delta |B| \Rightarrow$ Result. $\square$

$\delta = (1-\varepsilon)$

The rest is on proving Thm 1.

III Key density estimate

$\exists$ Cabré 1997 CPAM paper on Harnack estimate on M. $K_M \geq 0$

I have a paper — Kähler mfld. $\text{bi}K_M \geq 0$

[Contemp. Math., 367 2005, 149–165
ArXiv 2002]

$$P^u_{\Omega,\lambda} := \frac{|\{x \in \Omega \mid u(x) \geq \lambda\}|}{|\Omega|}.$$

$L = [a_{ij}]^2_{ij}$

$\lambda|\xi|^2 \leq a_{ij}\xi^i\xi^j \leq \Lambda|\xi|^2$

Density Lemma: $B_{2R}(x_0) \subset \Omega$, $Lu \leq 0$, $u \geq 0$. $u \in C^2(\Omega)$

Assume $\boxed{\inf_{B_R(x_0)} u \leq 1.}$ Then $\boxed{P_{B_{7R/4},\ \frac{33}{8} \cdot M_0} \leq 1-\varepsilon}$

for some $\varepsilon = \varepsilon(\lambda, \Lambda, n) > 0$.

Pf: It uses the idea of normal map in a clever way!

$y \in B_{R/4}(x_0)$. Consider $\boxed{F(x) = \frac{R^2}{4}u(x) + \frac{1}{2}|x-y|^2}$ Cabré's idea!

adding a moving quadratic

ob1: $\inf_{B_{2R}(x_0)} F = \inf_{B_{7R/4}(x_0)} F(x)$

The reason is: (i) $F(x) \geq \frac{I}{2}|x-y|^2 \geq \frac{1}{2}\cdot\frac{36}{16}R^2$ $\forall x \in B_{2R} \setminus B_{7R/4}$

(ii) $\inf_{B_R(x_0)} F(x) \leq \frac{R^2}{4} + \left(\frac{5}{4}R\right)^2 = \frac{33}{32}R^2$ $\left(\text{since } \inf_{B_R} u \leq 1\right)$.

ob2: Define $\mathcal{M} = \left\{ z \in B_{7R/4} \;\middle|\; F(z) = \inf_{y \in B_{R/4}} F(x), \right\}$

For such z $\underbrace{\frac{R^2}{4}\nabla u(z) + z - y = 0}$, & $\underbrace{\frac{R^2}{4}\nabla^2 u + id \geq 0}$

$\Rightarrow \boxed{y} = \underline{\frac{R^2}{4}\nabla u(z) + z}$ namely

$y \in \underbrace{\nabla\Big(\underbrace{\frac{R^2}{4}u + \frac{1}{2}|x|^2}_{\phi}\Big)}$ at z: or $y = \partial(\phi)(z)$.

Namely $\boxed{\partial\phi(\mathcal{M}) \supset B_{R/4}(x_0).}$

Combining maximum principle with the normal map

ob3 $\forall z \in \mathcal{M}$. we have

$$\frac{R^2}{4}u(z) + \frac{1}{2}|z-y|^2 \leq \frac{R^2}{4}u(x_1) + \frac{1}{2}|x_1-y|^2$$

$$\leq \frac{R^2}{4} + \frac{1}{2}\cdot\left(\frac{5}{4}R\right)^2 \Rightarrow$$

$u(x_1) = \inf_{x \in B_R} u$

$$\boxed{u(z)} \leq 1 + \frac{25}{8} = \frac{33}{8}.$$

$$\Rightarrow \mathcal{M} \subset \left\{ z \in B_{7R/4} \;\&\; \underline{u(z) \leq \frac{33}{8}} \right\}.$$

Now $\overset{\text{ob1-3}}{\Longrightarrow}$

$$\underline{|B_{7R/4}|} \le 7^n \underline{|B_{R/4}|} \le 7^n \int_{\Omega} \det(\nabla\phi)$$

$\underline{\partial\phi(m)}$

$$\le 7^n \int_{\{x\in B_{7R/4},\, u\le \frac{33}{8}\}} \boxed{\frac{\det(A(x))}{\det(A(x))} \det(\nabla\phi)}$$

$$\det(\underbrace{\tfrac{R^2}{4}\nabla^2 u + id}_{})$$

$$\le \frac{7^n}{\lambda^n}\frac{1}{n^n} \int_{\{x\in B_{7R/4},\, u\le\frac{33}{8}\}} \left[\mathrm{Tr}\left(\frac{R^2}{4}A\cdot\nabla^2 u\right) + \Lambda\, id\right]^n$$

$\underline{a^{ij}\nabla^2_{ij} u \le 0}$

$$\le \left(\frac{7\Lambda}{\lambda}\right)^n \underline{|\{x\in B_{7R/4},\, u\le \tfrac{33}{8}\}|}.$$

$$\left(\frac{7\Lambda}{\lambda}\right)^n \,\text{''}\, \frac{1}{\varepsilon}$$

Namely $\underbrace{|\{x\in B_{7R/4},\, u\le\frac{33}{8}\}|}_{} \ge \underbrace{\varepsilon}_{} |B_{7R/4}|$ $\square$

Clearly R is irrelevant. The proof can be done on a Riemannian mfld.

IV Helping Lemma & Putting Together.

Lemma 1: If $Lu\le 0$, $u\ge 0$ in $\Omega \supset B_{9R/4}(x_0)$ & $\inf_{B_R(x_0)} u \ge 1$

$\Longrightarrow$ $\inf_{B_{2R}(x_0)} u \ge \gamma > 0$ with $\gamma=\gamma(\lambda,\Lambda,n)$.

Another weak form of Harnack

Book [G] P44-47 Consider u^{ε} ← power of ε

Lemma 2: For M_0 & γ in Density Lemma, $\exists\, M_1$ such that

if $\inf_{B_{2R}(x_0)} u\le 1$, $P_{B_R,\,M_1} \le (1-\varepsilon)$

& if $\inf_{B_{2^kR}(x_0)} u\le 1 \Longrightarrow P_{B_R,\, M_0/\gamma^{k+1}} \le (1-\varepsilon)$

$M_1 = \frac{M_0}{\gamma^2}$

~ iterate the two results