Now we study a slightly more general problem:
$$\begin{cases} -\Delta u = f(u) \\ u|_{\partial\Omega} = 0 \end{cases} \qquad (SL)$$

① $|f(z)| \le c(1+|z|^p)$ — basic $\qquad 1 < p < \frac{n+2}{n-2}$

② $|f'(z)| \le c(1+|z|^{p-1})$ Lipschitz condition

Needed in checking & (PS)

③ $0 \le F(z) \le \boxed{\gamma} f(z) z$ $\qquad \boxed{\gamma < \frac{1}{2}}$ ~ Check for the special case

$F(z) = \int_0^z f(s)\,ds$, $\qquad a|z|^{p+1} \le \boxed{F(z) \le A|z|^{p+1}}$ $\Downarrow$ $f(0)=0$

Needed for (PS)

e.g. $\qquad f(z) = |z|^{p-1} z$

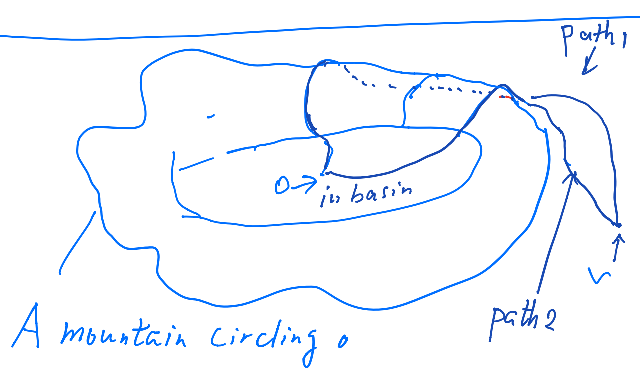
$$f'(z) = \begin{cases} p z^{p-1} & z > 0 \\ 0 & z = 0 \\ -[(-z)^p]' = p(-z)^{p-1} & z < 0 \end{cases}$$

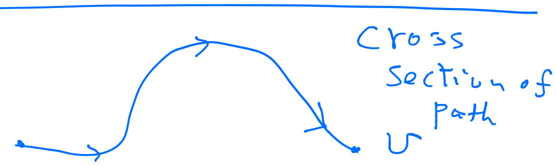
$$F(z) = \begin{cases} \frac{1}{p+1} z^{p+1} & z > 0 \\ 0 & z = 0 \\ \boxed{\frac{1}{p+1}} |z|^{p+1} & z < 0 \end{cases}$$

$\gamma = \frac{1}{p+1}$

$f(z)z = |z|^{p-1} z \cdot z = |z|^{p+1}$

Theorem (SL) admits a $u \in H_0^1(\Omega)$ $u \ne 0$.





$0 \in H$ — a Hilbert space

$H_0^1(\Omega)$ in our case

$$\begin{cases} ① \; I(0)=0, \quad ② \; I(u) \geq \alpha > 0, \ \forall \ \|u\|=r, \text{ for some } r>0 \\ \& \; ③ \; \exists \ v, \ \|v\|>r, \ I(v) \leq 0 \end{cases}$$

Namely 0 is circled by a Mountain with positive height ($I(u) \geq \alpha > 0$)

Define the paths

$$\Gamma := \left\{ g : C([0,1], H) \;\middle|\; \begin{matrix} g(0)=0 \\ g(1)=v \end{matrix} \right\}$$

Then $$c = \inf_{g \in \Gamma} \boxed{\max_{0 \leq t \leq 1} I(g(t))} \geq \alpha$$

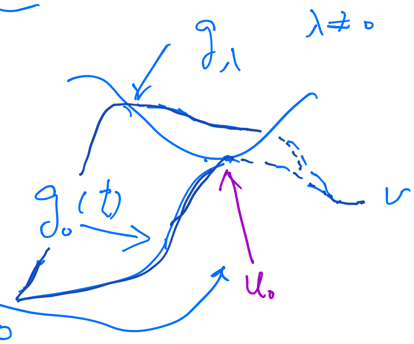
is a critical value of I

Namely $\exists \ u_0 \in H \qquad I(u_0)=c, \qquad I'(u_0)=0$

$\lambda \neq 0$

Intuitively this is true

u_0 is a "saddle point"



$I'(u_0)=0 \iff u_0$ Satisfies the PDE, — Euler Lagrangian equation of I

$c \geq \alpha \implies I(u_0) \neq 0$

$\overset{\shortparallel}{I(u_0)}$ & $I(0)=0$

Hence $u_0 \neq 0$

In our case

$$I(u) \doteqdot \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$$

$$I'(u_0) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega \frac{1}{2}\left|\nabla(u_0+\varepsilon\eta)\right|^2 - F(u_0+\varepsilon\eta)$$

$$= \int_\Omega \langle \nabla u_0, \nabla\eta\rangle - f(u_0)\,\eta$$

Hence $0 = I'(u_0) \iff \begin{cases} -\Delta u_0 = f(u_0) & \text{weakly} \\ u_0|_{\partial\Omega} = 0 \end{cases}$

It is easy to check Conditions: 1, 2, 3

if we choose a good norm on $H_0^1(\Omega)$

$$\boxed{\|u\|^2 := \int_\Omega |\nabla u|^2}$$

$H_0^1(\Omega) = H$ — a Hilbert space.

$$(u, v) = \int_\Omega \langle \nabla u, \nabla v\rangle$$

By Poincaré inequality this is a norm equivalent to the standard norm $\|u\|^2_{H^1} = \boxed{\int_\Omega u^2 + |\nabla u|^2} \leq C\int_\Omega |\nabla u|^2$

$$\|u\|^2 = r^2 \iff \int |Du|^2 = r^2$$

$$\underbrace{I(u) = \frac{1}{2} r^2 - \boxed{\int F(u)}} \quad \text{on } \|u\| = r$$

$|F(u)| \leq C|u|^{p+1}$

But
$$\left| \int_\Omega \underline{F(u)} \right| \leq \underset{\substack{\| \\ C(F)}}{C} \left(\int_\Omega |u|^{p+1} \right) \text{ by the assumptions}$$

$$\underbrace{\leq C \left(\int_\Omega |u|^{2^*} \right)^{\frac{p+1}{2^*}}}_{\text{monotonicity of } L^p} \leq C \left(\int_\Omega |Du|^2 \right)^{\frac{p+1}{2}}$$

$$= \|u\|^{p+1}$$

Sobolev embedding!

$$\boxed{\begin{gathered} 1 < p < \frac{n+2}{n-2} \\ \iff \\ 2 < p+1 < 2^* = \frac{2n}{n-2} \end{gathered}}$$

$$2^* = \frac{2}{1 - \frac{2}{n}} = \frac{2n}{n-2} > p+1$$

Here we use

(key)
$$\boxed{\left(\int_\Omega |u|^{2^*} \right)^{\frac{1}{2^*}} \leq C \left(\int_\Omega |Du|^2 \right)^{\frac{1}{2}}}$$

for $u \in H_0^1(\Omega)$. Namely the Sobolev inequality. Thm 3 of Ch 5.6.

$\forall u, \|u\| = r$

$$\Rightarrow \boxed{I(u) \geq \frac{1}{2} r^2 - C r^{\underline{p+1}}} = \frac{1}{2} r^2 \Big(\underbrace{1 - \boxed{C r^{p-1}}}_{\geq \frac{1}{2}} \Big)$$

$\leq \frac{1}{2}$

$$\geq \frac{1}{4} r^2 \doteq \alpha > 0$$

$\underline{p+1 > 2} \Rightarrow \underline{\text{For } r \ll 1}$

Moreover pick $u_0 \neq 0$ $u_0 \in H_0^1(\Omega)$

Consider

$$I(\lambda u_0) = \lambda^2 \frac{1}{2}\int_\Omega |\nabla u_0|^2 - \int_\Omega F(\lambda u_0)$$

$$= \alpha_1 \lambda^2 - \int_\Omega F(\lambda u_0)$$

$$\underbrace{F(\lambda u_0) \geq a|\lambda u_0|^{p+1}}= a\lambda^{p+1}|u_0|^{p+1}$$

$$-\int_\Omega F(\lambda u_0) \leq -a\lambda^{p+1}\int_\Omega |u_0|^{p+1}$$

$$\Rightarrow \boxed{I(\lambda u_0) \leq \alpha_1\lambda^2 - \alpha_2 \lambda^{p+1}} \to -\infty$$

as $\lambda \to +\infty$

Hence $\exists v$ $I(v) < 0$

Mountain Pass can be applied to get a $u_0 \neq 0$

u_0 is a critical point of I.

Unfortunately it has several other conditions to be checked for Mountain Pass principle.

① $I \in C^1$ [In fact the book only prove MPT for I' being locally Lipschitz (Thm 2 of Ch 8.5)

② H, I need to satisfy the so-called Palais-Smale condition. (to be explained)

Theorem (Mountain Pass Theorem). Assum: $I: H \to \mathbb{R}$

is C^1 & $I'(u)$ is Lipschitz — locally

(i) $I(0) = 0$

(ii) $\exists\, r, a > 0$ $\quad I(u) \geq a \quad \forall\, \|u\| = r$

(iii) $\exists\, v \in H$ $\quad \|v\| > r$ & $I(v) \leq 0$

$I(u)$ satisfies (PS) Condition

Then $c = \inf\limits_{g \in \Gamma} \max\limits_{0 \leq t \leq 1} I(g(t))$ is a critical value.

The plan is to explain these conditions and verify them for our case considered.

Then we look into MPT & its proof.

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u) = I_1(u) - I_2(u).$$

$I'(u)$ is defined to satisfy:

$$\boxed{I(w) = \underbrace{I(u) + \boxed{\langle I'(u), w-u \rangle}}_{\text{Linear approximation at } u} + o(\|w-u\|)} \quad \text{— definition}$$

(As in the finite dimension case) $\qquad \forall\ \|w-u\|$ small

$$I_1(u) = \frac{1}{2}\|u\|^2$$

$$I_2(u) \doteq + \int_\Omega F(u) \qquad I = I_1 - I_2$$

$$I_1(u+\underset{w'}{\overset{t}{w}}) = \frac{1}{2}\|u+w\|^2 = \frac{1}{2}\left(\|u\|^2 + 2\langle u, w\rangle + \|w\|^2\right)$$

$$= I_1(u) + \underbrace{\langle \boxed{u}, w'-u\rangle}_{= t\langle u, w\rangle} + \underbrace{\frac{1}{2}\|w'-u\|^2}_{\sim o(\|w'-u\|)}$$

$$\Rightarrow \underbrace{I_1'(u) = u}$$

Clearly it is Lipschitz (locally)

differential of I_1 at u

$$\underbrace{I_2(u) = \int_\Omega F(u)}_{+ t\eta}$$

$$\boxed{(I_2'(u), \eta) = \int_\Omega f(u)\,\eta}$$

$S(f(u))$

Here we use linear PDE result. $\int_\Omega \langle \nabla v, \nabla \eta\rangle = \varphi(\eta)$

$$\left\{\begin{array}{l} \Delta v = \boxed{\varphi} \\ v|_{\partial\Omega} = 0 \end{array}\right. \quad \text{is solvable } \forall \varphi \in H^{-1}(\Omega)$$

$v = S(\varphi)$ the dual of $H_0^1(\Omega)$

Observation 1: $|f(u)| \le A(1+|u|^p) \Rightarrow f(u) \in H^{-1}(\Omega)$

$$\Rightarrow \left|\int_\Omega f(u)\cdot \eta\right| \le \|f(u)\|_{q'} \|\eta\|_{2^* = q}$$

$\forall \eta \in H_0^1(\Omega)$

$u \in L^{2^*}$

$2^* = \frac{2n}{n-2}$

$\frac{1}{q} + \frac{1}{q'} = 1$

$q' = \frac{q}{q-1}$

$q' = \frac{2n}{n+2}$

$u \in H_0^1(\Omega) \Downarrow u \in L^{2^*}(\Omega)$

$$\Rightarrow \left(\int |f(u)|^{q'}\right) \le \int A(1+|u|^p)^{q'}$$

$$\le A' + A'\int_\Omega |u|^{p\cdot q'} < +\infty$$

$$p\cdot q' < \frac{n+2}{n-2}\cdot\frac{2n}{n+2} = \frac{2n}{n-2} = 2^*$$

Namely $f(u) \in H^{-1}(\Omega)$. $\boxed{\|f(u)\|_{H^{-1}} \le \|f(u)\|_{q'}}$ (H-1)

We need an improved version later

Observation 2 $v = S(\varphi)$ is obtained by Riesz Representation

$\exists! \tilde{u}$ $\quad (\tilde{u}, \eta) = \varphi(\eta) \quad \forall \eta \in H_0^1(\Omega)$

& $\quad \|\tilde{u}\| = \|\varphi\|$ linear functional norm

inner product on $H_0^1(\Omega)$ — using the norm $\int |\nabla \tilde{u}|^2 = \|\tilde{u}\|^2$

i.e

$$\int_\Omega \langle \nabla \tilde{u}, \nabla \eta \rangle = \varphi(\eta)$$

In case φ given by $f(u)$ $\quad \tilde{u} = \underline{S}(\underline{f(u)})$ satisfies

$$\boxed{\int \nabla \underbrace{S}(f(u)) \cdot \nabla \eta = \underbrace{\int f(u)\, \eta}_{= I_2'(u)}} \quad \forall \eta \in H_0^1(\Omega)$$

Now we use a calculus result

$$\left\{ \begin{aligned} \underline{F(a+b) - F(a)} &= \int_0^1 \underline{\tfrac{d}{ds} F(a+sb)}\, ds \\ &= -\int_0^1 \boxed{b\, f(a+sb)}\, d\boxed{(1-s)} \\ &= \underline{b\, f(a)} + \boxed{b^2 \int_0^1 (1-s)\, f'(a+sb)\, ds} \end{aligned} \right.$$

$$I_2(\underbrace{u+w}_{= w'}) = \int_\Omega F(u+w) = \int_\Omega F(u) + \boxed{w\, f(u)}$$

$w = (w'-u)$

$$+ \underbrace{\int_\Omega w^2 \int_0^1 (1-s)\, f'(a+sb)\, ds}_{= E} = I_2(u) + \boxed{\int \nabla S(f(u)) \cdot \nabla(w'-u) + E}$$

$$\int_\Omega \nabla S(f(u))\cdot \nabla(\underbrace{w'-u}_{\eta\in H_0^1}) = \int_\Omega f(u)\cdot(\underbrace{w'-u}_{\eta\in H_0^1(\Omega)}) \quad \leftarrow (2) \text{ above}$$

$$|E| \leq \int_\Omega A\,\underline{|w|^2}\,(\underbrace{1+|u|^{p-1}+|w|^{p-1}})$$

$$2^* = \frac{2}{1-\frac{2}{n}}$$

$$A\int_\Omega |w|^2 \leq A'\,\|\underbrace{w'-u}_{w}\|^2 \quad \text{trivially}$$

$$A\int |w|^2|u|^{p-1} \leq A\left(\|w\|_{2^*}\right)^{\frac{1}{q}\cdot 2^*} \left(\int |u|^{(p-1)\cdot q'}\right)^{\frac{1}{q'}}$$

$$\|w\|_{2^*} \leq \|w\|$$

$$2^* = \frac{2}{1-\frac{2}{n}} = 2\cdot q \qquad q = \frac{n}{n-2}$$

$$q' = \frac{\frac{n}{n-2}}{\frac{2}{n-2}} = \boxed{\frac{n}{2}}$$

$$\leq C A\,\|w\|^{\frac{2^*}{q} = 2}$$

only need to be finite

$$\underbrace{(p-1)\cdot\frac{n}{2}} < \left(\boxed{\frac{n+2}{n-2}}-1\right)\cdot\frac{n}{2} = \frac{4}{n-2}\cdot\frac{n}{2} = \frac{2n}{n-2} = 2^*$$

$$\Rightarrow \quad A\int |w|^2|u|^{p-1} \leq A'_u\,\|w'-u\|^2$$

Finally

$$A\int_\Omega |w|^{p+1} \leq A'\int_\Omega |w|^{2^*} \leq A'C\,\|w\|^{\boxed{2^*}} \leq A'C\,\|w'-u\|^2 \quad \text{as } \|w'-u\|\to 0$$

Namely $\boxed{E = O\left(\|w'-u\|^2\right) = o\left(\|w'-u\|\right)}$

This proves $\underbrace{I_2'(u) = S(f(u))}$.

Finally the Lipschitz part of $I' = I_1' - I_2'$

$$\| I_2'(u_1) - I_2'(u_2) \| = \| S(f(u_1)) - S(f(u_2)) \|$$

$$\leq \| f(u_1) - f(u_2) \|_{H^{-1}(\Omega)}$$

$$\leq C \| f(u_1) - f(u_2) \|_{L^{q'}} \quad (1)$$

$q' = \frac{2n}{n+2}$

$L^{q'} \supset H^{-1}(\Omega)$

$$\left(\int_\Omega |f(u_1) - f(u_2)|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \qquad \| f(u_1) - f(u_2) \|_{L^{q'}}$$

$q' = \frac{2n}{n+2}$

$$\leq \int_\Omega \left| \int_0^1 \frac{d}{ds} \big(f(s u_1 + (1-s) u_2) \big) \right|^{\frac{2n}{n+2}}$$

$$\leq \int_\Omega A \left(1 + |u_1|^{p-1} + |u_2|^{p-1} \right) |(u_1 - u_2)|^{\frac{2n}{n+2}} \sim L \quad (2)$$

Lipschitz can be checked via Hölder inequality &

$$\int |u_1 - u_2|^{\frac{2n}{n+2} \cdot \frac{n+2}{n-2}} = \left(\int |u_1 - u_2|^{2^*} \right)$$

$\frac{n+2}{n-2} = q_1$, $\quad q_1' = \frac{n+2}{4}$

$$\Rightarrow L^{\frac{n+2}{2n}} \leq \left(\| u_1 - u_2 \|_{L^{2^*}} \right)^{\frac{2^*}{q_1} \cdot \frac{n+2}{2n}} \left[\int \left(1 + |u_1|^{p-1} + |u_2|^{p-1} \right)^{\frac{n}{2}} \right]^{\frac{n+2}{2n} \cdot \frac{4}{n+2}} \quad (3)$$

$\frac{2n}{n+2} \cdot \frac{n+2}{4}$

$(p-1)\frac{n}{2}$

$$< \left(\frac{n+2}{n-2} - 1 \right) \cdot \frac{n}{2} = \frac{2n}{n-2} = \boxed{2^*}$$

$$\frac{2^*}{\frac{n+2}{n-2}} \cdot \frac{n+2}{2n} = \frac{\frac{\cancel{2n}}{n-\cancel{2}}}{\frac{\cancel{n+2}}{n-\cancel{2}}} \cdot \frac{\cancel{n+2}}{\cancel{2n}} = 1$$

Putting ①-③ together $\Rightarrow$

$$\|S(f(u_1))-S(f(u_2))\| \leq C\|f(u_1)-f(u_2)\| \leq C\|u_1-u_2\|_{L^{2^*}} \leq C\|u_1-u_2\| \quad (\ast)$$

$C(u_1,u_2)$

This proves the Local Lipschitz of $I_2'(u)$, hence $I'(u)$

· Palais-Smale Condition: $I(u)$ satisfies (PS):

if $\{u_k\}$ satisfies $\{I(u_k)\}$ is bounded ①

& $I'(u_k) \to 0$, then $\{u_k\}$ is precompact

$u_k - S(f(u_k))$ ②

②+③ $\Rightarrow$

$$\|f(u_1)-f(u_2)\|_{L^{\frac{2n}{n+2}}} \leq C\|u_1-u_2\|_{L^{2^*}} \Phi\left(\max\left(\|u_1\|, \|u_2\|\right)\right)$$

Or $\|f(u_1)-f(u_2)\|_{H^{-1}} \leq$ same estimate

$$\Rightarrow \|S(f(u_1))-S(f(u_2))\| \leq C\|u_1-u_2\|_{L^{2^*}} \Phi(\max\|u_i\|) \quad (\ast)$$

Some function Φ

Under the assumption

$$\boxed{I'(u_k) = u_k - S(f(u_k))} \to 0$$

$= I_2'(u_k)$

using $(\ast)$ and the Compactness of $L^{2^*} \hookrightarrow H_0^1 \Rightarrow$

$\|S(f(u_k)) - S(f(u_\ell))\|$

$\leq C(\sup\|u_k\|)$

$\|u_k - u_\ell\|_{L^{2^*}}$

which $\to 0$

This implies

$\{u_k\}$ is convergent

(After passing to a subsequence)

Key step: $\boxed{\{\|u_k\|\} \text{ is bounded.}}$

$\gamma < \frac{1}{2}$ plays a crucial role here!

Here we assumed $\{I(u_k)\}$ is bounded

i.e. $\left|\int \frac{1}{2}|Du_k|^2 - F(u_k)\right|$ is bounded

$\&$ $\quad \overset{J'(u_k)=}{u_k - S(f(u_k))} \longrightarrow 0 \quad$ in $H_0^1(\Omega)$

which is very strong

$$\Rightarrow \quad \left|\left(\boxed{u_k - S(f(u_k))}\,,\ u_k\right)\right| \le \varepsilon \|u_k\| \qquad k \gg 1$$

↑ by Cauchy-Schwarz

$$\left| \|u_k\|^2 - \int_\Omega f(u_k)\, u_k \right| \le \varepsilon \|u_k\|$$

$$\Rightarrow \quad \int_\Omega f(u_k)\, u_k \le \|u_k\|^2 + \varepsilon \|u_k\| \qquad (A)$$

On the other hand

$$\boxed{|I(u_k)| \le A} \Rightarrow$$

$$\Rightarrow \quad \frac{1}{2}\|u_k\|^2 - \int_\Omega F(u_k) \le A \qquad F(z) \le \gamma z f(z)$$

$$\Rightarrow \quad \|u_k\|^2 \le 2\int_\Omega \gamma\, u_k f(u_k) + A$$

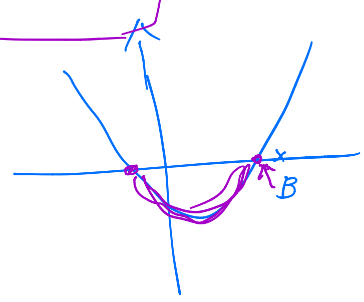
$$= 2\gamma\left(\|u_k\|^2 + \varepsilon \|u_k\|\right) + A$$

$$\Rightarrow \quad \boxed{(1-2\gamma)\underbrace{\|u_k\|}_{X}{}^2 \le 2\gamma\varepsilon\|u_k\| + A}$$

$$\Rightarrow \quad \exists B \qquad \|u_k\| \le B$$

↑ by simple quadratic inequality

$$(1-2\gamma)X^2 - 2\gamma\varepsilon X - A \le 0$$



$\Rightarrow \exists \{u_k\}$ convergent in L^{2^*} — after passing to a subsequence

$\Rightarrow$ $S(f(u_k))$ convergent in H_0^1

$\Rightarrow$ u_k convergent in H_0^1

This verifies the (PS) - condition.

The MPT is proved by a Morse Lemma where (PS) is needed.

[John Milnor Morse Theory — finite dimensional case]

"Only critical value matters in life" — Sard's theorem says: critical values are of zero measure

$\Downarrow$

Theorem: I satisfies the assumption $I: H \to \mathbb{R}$ is C^1 & I' is locally Lipschitz. I has (PS) property.

Assume c is NOT a critical value of I

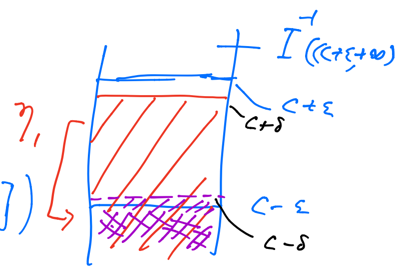
Then $\forall\, \varepsilon > 0$ $\underset{\varepsilon \ll 1}{}$ $\exists\, \delta > 0$. & η_s: $C([0,1]\times H, H)$

(i) $\eta_0 = id$

(ii) $\eta_1 \big|_{I^{-1}((-\infty, c-\varepsilon)) \cup I^{-1}((c+\varepsilon, +\infty))} = id$

(iii) $I(\eta_s(u)) \le I(u)$

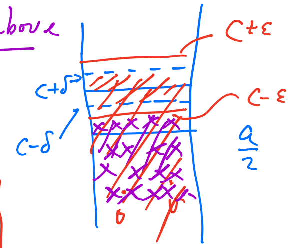
(iv) $\eta_1(I^{-1}((-\infty, c+\delta])) \subset I^{-1}((-\infty, c-\delta])$



Namely η_1 shrinks the "height"

η_s called isotopy

$c \geq a$ $\qquad$ $c-\varepsilon \geq \frac{a}{2} > 0$ $\qquad$ Conclusion above



Then $\boxed{I(\eta_1(g(t))) < c-\delta}$ for $g(t)$ chosen below:

$\max_{t\in[0,1]} I(g(t)) < \boxed{c+\delta}$ — $\exists$ such $g(t)$ by the definition $c = \inf_{g\in\Gamma} \max_{0\leq t\leq 1} I(g(t))$

$\Rightarrow$ $\quad \tilde{g}(t) = \eta_1(g(t))$ will satisfy

Also
$$\begin{cases} \tilde{g}(0) = \eta_1(g(0)) = \eta_1(0) = 0 \\ \tilde{g}(1) = \eta_1(g(1)) = \eta_1(v) = v \end{cases}$$
since $0 \in I^{-1}((-\infty, c-\varepsilon])$

$\tilde{g}(t)$ is an admissible path $\tilde{g}(0)=0$, $\tilde{g}(1)=v$

But $\max I(\tilde{g}(t)) = \boxed{c-\delta}$

& $\inf_{g\in\Gamma} \max(I(g(t))) = c$

these two facts are contradictory! $\Rightarrow$ $(\Rightarrow\Leftarrow)$