General principle: $\underline{\phi_s}$, $\underline{w_s}$ $\quad \left(\frac{\phi(s,x)}{w(s,x)}\right)$ — one parameter deformation of Ω & u

Assume $I(u) = \int_\Omega L(Du, u, x)\,dx$ is on $\mathcal{A}$ — some collection of functions

$\begin{cases} \phi - \text{a family of diffeomorphism defined in a neighborhood of } \Omega \\ \phi_0 = id \end{cases}$

$\boxed{v = \left.\frac{d\phi_s}{ds}\right|_{s=0}}$ — the vector fields generated the motion

$\frac{\partial\phi}{\partial s}$

$w \in \mathcal{A}$ — a family of functions $w_0(x) = u(x)$.

We say $\boxed{I(u) \text{ is invariant under } \phi_s \ w_s}$ if

$$\underbrace{\int_{\Omega'} L(Dw, w, x)\,dx}_{\approx I(w_s,\Omega)} = \int_{\phi_s(\Omega') \doteq \Omega'_s} L(Du, u, x)\,dx = I(u, \Omega_s) \quad \forall \Omega' \subset \Omega$$

$\boxed{m(x) = \left.\frac{\partial w}{\partial s}\right|_{s=0}}$ — m called a <u>multiplier</u>

<u>Theorem</u>: For invariant I (w.r.t ϕ w)

(1) $$\sum_{i=1}^n \left(\underbrace{m L_{p_i}(Du, u, x) - L(Du, u, x)v^i}\right)_{x_i}$$

$$= m \underbrace{\left(\sum \left(L_{p_i}(Du, u, x)\right)_{x_i} - L_z(Du, u, x)\right)}_{=0}$$

(2) If u satisfies the E-L

$$\Rightarrow \quad \operatorname{div}\left(\underbrace{mL_p - L\cdot v}_{X}\right) = 0$$

This is the theorem of Noether! — which implies many Conservation laws

$\exists$ Noether type conservation laws hidden more subtlely for, e.g. harmonic maps. — Stress-energy tensor

We discuss them if there are time left.

At beginning we do not worry about the smoothness

$$I(w) = \int_{\Omega'} \underline{L(Dw_s, w_s, x)}$$

Pf: Taking $\frac{\partial}{\partial s}\Big|_{s=0}$

$$LHS = \int_{\Omega'} L_{p_i} m_{x_i} + L_z m = \int_{\Omega'} \left(\underline{-div(L_p)} + L_z\right) m + \int_{\partial\Omega'} \underbrace{m \langle L_p, \nu\rangle}$$

$$\frac{d}{ds} w^s_{x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial s} w_s\right)_{s=0}$$

$$RHS = \int_{\underline{\underline{\partial\Omega'}}} L \langle V_s, \nu\rangle$$

$$\Rightarrow \int_{\Omega'} \left(-div(L_p) + L_z\right) \cdot m = \int_{\partial\Omega'} L\langle V, \nu\rangle - \int_{\partial\Omega'} m\langle L_p, \nu\rangle$$

$$= (\langle LV, \nu\rangle - \langle mL_p, \nu\rangle)$$

$$= \boxed{\langle L\vec{V} - m\vec{L}_p, \nu\rangle}$$

$$= \int_{\Omega'} \underbrace{div(-m\vec{L}_p + L\vec{V})}$$

$\square$.

If u is vector valued, the same result holds

$$LHS = \frac{d}{ds}\Big|_{s=0} \int_{\Omega'} L(D\vec{w}_s, \vec{w}_s, x)\,dx = \int_{\Omega'} L_{p^i_j} m^i_{x_j} + L_{z^i} m^i$$

$$= \int_{\Omega} \left[\left(-L_{p^i_j}\right)_{x_j} + L_{z^i} \right] m^i + \int_{\partial\Omega'} L_{p^i_j} m^i \nu^j$$

$$\mathrm{RHS} = \int_{\partial\Omega'} L \,\langle V, \nu \rangle$$

$$\Rightarrow \underbrace{\left[\left(-L_{p^i_j}\right)_{x_j} + L_{z^i} \right]}_{=0} m^i = \mathrm{div}\Big(\underbrace{-\sum_i L_{p^i_{(\cdot)}} m^i + VL}_{= X} \Big)$$

For Euler Lagrange solution u

$$\mathrm{div}\Big(\underbrace{-L_{p^i_{(\cdot)}} m^i + L\cdot V}_{X} \Big) = 0 \qquad (*)$$

$\underbrace{-L_{p^i_{(\cdot)}}}_{\text{a vector}} = \vec{W}$ $\qquad \underbrace{W^k}_{} = -L_{p^i_k} m^i$

Namely $(*)$ is

$$\sum_{k=1}^{n} \frac{\partial}{\partial x^k} \Big(\underbrace{-L_{p^i_k} m^i}_{} + L V^k \Big) = 0$$

Applications: (A) motions of particles

Motion of one partical

$$L(p, z) = \frac{1}{2} \tilde{m} |p|^2 - U(z)$$

$$L_p = \tilde{m} p$$

$$\underbrace{I(\vec{x}) = \int_{t_0}^{t_1}}_{} \underbrace{\frac{1}{2} \tilde{m} |\dot{\vec{x}}|^2}_{\text{kinetic energy}} - \underbrace{U(\vec{x})}_{\text{potential energy}}$$

mass

kinetic energy

potential energy.

$$\phi_s(t) = t + s \qquad v = 1 = \left.\frac{d\phi}{ds}\right|_{s=0}$$

$$\vec{x}(s,t) = \vec{x}(t+s) \qquad \left.\frac{d\vec{x}}{ds}\right|_{s=0} = \dot{\vec{x}} \overset{m}{=}$$

$$\int_{t_0}^{t_1} \underbrace{\frac{1}{2}\tilde{m}\,|\dot{\vec{x}}(\underbrace{t+s}_{\tau})|^2 - U(\vec{x}(\underbrace{t+s}_{\tau}))}\, dt \qquad \underline{L_p = \tilde{m}\dot{\vec{x}}}$$

$$= \boxed{\int_{t_0+s}^{t_1+s}} \frac{1}{2}\tilde{m}\,|\dot{\vec{x}}(\tau)|^2 - U(\vec{x}(\tau))\, d\tau$$

$L_p \cdot m$

$$(*) \Rightarrow \tilde{m}|\dot{\vec{x}}|^2 - \overset{L}{\left(\frac{1}{2} m |\dot{\vec{x}}|^2 - U(\vec{x})\right)}$$

$$= \frac{\tilde{m}}{2}|\dot{\vec{x}}|^2 + U(\vec{x}) \quad \text{is preserved.}$$

Energy Conservation

If $\exists$ k-particles $\Rightarrow$

$$\sum_{i=1}^{k} \frac{\tilde{m}}{2}|\dot{\vec{x}}_i|^2 + U(\vec{x}_1 \dots \vec{x}_k) \quad \text{is preserved.}$$

Back to 1-particle, if we use $\boxed{\frac{\tilde{m}}{2}\left[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2\right] - U(\vec{x})}$ $L(p, \dot{z})$

$$w\left(s, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+s \\ y \\ z \end{pmatrix} \Rightarrow m = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = 0$$

$\Rightarrow$ No motion in domain $[t_0, t_1]$

$\Rightarrow P_x = \tilde{m}\dot{x}$ is preserved.

x-momentum

$$\underbrace{L_p}_{\tilde{m}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}} \cdot m - \underset{0}{\underset{\parallel}{L_v}}$$

Simily $\quad P_y = \tilde{m}\dot{y}, \; P_z = \tilde{m}\dot{z} \quad$ preserved.

For k-particles

$$P_x = \sum \tilde{m}_i \dot{x}_i$$
$$P_y = \sum \tilde{m}_i \dot{y}_i$$
$$P_z = \sum \tilde{m}_i \dot{z}_i$$

Back to 1-particle case

$$w_s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos s\, x + \sin s\, y \\ -\sin s\, x + \cos s\, y \\ z \end{pmatrix} \qquad m = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

$$V = 0$$

$\boxed{\frac{\tilde{m}}{2}|\dot{\vec{x}}|^2}$ is preserved. $\quad L_p \cdot m$

$\Rightarrow$ (*) Implies $\quad \frac{\tilde{m}}{2}(\dot{x}y - \dot{y}x)$ is Constant

This is the angular momentum in the direction of z

Namely all classical conservation laws follows from
(for motions of particles)
Noether's theorem.

Multi-variables:

1. motion of a string $\quad \boxed{I_1(u) = \int_{t_0}^{t_1}\int_{x_0}^{x_1} \frac{1}{2}(u_t^2 - u_x^2)}$

2. motion of a membrane $\quad I_2(u) = \int_{t_0}^{t_1}\int_D \frac{1}{2}[u_t^2 - u_x^2 - u_y^2]$

3. Klein-Gordan

— Scalar field corresponding to uncharged particle of mass M. — $\quad I_3(u) = \int_{t_0}^{t_1}\int_{\Omega \subseteq \mathbb{R}^3} \frac{1}{2}(u_t^2 - u_x^2 - u_y^2 - u_z^2 - M^2u^2)$

B waves

$$\delta I_1 = \int_{t_0}^{t_1}\int_{x_0}^{x_1} (u_t \eta_t - u_x \eta_x) = 0 \quad \forall \eta = 0 \text{ on the boundary}$$

$$= -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \eta (u_{tt} - u_{xx}) = 0 \longrightarrow \boxed{u_{tt} - u_{xx} = 0}$$

1-dim wave equation

$$\delta I_3 = -\int_{t_0}^{t_1}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \left[u_{tt} - (\Delta u) + M^2 u\right]\eta = 0$$

$$\longrightarrow \underbrace{\boxed{u_{tt} - (\Delta u) + M^2 u}}_{\text{Klein-Gordon eq.}} = 0$$

Let's consider I_2 I_3

$$L_2(p) = \frac{1}{2}\left(|p_0|^2 - |p_1|^2 - |p_2|^2\right)$$

$$\phi(s, t) = t+s, \quad \phi(s, x) = x, \quad \phi(s, y) = y \qquad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$w(s, (t, x, y)) = u(t+s, x, y) \qquad m = \frac{\partial u}{\partial t}$$

$$RHS = \int_{t_0}^{t_1}\int_{\Omega = \mathbb{R}^2} \frac{1}{2}\left(\underbrace{u_t^2 - u_x^2 - u_y^2}_{u_t^2(t+s, x, y)}\right)$$

$u_x^2(t+s, x, y)$ 〃 〃

$$LHS = \int_{t_0+s}^{t_1+s}\int_{\mathbb{R}^2} L(p, u_x)$$

$$L = \frac{1}{2}\left(|p_0|^2 - |p_1|^2 - |p_2|^2\right)$$

$$\Rightarrow \operatorname{div}\left(\underbrace{u_t L_{p_i}} - L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$(*) \Longrightarrow \operatorname{div}\left(\begin{pmatrix} u_t p_0 - L \\ -u_t p_1 \\ -u_t p_2 \end{pmatrix}\right) = 0$$

$$\operatorname{div}\begin{pmatrix} \underbrace{u_t u_t - \frac{1}{2}(u_t^2 - u_x^2 - u_y^2)} = \underbrace{\frac{1}{2}(u_t^2 + u_x^2 + u_y^2)} \\ -u_t u_x \\ -u_t u_y \end{pmatrix} = 0 \iff e_t + \frac{\partial}{\partial x}(-u_t u_x) + \frac{\partial}{\partial y}(-u_t u_y) = 0$$

$$e_t = \operatorname{div}'\left((u_t u_x, u_t u_y)\right) \neq r$$

$$\int_{t_0}^{t_1}\int_{\Omega} \frac{d}{dt}(e) - \operatorname{div}^{(s)}(\underbrace{u_t \nabla^{(s)} u}_{= \gamma}) \qquad \nabla^{(s)} u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$\Omega \subset$ in xy-plane

$\Rightarrow \frac{d}{dt}\int_{\Omega} e$ is preserved — Namely the total energy is preserved.

$$= \int_{\Omega} e_t = \int_{\Omega} \operatorname{div}^{(s)}(\gamma) = 0$$

If $\phi(s,t) = t \quad \phi(s,x) = x+s, \quad \phi(s,y) = y$

$$w(s,(t,x,y)) = u(t, x+s, y)$$

$$\operatorname{div}(L_p \cdot m - L v) = 0, \qquad L_p = \begin{pmatrix} u_t \\ -u_x \\ -u_y \end{pmatrix}$$

$$v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad m = \frac{\partial u}{\partial x}$$

$\boxed{(*) \Rightarrow}$

$$\operatorname{div}\begin{pmatrix} u_x u_t \\ -u_x^2 - L \\ -u_x u_y \end{pmatrix} = \operatorname{div}\begin{pmatrix} \boxed{u_x u_t} \\ -\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 \\ -u_x u_y \end{pmatrix} = 0$$

$$\| \quad (u_x u_t)_t + (-\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2)_x + (-u_x u_y)_y = 0$$

If we assume u decay fast

$\Rightarrow \boxed{\int_{\Omega} u_x u_t}$ is preserved!

$$\begin{pmatrix} \int_{\Omega} u_x u_t \\ \int_{\Omega} u_y u_t \end{pmatrix} \text{— momentum vector}$$

Similarly $\int_{\Omega} u_y u_t$ is ———

Notice

$$\int_{\Omega} \underbrace{(u_x u_t)_t}_{P_x} = \boxed{\int_{\Omega} \operatorname{div}^{(s)}\Bigg(-\underbrace{\begin{pmatrix} -\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 \\ -u_x u_y \end{pmatrix}}_{\vec{J}_x}\Bigg)}$$

only in space

$$= \int u_{xt}u_t + u_x(u_{xx} + u_{yy})$$

This helps to solve HW #19

(Evans did NOT even define what are P_k $\vec{J}_k$

K-G We can have similar discussion.

(c) Nonlinear Wave & A Conformal transform.

$$I(u) = \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \frac{1}{2} u_t^2 - \left(\frac{1}{2} |Du|^2 + F(u) \right)$$

$f(z) = F'(z)$

Euler-Lagrange:

(NLW) $\boxed{u_{tt} - \Delta u + f(u) = 0}$

$$L = \frac{1}{2} |P_0|^2 - \frac{1}{2} |P|^2 - F(z)$$

Similarly $\phi(s,t) = t+s$ $\qquad v = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\phi(s,x) = x$

$W(s,(t\ x)) = u(t+s, x)$ $\qquad m = \frac{\partial u}{\partial t}$

$\mathrm{div}\left(L_p m - L v \right)$

(*)

$$\mathrm{div} \boxed{\begin{pmatrix} L_{P_0} u_t - L \\ \boxed{L_{P_k} u_t} \end{pmatrix}} = 0$$

$L_{P_0} = u_t$

$$L_{P_0} u_t - L = |u_t|^2 - \frac{1}{2} |u_t|^2 + \frac{1}{2} |Du|^2 + F(u)$$

$$= \frac{1}{2} \left(|u_t|^2 + |Du|^2 \right) + F(u) = \boxed{e}$$

Energy for NLW

$\int_\Omega e$ is preserved, since $\int_\Omega e_t = \int_\Omega \sum^n u_t u_{tt} + u_{x_i} u_{x_i t} + f(u) u_t$ $\quad u_t(\Delta u - f(u))$

$= - \int_\Omega \mathrm{div}^{(s)} \left(\boxed{L_{P_k} u_t} \right)$

$$\mathrm{div}^{(s)} \left(L_{P_k} u_t \right) = \left(- \begin{pmatrix} u_{x_1} u_t \\ \vdots \\ u_{x_n} u_t \end{pmatrix} \right) = - \left[u_{x_i x_i} u_t + u_{t x_i} u_{x_i} \right]$$

For the wave equation

$$I(u) = \int_{t_0}^{t_1} \int_\Omega \frac{1}{2}\left(|u_t|^2 - |\nabla u|^2\right)$$

$$L(p, z) = \frac{1}{2}\left(p_0^2 - p_1^2 \cdots - p_n^2\right)$$

$$\boxed{V = \begin{pmatrix} t \\ x \end{pmatrix}}$$

$$\phi(s, (t, x)) = (e^s t, e^s x)$$

$$w(s, (t, x)) = e^{\frac{n-1}{2}s} \cdot u(e^s t, e^s x)$$

$$\boxed{m = t u_t + \langle x, \nabla u\rangle + \frac{n-1}{2} u}$$

$$\Rightarrow \quad I(w) = \int_{e^s t_0}^{e^s t_1} \int_{\Omega(s)} \frac{1}{2}\left(u_t^2 - |\nabla u|^2\right) dx\, dt$$

$\Omega(s) = e^s \Omega$

$$= \int_{t_0}^{t_1} \int_\Omega \frac{1}{2}\left(w_t^2 - |\nabla w|^2\right) dx\, dt$$

$(e^s t, e^s x)$, $e^s t = \tilde{t}$, $e^s x = y$

$$\text{Then } (*) \Rightarrow \quad = \int_{t_0}^{t_1} \int_\Omega e^{(n-1)s} e^{2s} \frac{1}{2}\left[u_t^2 - |\nabla u|^2\right] dx\, dt$$

$$\operatorname{div}\left(L_p \cdot m - L V\right)$$

$$L_p = \begin{pmatrix} u_t \\ -u_{x_1} \\ \vdots \\ -u_{x_n} \end{pmatrix} \cdot m$$

$$= \left(u_t m - L t\right)_t + \operatorname{div}^{(s)}\left(m L_{p_x} - L \cdot x\right)$$

$m L_{p_x} = -m \nabla u$, $L \cdot x = \frac{1}{2}\left(|u_t|^2 - |\nabla u|^2\right) x$

$$\left[u_t\left(t u_t + \langle x, \nabla u\rangle + \frac{n-1}{2} u\right) - \frac{1}{2}\left(|u_t|^2 - |\nabla u|^2\right) t\right]_t$$

$$= \Big[\underbrace{\frac{1}{2}(u_t)^2 t + u_t \langle x, \nabla u\rangle + \frac{1}{2}|\nabla u|^2 t + \frac{n-1}{2} u}_{P}\Big]_t$$

$$- \operatorname{div}^{(s)}\left(m \nabla u + \frac{1}{2}\left(u_t^2 - |\nabla u|^2\right) x\right)$$

$$\Rightarrow \quad \int_\Omega P \text{ is preserved.}$$

Not sure the geometric meaning of P

· The Semi-linear Elliptic PDE

For $I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$ $F(u) = \frac{1}{p+1}|u|^{p+1}$

$-\Delta u = |u|^{p-1} u$

Non existence result ch 9.4 in the last Lecture

One can perform

$$\begin{cases} \phi(s,x) = e^{\beta s} x \\ w(s,x) = e^{\alpha s} u(e^{\beta s} x) \end{cases}$$

$\left.\frac{\partial\phi}{\partial s}\right|_{s=0} \longrightarrow V = \beta x$

$m = \alpha u + \beta \langle \nabla u, x\rangle$

to have the condition of Theorem → Invariance of I

$$\& \begin{cases} 2\alpha + 2\beta = (p+1)\alpha \\ (p+1)\alpha = \beta n \end{cases} \Rightarrow \begin{cases} 2\beta = (p-1)\alpha \\ n\beta = (p+1)\alpha \end{cases}$$

$\Rightarrow p = \boxed{\frac{n+2}{n-2}}$ — the critical power

$L_p \cdot m - L \cdot V$

$L = \frac{1}{2}|P|^2 - F(z)$

$L_p = P$

This m suggests $\langle \nabla u, x\rangle$ - factor in the discussion of last week.

$$\operatorname{div}\left[\underbrace{(\alpha u + \beta\langle \nabla u, x\rangle)}_{m} \nabla u - \beta\left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right)x\right] = 0$$

$\langle \nabla u, x\rangle \Delta u$ is one of them

$$\operatorname{div}\left(\frac{1}{p+1}|u|^{p+1} x\right) = \frac{1}{p+1}|u|^{p+1} n + \boxed{|u|^{p-1} u \langle \nabla u, x\rangle}$$

another one

(D) Monotonicity formulae

Ex 1. $I(u) = \int_\Omega |\nabla u|^p \, dx$

p-harmonic functions

$\phi(s, x) = e^s x$ $\qquad v = x$

$w(s, x) = e^{\frac{n-p}{p}s} u(e^s x)$ $\qquad m = \frac{n-p}{p} u + \langle \nabla u, x \rangle$

$|\nabla w|^p = e^{(n-p)s} e^{ps} |\nabla u|^p (e^s x)$ $\qquad L = |P|^p$, P vector

$\Rightarrow I_\Omega(w) = \int e^{ns} |\nabla u|^p (e^s x) \, dx$ $\qquad L_P = p|P|^{p-2} P$

$= \int_{e^s\Omega} |\nabla u|^p (y) \, dy = I_{\Omega^s}(u)$ $\qquad P = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}$

Noether's theorem $\qquad |P| = \sqrt{P_1^2 + \dots + P_n^2}$

$(*) \Rightarrow \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ — E-L

$0 = \operatorname{div}[L_P m - L v] = \operatorname{div}\left[p|\nabla u|^{p-2} \nabla u \left(\underbrace{\frac{n-p}{p} u}_{1} + \underbrace{\langle \nabla u, x \rangle}_{2} \right) \underbrace{- |\nabla u|^p x}_{3} \right]$

Since p-harmonic $\Rightarrow$ $\qquad \operatorname{div}(\underbrace{(n-p)|\nabla u|^{p-2} \nabla u}_{X} \cdot u)$

$\Rightarrow \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ $\qquad \operatorname{div}(X \cdot u)$

$(n-p) \operatorname{div}((|\nabla u|^{p-2} \nabla u) u) = (n-p) |\nabla u|^p$ $\qquad = \underbrace{\operatorname{div}(X)}_{=0} . u + \langle X, \nabla u \rangle$

$\Rightarrow \int_\Omega (n-p) |\nabla u|^p + \int_{\partial\Omega} \underbrace{p|\nabla u|^{p-2} \langle \nabla u, x \rangle \langle \nabla u, \nu \rangle}_{= p|\nabla u|^{p-2} r |u_\nu|^2} - |\nabla u|^p \langle x, \nu \rangle = 0$

$x = r\nu$

For $\Omega = B(0, r)$ $\Rightarrow$ $\int_{B(0,r)} (n-p) |\nabla u|^p = r \int_{\partial B(0,r)} |\nabla u|^p - pr \int_{\partial B(0,r)} |\nabla u|^{p-2} (u_r)^2$

$\boxed{x = \nu r}$ on $\partial B(0, r)$

$r = |x|$ $\qquad n=p$ $\qquad \frac{d}{dr} \int_{B(0,r)} |\nabla u|^p = \int_{\partial B(0,r)} |\nabla u|^p = pr \int_{\partial B(0,r)} |\nabla u|^{p-2} (u_r)^2$

$$G(r) = \int_{B(0,r)} |\nabla u|^p$$

$$G'(r) = \int_{\partial B(0,r)} |\nabla u|^p$$

$$\downarrow$$

$$(n-p)\, G(r) = r\, G'(r) - p r \int_{\partial B(0,r)} \underbrace{|\nabla u|^{p-2} u_r^2}_{\geq 0}$$

$$p r \int_{\partial B(0,r)} |\nabla u|^{p-2} u_r^2 = r\, G'(r) - (n-p)\, G(r)$$

$$p\, r^{-(n-p)} \int_{\partial B(0,r)} |\nabla u|^{p-2} u_r^2 = r^{-(n-p)} G'(r) - (n-p)\, r^{-(n-p)-1} G(r)$$

$$= \frac{d}{dr}\left(r^{-(n-p)} G(r) \right). \geq 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r^{-(n-p)} \int_{B(0,r)} |\nabla u|^p \right) = \cdot \; p \, r^{-(n-p)} \int_{\partial B(0,r)} |\nabla u|^{p-2} (u_r)^2 \geq 0$$

Namely

$$J(r) = \frac{\int_{B(0,r)} |\nabla u|^p}{r^{n-p}}$$ is monotone non-decreasing.

It holds for harmonic maps, But if $u : \Omega \to M \subset \mathbb{R}^K$ it is a variational problem with constraint. It's proof needs a new ingredient.

(E) Stress-energy tensor

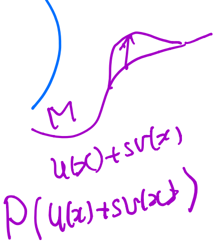
Let $u \in H^1(\Omega, \mathbb{R}^K)$ $\quad u(x) \in M$ $\quad v(x) \subset C_c^\infty(\Omega, \mathbb{R}^K)$

be a critical point of $\quad \underbrace{I(u) = \int_\Omega \frac{1}{2} |\nabla u|^2}$

$\underline{u_s(x)} \in M$ $\qquad \dfrac{u(x)+sv(x)}{|u(x)+sv(x)|}$

We define $\quad S_{ij} = \left(|\nabla u|^2 \delta_{ij} - 2 D_i u \, D_j u \right)$

Stress-energy tensor



Theorem: If u is a critical point w.r.t "variations of the domain" then $\underbrace{\operatorname{div}(S) = 0}$.

[Different from Noether's theorem which asserts $\operatorname{div}(X) = 0$ for some vector]

Pf: Weakly it means $\frac{\partial S_{ij}}{\partial x^i} = 0 \quad \forall\, 1 \le j \le n$

$\operatorname{div}(S) = 0$

$$\int_\Omega \left(|\nabla u|^2 \delta_{ij} - 2 \nabla_i u \cdot \nabla_j u \right) \nabla_i \xi^j = 0 \qquad (*)$$

$\nabla_i u \cdot \nabla_j u = \langle \nabla_i u, \nabla_j u \rangle$

$\forall \xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix} \in C_c^\infty(\Omega)$. $S = |\nabla u|^2 id - 2 \nabla u \cdot (\nabla u)^{tr}$

Consider the variation $y(x) = x + s\xi(x) = id + s\xi(x)$

$\xi \in C_c^\infty(\Omega, \mathbb{R}^n)$ Not $\mathbb{R}^K$

$\Omega \subset \mathbb{R}^n$

$$\left(\frac{\partial y^i}{\partial x^j}\right) = id + s\left(\frac{\partial \xi^i}{\partial x^j}\right)$$

$= \nabla y$

$\Omega \to \mathbb{R}^n$

$u: \Omega\ \mathbb{R}^n \to \mathbb{R}^K$

$$J(\nabla y) = \det\left(id + s\left(\frac{\partial \xi^i}{\partial x^j}\right)\right)$$

$$\left.\frac{\partial}{\partial s}\right|_{s=0} J^{-1} = \frac{\partial}{\partial s}\left(e^{-\log J}\right) = -J^{-1}\Big|_{s=0} \operatorname{trace}\left(\frac{\partial \xi^i}{\partial x^j}\right)$$

$= \frac{d}{ds}\log J$

$$= -J^{-1}\left(\sum_{i=1}^n \nabla_i \xi^i\right)$$

$$I(u_s) = \int \frac{1}{2} |\nabla u_s|^2$$

$$\boxed{u_s(x) = u(\underbrace{x + s\xi(x)}_{y(x)})}$$

$y: \Omega \to \Omega$ if $s \ll 1$

$$\frac{\partial u_s}{\partial x_i} = \frac{\partial u}{\partial y_j}\left(\delta_{ji} + s\frac{\partial \xi^j}{\partial x^i}\right) \implies \nabla u_s(x) = \nabla u(x + s\xi(x)) + s\, \nabla u \cdot \nabla \xi$$

$$\Rightarrow \quad \frac{1}{2}|\nabla u_s|^2 = \underbrace{\frac{1}{2}|\nabla u|^2}(x+s\xi(x)) + \boxed{s}\langle \underbrace{\nabla u}_{y}, \nabla u \cdot \nabla \xi\rangle + O(s^2)$$

$$\langle \nabla u, \nabla u \circ \nabla \xi\rangle = \nabla_{x_i} u \quad \nabla_{x_j} u \quad \nabla_{x_i} \xi^j$$

$$\Rightarrow 0 = \frac{d}{ds}\Big|_{s=0} I(u_s) = \int \frac{d}{ds}\Big|_{s=0} \underbrace{\frac{1}{2}\left(|\nabla u|^2 (x+s\xi(x))\right)}_{y}$$

$$+ \int \underline{\nabla_i u \; \nabla_j u \; \nabla_i \xi^j}$$

$\Rightarrow$ First term on RHS

determinant

$$= \int_\Omega \underbrace{\frac{1}{2}|\nabla u|^2(y)} \boxed{\frac{d}{ds}\Big|_{s=0}\left(\underbrace{\left|\frac{\partial x}{\partial y}\right|}_{J^{-1}}\right)} dy$$

$$= \int_\Omega \frac{1}{2}|\nabla u|^2(y)\left(-\underline{\sum \nabla_i \xi^i}\right) \underline{J\, dy}$$

$$= \int_\Omega \frac{1}{2}|\nabla u|^2(x)\left(-\sum_{i=1}^n \nabla_i \xi^i\right) dx$$

Putting together $\Longrightarrow$

$$0 = \int_\Omega \left[2 \nabla_i u \, \nabla_j u \, D_i \xi^j - |\nabla u|^2 \delta_{ij} \, D_i \xi^j \right]$$

Hence $(*)$.

The other advantage is that $(*)$ only requires $u \in H^1_0(\Omega, \mathbb{R}^K)$ being a critical point. The regularity requirement is lower. $u \in M \subset \mathbb{R}^N$

A good reference: Hélein: $\boxed{\text{Harmonic Maps, Conservation Laws, and Moving frames.}}$

$(*)$ was proved for smooth maps by Baird-Eells (1981).

$u : \Omega \to M$

$du = Du$

It follows from

$$S = \langle \cdot , \cdot \rangle \frac{1}{2} |du|^2 - \langle du(\cdot), du(\cdot) \rangle, \quad Du \cdot (Du)^{tr}$$

$$S_{ij} = \left[\frac{1}{2} |du|^2 \boxed{g_{ij}} - \langle du(e_i), du(e_j) \rangle \right] e_i^* \otimes e_j^*$$

$\nabla_{e_j} du(e_j) = 0$ if u is harmonic

$$\boxed{\nabla_j S_{ij}} = \delta_{ij} \langle \nabla_{e_j} du(e_k), du(e_k) \rangle - \langle \nabla_{e_j} du(e_i), du(e_j) \rangle$$

$$= \langle \nabla_{e_i} du(e_k), du(e_k) \rangle - \langle \nabla_{e_i} du(e_j), du(e_j) \rangle = 0.$$

orthogonal frame

Let $\xi = \dfrac{x - x_0}{|x - x_0|}$ $\quad (*) \Longrightarrow$ The monotonicity formula

for harmonic map from $B(x_0, R)$