MA equation $\det(\nabla^2 u) = f$. Monge - 1795, 1st book on DG

Monge — Ampère

For linear $Lu = f$ — H^1 is the right space

For MA $C(\bar{\Omega})$ & convex is the weak solution admissible set.

(0) A little introduction:

$$(\Omega_1, \underbrace{f(x)\,dx}_{d\mu_1}) \longrightarrow (\Omega_2, \underbrace{g(y)\,dy}_{d\mu_2})$$

If $\exists$ map given by ∇u $y = \nabla u(x)$

$$\int \varphi\, g(y)\, dy = \int \varphi(\nabla u(x))\, g(\nabla u(x)) \det(\nabla^2 u)\, dx$$

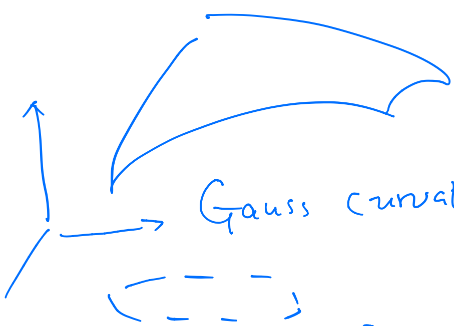
$$= \int \varphi(\nabla u(x))\, f(x)\, dx$$

Then u satisfies $\boxed{\det(\nabla^2 u) = \frac{f(x)}{g(\nabla u(x))}}$

Arises optimal transport problem! —— Monge: $I(T)$
$= \int c(x, T(x))\, d\mu(x)$

$x_{n+1} = \varphi(x)$

Minimize T. $T \# \mu = \nu$, $\nu(B) = \mu(T^{-1}(B))$



$$\text{Gauss curvature} = \frac{\det(D^2\varphi)}{\left(\sqrt{1+|\nabla\varphi|^2}\right)^{\frac{n+2}{2}}}$$

Some calculation using definition

$\Rightarrow$ $\det(D^2\varphi) = \left(\sqrt{1+|\nabla\varphi|^2}\right)^{\frac{n+2}{2}} f(x)$ is the prescribing Gauss curvature equ.

$\det(\nabla^2 u) = B^{ij}\partial^2_{x_i x_j} u$

$n \det(A) = a_{ij} A^{ij}$

We shall introduce two solutions to MA equation

Aleksandrov

① A-solution, Alexandrov — Our focus

② V solution, Viscosity.

$u(tx_1+(1-t)x_2) \leq t\,u(x_1)+(1-t)u(x_2)$

Tool one:

Radon measures: Check Folland Ch 7.

① Outer regular on Borel
② Inner regular on open
③ $\mu(K)<+\infty$

— Borel measure + ①, ②, ③ Called Radon

But

Theorem 7.8 X LCH, 2nd Countable

$\Rightarrow \forall \mu$, $\mu(K)<+\infty \Rightarrow \mu$ is Radon

Key: Radon measure is dual to $C_0(X)$.

L^p $(L^p)^* = L^q$ $1<p<+\infty$

(L^1 Not the dual of L^∞, $(L^1)^* = L^\infty$ - for μ σ-finite)

Good dual for L^∞ is Radon measures
$\cup$
$C_0(X)$

$\exists$ good compactness result!

If $\sup_k \mu_k(K) <+\infty$, $\forall K$ compact
then $\mu_{k_j} \rightharpoonup \mu$
$\Leftrightarrow \int f(x)\,d\mu_k(x) = \int f\,d\mu$ $\forall f \in C_c(\Omega)$

(I) MA-measure for $u \in C(\Omega)$. u is convex.

$M(u)$ is called the M-A measure

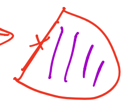
If $u \in C^2(\Omega)$, $\nabla^2 u \geq 0$, Ω convex

$$M(u)(E) = \int_E \underbrace{\det(\nabla^2 u)}_{J(\nabla u)}\,dx \leq |\nabla u(E)|$$

Goal: Theorem: $\Omega \subset \mathbb{R}^n$ $\boxed{\text{strictly}}$ convex. open bounded

$\longrightarrow \forall x, y \in \bar{\Omega}$

Interior of $\overline{xy} \subset \Omega$

$\longrightarrow$ Not strictly $\rightarrow$ 

ν Borel. measure $\nu(\Omega) < +\infty$.

$g|_{\partial\Omega}$ continuous $\exists!\ u \in C(\bar{\Omega})$ convex

$$\begin{cases} M(u) = \nu \\ u|_{\partial\Omega} = g \end{cases}$$

We need to enlarge the scope of the "solution" concept

by defining $M(u)$ for u continuous — Ampère is a student of Monge

$\uparrow$ MA measure

"Uncertainty principle": Existence $\times$ Uniqueness $=$ Const

difficulty level

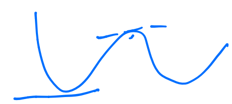
Perron

The trick is to find the "sweet spot" to balance it.

Tool two: Basics on Convex sets! [P. Lax Linear Algebra]

Convex functions

We build up this a little bit!



Def: Normal map

$\partial u(x_0) : x_0 \longrightarrow S$ — a set $\subset \mathbb{R}^n$

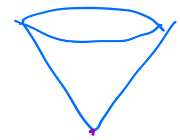
$$\| \quad \{p \mid \forall x \in \Omega,\ u(x) \geq \underbrace{u(x_0) + \langle p, x - x_0 \rangle}_{\ell_p(x)}\}$$

$\ell_p(x)$ — called the support linear function.

p — subdifferential.

Example: $u(x) = \frac{h}{R}|x-x_0|$

$\Omega = B_R(x_0)$



$(x_0, 0)$

$x_1 \neq x_0$. $\quad \partial u(x_1) = \boxed{\frac{h}{R}\frac{x_1-x_0}{|x_1-x_0|}} \in \partial B_{\frac{h}{R}}(0)$

$\wr$

$\nabla u(x_1)$

$|\partial u(B_R(x_0)\setminus\{x_0\})| = 0$

Lemma. If u is C^1 at x_1, $\partial u(x_1) \neq \phi$, $\partial u(x_1) = \nabla u(x_1)$

Pf: $u(x) = u(x_1) + \langle \nabla u(x_1), x - x_1\rangle + o(|x-x_1|)$

$\geq u(x_1) + \langle p, x - x_1\rangle$

$\Rightarrow \langle \nabla u(x_1) - p, x - x_1\rangle + o(|x-x_1|) \geq 0 \quad \forall x \in B_\delta(x_1)$

This forces $\nabla u(x_1) = p$.

Otherwise $\nabla u(x_1) - p = w \neq 0$ let $x - x_1 = -\eta \frac{w}{|w|}$, $|x-x_1| = \eta$

$\Rightarrow$ LHS $= \langle w, -\eta\frac{w}{|w|}\rangle \neq o(\eta)$

$= -|w|\eta + o(\eta) < 0$ if $\eta \ll 1$.

At x_0. $\quad \forall p \in \overline{B_{\frac{h}{R}}(0)} \qquad$ Claim $\overline{B_{\frac{h}{R}}(0)} = \partial u(x_0)$

$\Rightarrow u(x) = \frac{h}{R}|x-x_0|$

$u(x_0) = 0 \quad \ell_p(x) = u(x_0) + \langle p, x-x_0\rangle$

$\leq |p|\,|x-x_0| \leq \frac{h}{R}|x-x_0| = u(x)$.

$\overline{B_{\frac{h}{R}}(0)} \subset \partial u(x_0)$.

$\forall p \quad |p| > \boxed{h/R}$ $\qquad$ $\underline{p = |p|\,w}$ $\quad w \in S^{n-1}(1)$

$\Rightarrow \quad l_p(x) = u(x_0) + \langle p, x - x_0 \rangle$

$= \underset{\parallel_0}{u(x_0)} + |p| \langle w, x - x_0 \rangle$

$\boxed{x - x_0 = (R-\varepsilon) w} \Rightarrow \quad l_p(x) = |p| (R-\varepsilon)$ $\quad$ if $\varepsilon \ll 1$

$\geq \frac{h}{R-\varepsilon}(R-\varepsilon) = h$ $\quad$ $u(x) = \frac{h}{R}|x - x_0|$

But $\quad u(x) \leq h \quad (\Rightarrow \Leftarrow)$.

Hence $\quad \partial u(x_0) = B_{\frac{h}{R}}(0)$ $\quad \Rightarrow \quad M(u)(E) \doteq |\partial u(E)| \quad \forall E \subset \underset{\Omega}{\overset{B_R(x_0)}{\shortparallel}}$

$M(u) = |B_{\frac{h}{R}}(0)| \, \delta_{x_0}(x)$

- Monge Ampère measure

$M(u)(E) := |\partial u(E)|$. $\quad \forall E \subset \mathbb{R}^n$ or $E \subset \Omega$ $\Big\}$ Borel.

At this point it is simply a function ≥ 0

(1) $M(u)(K) < +\infty$.

Lemma: $\forall K \subset\subset \Omega$, $\partial u(K)$ is compact!

pf: Assume $\{p_k\} \subseteq \partial u(x_k)$ $\quad x_k \in K$

Pick $\quad c > 0 \qquad K_c \subset\subset \Omega \qquad K_c = \{x \mid d(x, K) \leq c\}$

$\boxed{u(x) \geq u(x_k) + \langle p_k, x - x_k \rangle}$

Let $\quad x = x_k + c \frac{p_k}{|p_k|} \Rightarrow$

$\boxed{u(x) \geq u(x_k) + c|p_k|}$ $\qquad \Rightarrow \quad \boxed{|p_k| \leq \frac{1}{c}\left[\max_{K_c} u - \min_{K} u\right]}$

$u(x) \geq u(x_0) + \langle p, x - x_0 \rangle$

$x = x_0 + c \frac{p}{|p|}$

$\Rightarrow \quad \geq u(x_0) + c|p|$

$\Rightarrow |p| \leq \frac{1}{c}\left[\max_{\Omega} u - \min_{\Omega} u\right]$

$c \leq d(K, \Omega)$

$\forall p \in \partial u(K)$

Hence $\{x_k\}$, $\{p_k\}$ are bounded sequence

$\exists$ convergent subsequence $\quad x_k \to x_0 \in K$

$p_k \to p$

$\Rightarrow u(x) \geq u(x_0) + \langle p, x - x_0 \rangle \quad \forall x \in \Omega \Rightarrow p \in \partial u(x_0)$

$x_0 \in K$. $\qquad \square$

Corollary: $M(u)(K) < +\infty$.

Next we show $M(u)$ is a measure.

Fact: (Ex 15).

$$\partial u(\Omega\setminus E) = \left(\partial u(\Omega)\setminus \partial u(E)\right) \cup \left(\partial u(\Omega\setminus E)\cap \partial u(E)\right) \qquad (*)$$

Key difference: $\partial u : \Omega \longrightarrow S \subset \mathcal{P}(\mathbb{R}^n)$. Namely ∂u maps a pt into a set!
$\quad x_0 \mapsto \partial u(x_0)$

E.g. $\partial u(x_0)$ is mearable

$$\begin{cases} u(x) \geq u(x_0) + \langle p, x-x_0\rangle \\ u(x) \geq u(x_0) + \langle p', x-x_0\rangle \end{cases}$$
$$\Rightarrow u(x) \geq u(x_0) + \langle tp + (1-t)p', x-x_0\rangle \quad \forall t\in[0,1]$$

$\Rightarrow \partial u(x_0)$ is a convex subset!

[It is a m-measurable].

direct checking!

(*) can be proved!

The key to show $M(u)(E)$ is a measure is the Lemma below:

Lemma: $u \in C(\Omega)$

$$S := \{ p \mid \exists x \neq y,\ p \in \partial u(x) \cap \partial u(y) \}$$

Then $|S| = 0$.

Sard's theorem for continuous functions

pf: use two key ingredients.

① Rademacher's theorem: A Lipschitz function is a.e differentiable. ← Evans–Gariepy

② Legendre transform & convex functions are locally Lipschitz.

Assuming the above. $\sigma := \{ S \subset \Omega \mid \partial u(S) \text{ is Lebesgue measurable} \}$

$$(\partial u)(\Omega \setminus E) = [\partial u(\Omega) \setminus \partial u(E)] \cup (\underbrace{\partial u(\Omega \setminus E) \cap \partial u(E)}) \quad m(\square) = 0$$

$$\Rightarrow \quad |\partial u(\Omega \setminus E)| = |\underline{\partial u(\Omega)} \setminus \underline{\partial u(E)}| \quad \Rightarrow \quad \Omega \setminus E \text{ is } \in \sigma \text{ if } E \in \sigma$$

provided (1) Ω is $M(u)$ measurable

Since $\Omega = \bigcup_n K_n$ $\Rightarrow$ $\boxed{\partial u(\Omega) = \bigcup_{n=1}^{\infty} \partial u(K_n)}$ K_n is $M(u)$ measurable

Same holds for any U open $U \subset \Omega$.

Namely σ contains all open subsets of Ω

(2) If $E \in \sigma \Rightarrow \Omega \setminus E \in \sigma$ [See above]

(3) Moreover, σ is a σ-algebra.

Hence $M(u)$ is a Radon measure. (A Borel measure with $u(K) < +\infty$)

by Lemma

$m(\partial u(\Omega \setminus E) \cap \partial u(E)) = 0$

For (2) $\boxed{\partial u(\Omega \setminus E) = [\partial u(\Omega) \setminus \partial u(E)] \cup [\partial u(\Omega \setminus E) \cap \partial u(E)]}$

$\in \mathcal{M}$; $E \in \sigma$; $\in \mathcal{M}$

$\mathcal{M}$- Lebesgue measurable

$\Rightarrow \Omega \setminus E \in \sigma$ $\in \sigma$

For (3) $\boxed{\bigcup_{i=1}^{\infty} \partial u(E_i) = \partial u\left(\bigcup_{i=1}^{\infty} E_i\right)}$

$\in \mathcal{M}$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \sigma$

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(4) $M(u)$ is a measure on σ

Namely $\boxed{M(u)\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} M(u)(E_i)}$ (**)

if $E_i \in \sigma$ & $E_i \cap E_j = \emptyset$ $\forall i \neq j$

Let $H_i = \partial u(E_i)$

$$LHS = \left|\bigcup_{i=1}^{\infty} H_i\right|, \quad RHS = \sum_{i=1}^{\infty} |H_i|$$

$$\left|\partial u\left(\bigcup_{i=1}^{\infty} E_i\right)\right| = \left|\bigcup_{i=1}^{\infty} \partial u(E_i)\right|$$

Clearly

$$\bigcup_{i=1}^{\infty} H_i = H_1 \cup (H_2 \setminus H_1) \cup (H_3 \setminus (H_1 \cup H_2)) \cdots$$

$$\left|\bigcup_{i=1}^{\infty} H_i\right| = |H_1| + \underbrace{|H_2 \setminus H_1|}_{=|H_2|} + \underbrace{|H_3 \setminus (H_1 \cup H_2)|}_{|H_3|} + \cdots$$

By the Lebesgue measure property

Claim: $|H_n| = |H_n \setminus (H_1 \cup H_2 \cup \cdots \cup H_{n-1})|$

$$\boxed{H_n = [H_n \setminus (H_1 \cup H_2 \cdots \cup H_{n-1})] \cup [H_n \cap (H_1 \cup H_2 \cdots \cup H_{n-1})]}$$

measure 0

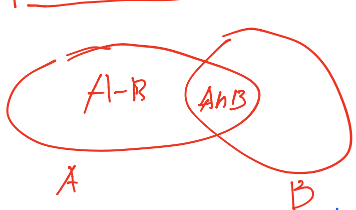
$H_n = \partial u(E_n)$, $\quad H_1 \cup H_2 \cdots \cup H_{n-1} = \partial u(E_1) \cup \cdots \cup \partial u(E_{n-1})$

$|\cdot| = 0$ by the Lemma.

$E_i \cap E_j = \phi$

$p = \partial u(x) \quad x \in E_n$

$= \partial u(y) \quad y \in \bigcup_{i=1}^{n-1} E_i$



Now we prove the Lemma:

(1) $\quad u(y) \geq u(x) + \langle P_x, y - x\rangle \qquad P_x \in \partial u(x)$

$\Rightarrow \quad u(y) - u(x) \geq -|P_x|\,|y-x|$

& similarly $\quad u(x) - u(y) \geq -|P_y|\,|x-y|$

$$\Rightarrow \boxed{|u(x) - u(y)| \leq \max\{|P_x|, |P_y|\}\,|x-y| \leq C_K |x-y| \text{ if } x, y \in K}$$

$\Rightarrow$ u is Lipschitz.

Here convexity is used since $\partial u(x) \neq \phi$.

$\Rightarrow u(y) \geq l_p(y), \ \forall y \in \Omega.$

if u is convex

$epi(u) = \{(x, a) \mid a \geq u(x)\}$

$\in epi(u)$

$l_p(x) \doteq u(x_0) + \langle P, x - x_0\rangle$

(2) Legendre transform:

$$u^*(p) := \sup_{x\in\Omega} \{ \underbrace{\langle p, x\rangle - u(x)}_{=L_{u,x}(p)} \} \quad \forall p \in \mathbb{R}^n$$

$L_{u,x}(p)$ — linear in p

Hence $-L_u(p)$ is convex in p

$\Rightarrow$ $u^*(p)$ is always convex.

$\Rightarrow$ $u^*(p)$ is locally Lipschitz

By Radmacher's theorem

$\Rightarrow$ $\mathcal{N} := \{ p_0 \mid u^*(p) \text{ is Not differentiable at } p_0 \}$ is of zero measure!

Lemma follows if we show that

$S \doteq \{ p \in \mathbb{R}^n \mid p \in \partial u(x) \cap \partial u(y), \ x \neq y \in \Omega \}$

$\boxed{S \subset \mathcal{N}}$

$\forall p \in S$ (*) $u(z) \geq u(x) + \langle p, z - x\rangle$ $\quad \forall z, z' \in \Omega$

$u(z') \geq u(y) + \langle p, z' - y\rangle$ $\quad$ for some $x \neq y$

$0 \geq -u(z) + \langle p, z\rangle + u(x) - \langle p, x\rangle$ $\quad \forall z \in \Omega$

$\Rightarrow \langle p, x\rangle - u(x) \geq \langle p, z\rangle - u(z)$

$\Rightarrow u^*(p) = \langle p, x\rangle - u(x)$

Similarly $u^*(p) = \langle p, y\rangle - u(y)$

If $p \in \partial u(x)$

$\Rightarrow u^*(p) = \langle p, x\rangle - u(x)$

& $x \in \partial u^*(p)$

But $u^*(q) = \sup_{z\in\Omega} (\langle q, z\rangle - u(z))$

$\geq \langle q, x\rangle - u(x) = u^*(p) + \langle x, q-p\rangle$

$q = p + q - p$

$\langle p, x\rangle - u(x)$

Similarly $u^*(q) \geq u^*(p) + \langle y, q-p \rangle$

Namely $x, y \in \partial u^*(p)$

$\Rightarrow$ u^* can not be differentiable at p, since $x \neq y \in \partial u^*$

Def: A convex continuous function u is called a generalized solution (A-solution) if $M(u)(E) = \nu(E)$ for ν a Radon measure. $\Leftrightarrow M(u) = \nu$

Reason: if $u \in C^2(\Omega)$, u is convex. & Ω is convex

claim: $\boxed{M(u)(E) = \int_E \det(\nabla^2 u)\, dx}$ is called Monge-Ampère, $\forall E$

Namely $|\partial u(E)| =$

$$
\begin{aligned}
|M(u)(E)| &= |\partial u(E)| \\
&= |\partial u(E \setminus S_0) \cup \partial u(S_0)| \\
&= |\partial u(E \setminus S_0)|
\end{aligned}
$$

$$\begin{cases} \partial u(x) = \nabla u(x) \\ \& \ \nabla u : \Omega \to \mathbb{R}^n \quad C^1\text{-map} \end{cases}$$

$$S_0 := \{ x \in \Omega,\ | \det(\nabla^2 u) = 0 \} \quad (\Leftrightarrow \nabla u \text{ is singular})$$

$\boxed{|\partial u(S_0)| = 0}$ $\uparrow$ C^1- Sard's theorem

$$\Rightarrow M(u)(E) = M(u)(E \setminus S_0) = |\underbrace{\partial u}_{\nabla u}(E \setminus S_0)| \overset{\text{area formula}}{=} \int_{E \setminus S_0} \underbrace{\det(\nabla^2 u)}_{J(\nabla u)} = \int_E \det(\nabla^2 u)$$

∇u is 1-1 on $\{\nabla^2 u > 0\}$

$\det(\nabla^2 u) = 0$ on S_0

(II) Uniqueness

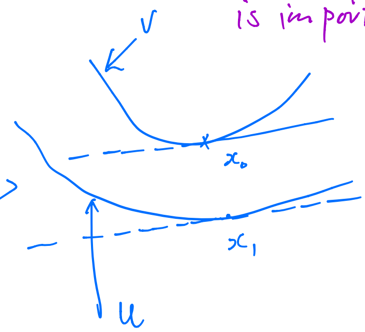
We study ① maximum principles $\begin{cases} \text{(i) Alexandrov} \\ \text{(ii) ABP, Alexandrov-Bakelman-Pucci} \end{cases}$

② Comparison principles $\Longrightarrow$ the uniqueness

Comparison (I): If $\boxed{u - v\big|_{\partial\Omega} = 0}$ $\quad v \geq u \quad$ in Ω $\quad$ Assumption is important!

Then $\boxed{\partial v(\Omega) \subset \partial u(\Omega)}$

$\Rightarrow \mathcal{M}(v)(\Omega) \leq \mathcal{M}(u)(\Omega)$



Pf: If $p \in \partial v(\Omega)$

$\Rightarrow \exists\, x_0 \in \Omega \quad v(x) \geq \underbrace{v(x_0) + \langle p, x - x_0 \rangle}_{= \ell_p^v(x)} \quad \forall x \in \Omega$

Let $\quad F(x) \doteq \underbrace{v(x_0) + \langle p, x - x_0 \rangle}_{\text{linear}} - \underbrace{u(x)}_{u}$

Try to maximize it!

$$F(x_0) = v(x_0) - u(x_0) \geq 0$$

$$F\big|_{\partial\Omega} = v(x_0) + \langle p, x - x_0 \rangle - v(x) \leq 0$$

what is needed is

$-u(x)\big|_{\partial\Omega} \leq -v(x)\big|_{\partial\Omega}$

$\Leftrightarrow \quad u(x)\big|_{\partial\Omega} \geq v(x)\big|_{\partial\Omega}$

But $u(x) \leq v(x)$ in Ω

$\Rightarrow \boxed{u(x) = v(x) \text{ on } \partial\Omega}$

Hence, either (1) $\quad F(x) \leq 0 \quad (v(x_0) = u(x_0))$ $\quad \Rightarrow \quad \boxed{u(x) \geq v(x_0) + \langle p, x - x_0 \rangle \geq u(x_0) + \langle p, x - x_0 \rangle}$

$\Rightarrow p \in \partial u(x_0)$

or (2) $\quad \max_{x \in \bar{\Omega}} F(x) > 0.$ $\quad$ [This is the case if $v(x_0) > u(x_0)$]

Let $x_1 \in \Omega$ satisfy $\quad F(x_1) = \max_{\bar{\Omega}} F(x)$

$\Rightarrow \quad F(x) \leq F(x_1)$

$\Rightarrow \quad \cancel{v(x_0)} + \langle p, x - \cancel{x_0} \rangle - u(x) \leq \cancel{v(x_0)} + \langle p, x_1 - \cancel{x_0} \rangle - u(x_1)$

$\Leftrightarrow \quad u(x) \geq u(x_1) + \langle p, x - x_1 \rangle$

$\Rightarrow \quad p \in \partial u(x_1). \quad \Rightarrow \quad p \in \partial u(\Omega).$ $\quad \square$

Comparison principle (2). $u, v \in C(\bar\Omega)$ v is convex.

$|\partial u(E)| \leq |\partial v(E)|$ $\forall$ E Borel $\Leftrightarrow$ $M(u) \leq M(v)$

Then $$\min_{x \in \bar\Omega} \left(u(x) - v(x) \right) = \min_{x \in \partial\Omega} \left(u(x) - v(x) \right) \quad (*)$$

[Linear analogue $\Delta u \leq \Delta v \Leftrightarrow \Delta(u-v) \leq 0$ $u-v$ - Sup-harmonic. Then $(*)$ holds]

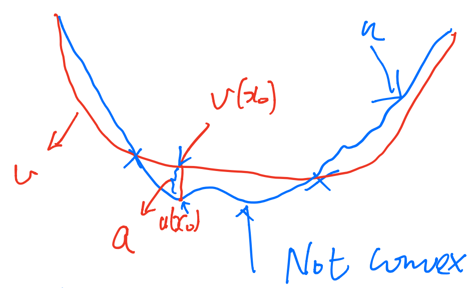
This proves the uniqueness in the main theorem!
We worry about the existence later!

Pf: W.L.G assume $\min_{x \in \partial\Omega} \left(u(x) - v(x) \right) = 0$

If $\min_{x \in \bar\Omega} \left(u(x) - v(x) \right) = -a < 0$ for some $a > 0$

let x_0 be the point the interior minimum is attained

$$u(x_0) - v(x_0) = -a$$
$$u(x_0) = v(x_0) - a$$



We construct

$$\boxed{w(x) := v(x) + \delta |x - x_0|^2 - \frac{a}{2}}$$

δ to be chosen

$$w(x_0) = v(x_0) - \frac{a}{2} > u(x_0)$$

$\Downarrow$

Hence $E \doteq \{ w(x) > u(x) \} \neq \phi$

open

However: $w\Big|_{\partial\Omega} \le v(x)\Big|_{\partial\Omega} + \delta\,\mathrm{Diam}^2(\bar\Omega) - \frac{a}{2}$

$\boxed{u(x) \ge v(x) \text{ on } \partial\Omega} \to \;\le\; u(x)\Big|_{\partial\Omega} + \delta D^2 - \frac{a}{2} \le u(x) - \frac{a}{4}$

$< u(x)\big|_{\partial\Omega}$

if $\boxed{\delta D^2 - \frac{a}{2} < -\frac{a}{4}}$ — we pick this δ.

$\delta D^2 < \frac{a}{4}$

$\Rightarrow \partial E \cap \partial\Omega = \phi \Rightarrow E \subset \Omega$

Now we apply the Comparison I to u & w on E

CP I

$\Rightarrow |\partial w(E)| \le |\partial u(E)|$

But this is a contradiction since

$w = v + \underbrace{\boxed{\delta|x-x_0|^2 - \frac{a}{2}}}_{\phi_2}$, $v = \phi_1$

$$\boxed{|\partial w(E)| \ge |\partial v(E)| + (2\delta)^n |E|} \quad (2)$$

$\|$

$|\partial(\phi_1+\phi_2)(E)| \ge |\partial\phi_1(E)| + |\partial\phi_2(E)|$ — ϕ_i convex

$M(\phi_2)(E)$

Here v is convex is needed!

[Concavity of $\det^{\frac{1}{n}}$ — Brun-Minkowski inequality]

Baby version: $\boxed{\det^{\frac{1}{n}}(A+B) \ge \det^{\frac{1}{n}}(A) + \det^{\frac{1}{n}}(B)}$ A B symmetric positive definite

$\Downarrow$

$\det(A+B) \ge \det(A) + \det(B)$

Details can be seen on P.18 of [Gu].