(c) - first

$$u : \Omega \to S^{m-1} \subset \mathbb{R}^m \qquad I(u) \doteq \frac{1}{2}\int_\Omega |Du|^2$$

$$\mathcal{A} = \left\{ u \in H^1(\Omega, \mathbb{R}^m) \;\middle|\; \begin{array}{l} |u|^2 = 1, \text{ a.e} \\ u|_{\partial\Omega} = g : \to S^{m-1} \end{array} \right\}$$

Assume such set is NOT empty.

Theorem: Minimizer exists & satisfies weakly

$$\begin{cases} -\Delta u = |Du|^2 u, \\ u|_{\partial\Omega} = g \end{cases} \qquad \text{i.e.} \quad \int \langle Du, Dv\rangle \xrightarrow{\forall v \in H_0^1(\Omega, \mathbb{R}^m)} = \int |Du|^2 \langle u, v\rangle$$

We use the notation from the last lecture

$$\langle Du, Dv\rangle = \sum u^i_{x_j} v^i_{x_j} = \operatorname{trace}\left[(Dv)^t \cdot Du \right]$$

Proof: $L(p) = \frac{1}{2}|p|^2 \geq \alpha |p|^2 - \beta$

& convex. $\Rightarrow$ Minimizing sequence $\{u_k\}$

has $\|u_k\|^2_{H^1} \leq A$

$\Rightarrow \exists \; u_k \rightharpoonup u_\infty$

& $\quad I(u_\infty) \leq \liminf_{k\to\infty} I(u_k) = \inf_{v \in \mathcal{A}} I(v).$

$$u_k - u_1 \in H_0^1 \implies u_\infty - u_1 \in H_0^1$$

by Mazur's theorem (weak closeness)

Moreover it $\implies$ $u_\infty\big|_{\partial\Omega} = u_1\big|_{\partial\Omega} = g$

Since $|u_k|^2 = 1$ a.e

$\implies$ $|u_\infty|^2 = 1$ a.e $\implies$ Minimizer exists

Now we show it satisfies the Euler-Lagrangian with the multiplier $|\nabla u|^2$.

Assume u is the minimizer. Now $\forall v \in H_0^1(\Omega, \mathbb{R}^m)$
$v \in C_c^\infty(\Omega)$

$\underbrace{u+\tau v}\big|_{\partial\Omega} = g$ $\quad w(\tau) \doteq \dfrac{u+\tau v}{|u+\tau v|}$ has the property $|w|^2 = 1$ a.e

$u+\tau v \neq 0$ a.e

$|u+\tau v| \geq |u| - |\tau||v| = 1-|\tau||v| \geq \frac{1}{2}$ a.e

If we are very careful. $v \in C_c^\infty(\Omega)$ works

$$\nabla w = \frac{\nabla u + \tau \nabla v}{|u+\tau v|} - \frac{(u+\tau v)\langle \nabla u+\tau \nabla v,\, u+\tau v\rangle}{|u+\tau v|^3}$$

$$I(\tau) \doteq \frac{1}{2}\int |\nabla w(\tau)|^2$$

$$|\nabla w|^2 = \frac{|\nabla u+\tau\nabla v|^2}{|u+\tau v|^2} - \cancel{2}\frac{\langle \nabla u+\tau\nabla v,\, u+\tau v\rangle^2}{|u+\tau v|^4} + \cancel{\frac{\langle \nabla u+\tau\nabla v,\, u+\tau v\rangle^2}{|u+\tau v|^{6}}\, 4\, |u+\tau v|^{2}}$$

$I(0) = I(u)$ — the minimum.

$$0 = \frac{dI}{d\tau}\bigg|_{\tau=0} = \frac{1}{2}\int_\Omega \frac{d}{d\tau}\frac{|\nabla u+\tau\nabla v|^2}{|u+\tau v|^2}\bigg|_{\tau=0} = \int \frac{\langle \nabla u \;\; \nabla v\rangle}{|u|^2}$$

$$\Rightarrow \quad - \int \frac{|\nabla u|^2 \langle u \;\; v\rangle}{|u|^4}$$

$$\Rightarrow \quad 0 = \int_\Omega \langle \nabla u \;\; \nabla v\rangle - |\nabla u|^2 \langle u \;\; v\rangle \qquad \forall v \in C_c^\infty(\Omega, \mathbb{R}^3)$$

$$\Rightarrow \quad \begin{cases} -\Delta u = |\nabla u|^2 u & \text{weakly} \\ u|_{\partial\Omega} = g \end{cases} \qquad \square$$

---

d <u>Stokes problem</u> . $\left\{ \begin{array}{l} \to I(u) = \frac{1}{2}\int |\nabla u|^2 - \int f \cdot u \\ \underline{A = \{ u \in H_0^1(\Omega, \mathbb{R}^3) \mid \operatorname{div} u = 0 \ \text{a.e} \}} \end{array} \right\}$

<u>Theorem:</u> $\exists$ minimizer $u \in H_0^1(\Omega, \mathbb{R}^3)$

<u>& u satisfies:</u> $\exists\, p$ — a pressure function $\in L^2_{loc}(\Omega)$

$\forall v \quad \int \langle \nabla u, \nabla v\rangle = \int f \cdot v + p \cdot \operatorname{div} v \quad \forall v \in H_0^1(\Omega)$

(vector valued).

$$\& \quad \boxed{\begin{cases} -\Delta u = f - \nabla p \\ u|_{\partial\Omega} = 0 \end{cases}}$$

i.e. weakly

pressure is caused by the constrain A – requiring $\operatorname{div}(\cdot) = 0$

<u>Existence of the Minimizer:</u> As before, we may prove the existence of the minimizer since

$$L(p, z) = \frac{1}{2}|p|^2 - \underline{f \cdot z}$$

$\uparrow$ does not really cause any issue

$\{u_k\}$ is a minimizing sequence $\qquad \langle f, u_k \rangle \overset{\shortparallel}{=} f \cdot u_k$

$$\Rightarrow \quad I(u_k) \leq A$$

$$\Rightarrow \int_\Omega \left(\tfrac{1}{2}|\nabla u_k|^2 - f \cdot u_k \right) \leq A$$

$$\Rightarrow \quad \frac{1}{2}\int_\Omega |\nabla u_k|^2 \leq A + \int_\Omega f u_k$$

$$\leq A + \|f\|_{L^2} \|u_k\|_{L^2}$$

$$\leq A + \varepsilon \|u_k\|_{L^2}^2 + \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

$$\underset{\text{Poincaré ineq}}{\leq} A + \underbrace{\varepsilon \boxed{C}}_{= \frac{1}{4}} \|\nabla u_k\|_{L^2}^2 + \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

$$\Rightarrow \quad \frac{1}{4}\int |\nabla u_k|^2 \leq A(\Omega, f).$$

The rest is easy! $\left\{ \begin{array}{l} u_\infty \text{ is a minimizer} \\ \operatorname{div}(u_\infty) = 0, \ \text{a.e.} \end{array} \right.$

Now assume u is the minimizer then $\underline{\forall\, w \in \mathcal{A}}$

$w \in H_0^1(\Omega, \mathbb{R}^3)$

$\operatorname{div} w = 0$

$$\boxed{0 = \frac{d}{d\tau}\Big|_{\tau=0} \int \frac{1}{2}|\nabla(u + \tau w)|^2 - f(u + \tau w)}$$

$$\boxed{= \int_\Omega \langle \nabla u, \nabla w \rangle - f w} \quad \forall\, w \in H_0^1(\Omega, \mathbb{R}^3), \ \operatorname{div} w = 0$$

$\Rightarrow \forall w$ satisfies $\operatorname{div} w = 0$

$$\int_\Omega \langle \nabla u, \nabla w \rangle - f w = 0$$

curl ξ, $\forall \xi$

Formally

$$\boxed{\int_\Omega \langle -\Delta u - f, w \rangle = 0}$$

$\forall w$ with $\operatorname{div} w = 0$

dw

$\overline{\operatorname{div} w = 0}$ $\Longleftrightarrow$ $\downarrow$ use $H^2_{DR}(\Omega) = \{0\}$ $\exists \xi \quad \operatorname{curl}(\xi) = w$

$\left\{ \begin{array}{l} w_1\, dx^2 \wedge dx^3 \\ + w_2\, dx^3 \wedge dx^1 \\ + w_3\, dx^1 \wedge dx^2 \end{array} \right.$

$$\Rightarrow \int_\Omega \langle (-\Delta u - f), \operatorname{curl}(\xi) \rangle = 0$$

$\frac{\partial w_1}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3$

$\frac{\partial w_2}{\partial x^2} dx^2 \wedge dx^3 \wedge dx^1$

$+ \frac{\partial w_3}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2$

d - exterior differential

$$\Longleftrightarrow \int_\Omega \langle \underbrace{(-\Delta u - f)}_{\text{1-form}}, \underbrace{* d(\xi)}_{\text{1-form}} \rangle = 0$$

conjugate of d

$\boxed{d^* = * d * \text{ modulo a sign}}$

Viewed as 1-form

$*^2 = id$

$$0 = \int_\Omega \langle \underbrace{-\Delta u - f}_{\in \Omega^2}, \underbrace{d^*(*\xi)}_{\rightsquigarrow \Omega^2} \rangle$$

$\alpha = \alpha^1 dx^1 + \alpha^2 dx^2 + \alpha^3 dx^3$

Namely $$\int_\Omega \langle \underset{0}{\underset{\|}{\underline{d(-\Delta u - f)}}}, \underset{\forall \xi}{\boxed{*\xi}} \rangle = 0$$

$d\alpha = 0 \Leftrightarrow \alpha$ is d-closed

i.e. $\frac{\partial \alpha^1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \alpha^1}{\partial x^3} dx^3 \wedge dx^1$

$+ \ldots$

$\Rightarrow -\Delta u - f$ is d-closed

$\Rightarrow \exists p$ $\quad \boxed{-\Delta u - f = -dp}$

or equivalently $\quad \Delta u = Dp - f$.

The arguement in the book simply is to put the above out rigorously!

Hence, Hodge theory was used implicitly, with the L^2 estimate associated with it

Result of Hodge theory was formulated by Hodge, proved by H. Weyl, & K. Kodaira in 1949.

See Morrey Ch 6 for boundary value case.

First $\quad u \in H^1_0$

$$\int \underbrace{\langle \nabla u, \nabla v\rangle - \langle f, v\rangle}_{v_\varepsilon} = 0 \qquad \left\{ \begin{array}{l} \forall v \in C_0^\infty(\Omega, \mathbb{R}^3) \\ \boxed{\operatorname{div} v = 0} \end{array}\right.$$

$\Rightarrow$
$$\int \langle \nabla u_\varepsilon, \nabla v\rangle - \langle f_\varepsilon, v\rangle = 0 \qquad (\text{Apply to } v_\varepsilon)$$

$\Rightarrow$ $\quad -\Delta u_\varepsilon - f_\varepsilon$ does satisfies the above discussion

$\Rightarrow$ $\quad \exists P_\varepsilon$ such that

$$-\Delta u_\varepsilon - f_\varepsilon = -\nabla P_\varepsilon, \qquad \fint_\Omega P_\varepsilon = 0$$

$\Rightarrow$ Namely $\boxed{\int \langle \nabla u_\varepsilon, \nabla v\rangle - \langle f_\varepsilon\ v\rangle = -\int \langle \nabla p_\varepsilon,\ v\rangle \quad \int p_\varepsilon \operatorname{div} v}$ (*)

The idea is to get a subsequence out of $p_\varepsilon \to p$

This needs some estimate.

Pick p_ε so that $\int_\Omega p_\varepsilon = 0$

$\Rightarrow \exists\ v_\varepsilon$ such that $\begin{cases} \boxed{\operatorname{div} v_\varepsilon = p_\varepsilon} \\ v_\varepsilon \in H_0^1 \end{cases}$

Strictly speaking we need to solve a d-problem $d v_\varepsilon = p_\varepsilon$ v_ε - 2-form with Dirichlet boundary.

$\int p_\varepsilon = 0$ is to ensure p_ε is in the image

$\boxed{\begin{cases} \Delta \varphi_\varepsilon = p_\varepsilon \\ \varphi_\varepsilon \in H_0^2 \end{cases} \Rightarrow v_\varepsilon = \nabla \varphi_\varepsilon \text{ will do}}$

$v_\varepsilon \in H_0^1$ is NOT ensured!

$\exists$ estimate $\boxed{\|v_\varepsilon\|_{H^1} \leq C \|p_\varepsilon\|_{L^2}}$

Then by (*) $\|p_\varepsilon\|_{L^2}^2 = \int p_\varepsilon^2 = \int \operatorname{div} v_\varepsilon\ p_\varepsilon = -\int \langle v_\varepsilon\ \nabla p_\varepsilon\rangle$

Perhaps more subtle. See the paper of DM cited by the book

$= \int \underbrace{\langle \nabla u_\varepsilon, \nabla v_\varepsilon\rangle - \langle f_\varepsilon, v_\varepsilon\rangle}$

$\leq \|\nabla v_\varepsilon\|_{L^2} \|\nabla u_\varepsilon\|_{L^2} + \|f_\varepsilon\|_{L^2} \|v_\varepsilon\|_{L^2}$

$\leq \|v_\varepsilon\|_{H^1} \left(\|\nabla u_\varepsilon\|_{L^2} + \|f_\varepsilon\|_{L^2}\right) \leq C \|p_\varepsilon\|_{L^2} \left(\|\nabla u\|_{L^2} + \|f\|_{L^2}\right)$

$\Rightarrow \exists$ uniform up-bound of $\|p_\varepsilon\|_{L^2}$ $\Rightarrow \exists\ p$ as $\varepsilon \to 0$ $p_\varepsilon \to p$ in L^2

$\Rightarrow$ (*) Again $\int \langle \nabla u_\varepsilon, \nabla v\rangle - \langle f_\varepsilon\ v\rangle = -\int \langle \nabla p_\varepsilon\ v\rangle = \int p_\varepsilon \operatorname{div} v$ $\forall v \in H_0^1(\Omega)$

$\Rightarrow \int \langle \nabla u, \nabla v\rangle - \langle f\ v\rangle = -\int \langle \nabla p,\ v\rangle = \int p \operatorname{div} v$

$\square$

(b) Variational inequality.

$$\mathcal{A} = \{ u \in H_0^1(\Omega) \mid u \geq h \text{ a.e.} \}$$

$$I(u) = \int \tfrac{1}{2}|\nabla u|^2 - fu$$

Theorem: $\exists!$ minimizer u

Moreover u satisfies

$$\int_\Omega \langle \nabla u, \nabla(w-u) \rangle \geq \int_\Omega f \cdot (w-u) \qquad \forall w \in \mathcal{A}$$

Pf: $\mathcal{A}$ is convex. $L(p,z) = \underbrace{\tfrac{1}{2}|p|^2}_{L'(p)} - fz$ ↑ linear term does not matter much

Existence ~ exactly as the previous cases

If u_1, u_2 two minimizers

$v = \frac{u_1+u_2}{2}$ $\qquad v \geq h \qquad v \in \mathcal{A}$

$$I(v) = \int \tfrac{1}{2}\left|\nabla\left(\tfrac{u_1+u_2}{2}\right)\right|^2 - f\cdot\left(\tfrac{u_1+u_2}{2}\right)$$

$$= \int \tfrac{1}{2}\Big|\nabla\Big(\underbrace{u_1 + \tfrac{u_2-u_1}{2}}_{L'(\nabla v)}\Big)\Big|^2 - f\cdot\left(u_1 + \tfrac{u_2-u_1}{2}\right)$$

L is convex in p

$$\leq \int \underset{(1)}{\tfrac{1}{2}|\nabla u_1|^2} - \boxed{\underset{(2)}{L_p(\nabla v)\cdot\nabla\tfrac{(u_1-u_2)}{2}}} + \tfrac{1}{2}\int \underset{(3)}{\frac{|\nabla u_1 - \nabla u_2|^2}{4}}$$

Used $\underset{(1)}{L'}(\underbrace{\nabla v + \nabla\tfrac{u_1-u_2}{2}}_{=\nabla u_1}) \geq L'(\nabla v) + \underset{(2)}{L_p'(\nabla v)\cdot\nabla\tfrac{(u_1-u_2)}{2}} + \underset{(3)}{\tfrac{1}{2}\left|\tfrac{\nabla u_1 - \nabla u_2}{2}\right|^2}$

$$I(v) \leq \int \tfrac{1}{2}|\nabla u_2|^2 - \boxed{L_p(\nabla v)\,\nabla\tfrac{(u_2-u_1)}{2}} - f\left(u_2 + \tfrac{u_1-u_2}{2}\right) - \tfrac{1}{2}\int\left|\tfrac{\nabla u_1 - \nabla u_2}{2}\right|^2$$

$v + \frac{u_2-u_1}{2} = u_2$

$$\Rightarrow \boxed{2\, I(v) \le I(u_1) + I(u_2) - \frac{1}{4}\int |\nabla u_1 - \nabla u_2|^2}$$

$\underset{I(u_1)+I(u_2)}{\vee\!\!/}$

Namely uniqueness is the same as before!

Now $u + \tau(\underbrace{w-u}_{w-u \in H_0^1(\Omega)}) = (1-\tau)u + \tau w \ge h$ a.e

$$\Rightarrow \boxed{I(u+\tau(w-u)) \ge I(u)} = I(\tau)\Big|_{\tau=0}$$

$\tau \ge 0$

$$\Rightarrow I'(0)\Big|_{\tau=0}^{+} = \lim_{\substack{\tau\to 0\\ \tau>0}} \frac{I(\tau)-I(0)}{\tau} \ge 0$$

$\|$

$$\Rightarrow \boxed{\int \langle \nabla u, \nabla(w-u)\rangle - f\,(w-u)} \ge 0. \qquad \square$$

Next we study the PDE: (SL) $\begin{cases} -\Delta u = \underbrace{|u|^{p-1}u} \\ u|_{\partial\Omega} = 0 \end{cases}$

$n \ge 3$

for (1) slightly more general $f(u)$

$p > 1$

$\int (RHS)^{\frac{2^*}{p+1}} < +\infty$

Theorem (1) If $1 < p < \boxed{\frac{n+2}{n-2}}$ (SL) admits a non zero solution (H_0^1)

$p+1 < \frac{2n}{n-2} = \frac{2}{1-\frac{2}{n}} = 2^*$

$p^* = \frac{p}{1-\frac{p}{n}}$ $p<n$

$2^* = \frac{2}{1-\frac{2}{n}}$

$u \in H_0^1(\Omega) \Rightarrow u \in L^{2^*}$

(2) If Ω is star-shaped $C^1\ \partial\Omega$ $p > \frac{n+2}{n-2} \Rightarrow$ (SL) can only have zero solution (C^2-solution)

Ch 9.4

$|f(u)| \sim |u|^{p+1}$

by Sobolev $u \in L^{2^*} \Rightarrow |f(u)| \in L^{\frac{2^*}{p+1}}$

For the part (1) we need to use a method called Mountain pass.

Two gaps (i) $p = \frac{n+2}{n-2}$

(ii) smooth v.s. H_0^1

could be easy

We prove ② first since it is easier

$$-\Delta u = |u|^{p-1}u$$

Ⓐ $$\int (-\Delta u)\, u = \int |u|^{p-1} u^2$$

$$\Rightarrow \boxed{\int |\nabla u|^2 = \int |u|^{p+1}}$$

Ⓑ $$\int_\Omega (-\Delta u)\langle \nabla u, x\rangle = \int_\Omega |u|^{p-1}u\langle \nabla u, x\rangle$$

Claims

ⓘ $$\text{LHS} = \underbrace{\frac{2-n}{2}\int_\Omega |\nabla u|^2}_{①} - \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \langle x, \nu(x)\rangle$$

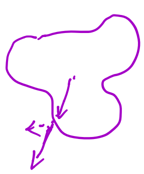
$\nu(x)$ exterior unit normal.

ⓘⓘ $$\text{RHS} = -\underbrace{\frac{n}{p+1}\int_\Omega |u|^{p+1}}_{②}$$

Result follows from combining claim

& $$\boxed{\int_{\partial\Omega} |\nabla u|^2 \langle x, \nu(x)\rangle \geq 0 \text{ if } \Omega \text{ is star-shaped}}$$

Then $$\boxed{\frac{2-n}{2}\int_\Omega |\nabla u|^2 \geq -\frac{n}{p+1}\int_\Omega |u|^{p+1}}$$



$$\frac{2-n}{2}\int |u|^{p+1} \Rightarrow \left(\int |u|^{p+1}\right)\left(\frac{n-2}{2} - \frac{n}{p+1}\right) \leq 0$$

$$\frac{1}{2(p+1)}\left((p+1)(n-2) - 2n\right)$$

$$= \frac{(n-2)}{2(p+1)}\left(p+1 - \frac{2n}{n-2}\right)$$

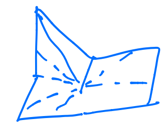
$$= \frac{n-2}{2(p+1)}\left(p - \frac{n+2}{n-2}\right)$$

$\Rightarrow$ If $n \geq 3$ & $p > \frac{n+2}{n-2}$

$\Rightarrow u \equiv 0$

$\langle x, \nu(x)\rangle \geq 0 \quad \forall \ x \in \partial\Omega$

Star-shaped



with respect to 0

$\forall x \in \Omega$

$\lambda x \in \Omega \quad \forall 0 \le \lambda \le 1$

Geometric clear.

We worry about the proof later!

(ii) is easy:

$$\int_\Omega |u|^{p-1}u\,\langle \nabla u, x\rangle = \int_\Omega |u|^{p-1}u\, u_{x_j} x_j$$

$$= \int_\Omega \left(\frac{|u|^{p+1}}{p+1}\right)_{x_j} x_j$$

$$= -\int_\Omega \frac{|u|^{p+1}}{p+1}\boxed{(x_j)_{x_j}} = -\frac{n}{p+1}\int |u|^{p+1}$$

(i)

$$\int_\Omega -\Delta u\,\langle \nabla u, x\rangle = \int_\Omega -u_{x_i x_i}\boxed{u_{x_j} x_j}$$

$$= \int_\Omega u_{x_i}\underbrace{(u_{x_j} x_j)_{x_i}}_{u_{x_j x_i} x_j + u_{x_j}\delta_{ij}} \quad - \int_{\partial\Omega} \langle u_{x_i}, \nu_i\rangle \boxed{u_{x_j} x_j}$$

ν – j component of ν exterior normal

$$= \int_\Omega \frac{1}{2}(u_{x_i}^2)_{x_j} x_j + \delta_{ij} u_{x_i}^2 + \int_{\partial\Omega} \pm |Du| \langle Du, x\rangle$$

$$\langle Du, x\rangle = \pm|Du|\,\langle \nu, x\rangle$$

$$= \int_\Omega |Du|^2 - \frac{n}{2}|Du|^2 + \frac{1}{2}\int_{\partial\Omega} u_{x_i}^2 \langle \nu_j x_j\rangle \qquad -\int_{\partial\Omega} |Du|^2 \langle \nu, x\rangle$$

$$\boxed{D_i u = \pm |Du|\,\nu_i}$$

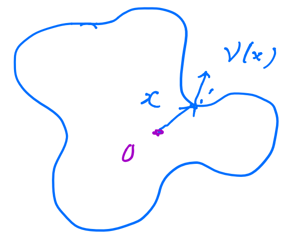
$\Rightarrow \langle u_{x_i}, \nu_i\rangle = \pm|Du|$

Putting together we have (ii)

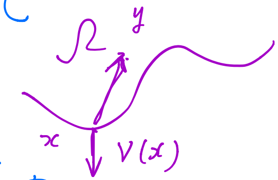
Lemma $\forall x \in \partial\Omega$

$$\left\langle \nu(x), \frac{x}{|x|} \right\rangle \geq 0$$

$$\Rightarrow \langle \nu(x), x \rangle \geq 0$$



Ω C^1

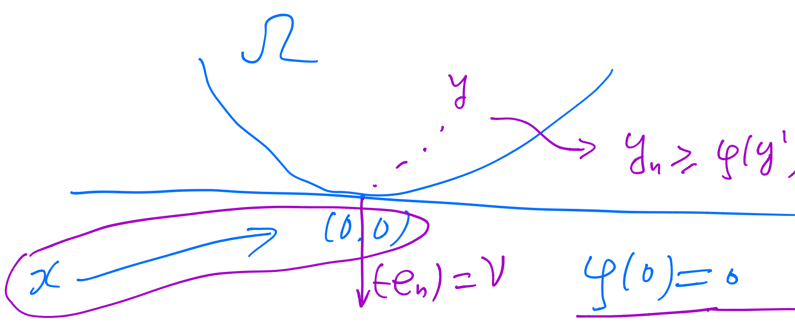


Proof:

Claim:

$$\boxed{\lim_{\substack{y \to x \\ y \in \Omega}} \left\langle \nu(x), \frac{y-x}{|y-x|} \right\rangle} \leq 0 \quad (*)$$

near x — change of coordinates



locally

$y_n \geq \varphi(y')$ $\quad \partial\Omega = \{x_n = \varphi(x')\}$

$\nu(x) = \nu(0,0) = -e_n$

$\varphi(0) = 0 \quad \nabla\varphi(0) = 0$

$$\lim \left\langle \underset{-e_n}{\underset{\parallel}{\nu(x)}}, \underset{y/|y|}{\underset{\parallel}{\frac{y-x}{|y-x|}}} \right\rangle = \lim_{y \to 0} \frac{-y_n}{|y|} \leq \lim_{y \to 0} \frac{-\varphi(y')}{|y|}$$

$$y \in \bar{\Omega} \iff y_n \geq \varphi(y') \qquad \lim_{y \to 0} \frac{-\varphi(y')/|y'|}{|y|/|y'|} = 0$$

Now $\left\langle \nu(x), \frac{x}{|x|} \right\rangle = -\boxed{\left\langle \nu(x), \frac{\lambda x - x}{|\lambda x - x|} \right\rangle}$ $\quad \lambda < 1$

$y = \lambda x \in \Omega \quad \lambda \to 1 \quad y \to x \quad \Rightarrow \lim_{\lambda \to 1^-} \text{RHS} \geq 0$ $\square$