① A – maximum principle
Aleksandrov / Alexandrov

Theorem $u \in C(\bar{\Omega})$, Ω convex

u is convex $\quad u|_{\partial\Omega} = 0$

$$|\underset{\uparrow}{u(x_0)}|^n \le C_n \boxed{\underset{\text{diameter}}{D}^{n-1} \operatorname{dist}(x_0, \partial\Omega)} |\partial u(\Omega)|$$

Pf ① Construct a cone & cone function

$$\begin{cases} v(x) = t\,u(x_0) \\ \text{if } x = t x_0 + (1-t) x_1 \\ \text{with } x_1 \in \partial\Omega \end{cases}$$

$$v(x) \ge u(x) = u(t x_0 + (1-t) x_1) \quad \begin{cases} \le t u(x_0) \\ \quad + (1-t) u(x_1) \\ \text{by convexity} \end{cases}$$

Here $u(x_1) = 0$ is used

② $\partial v(\Omega) = \partial v(x_0)$

This is easy: $\forall\, p \in \partial v(\Omega)$

$$v(x) \ge v(x_1) + \langle p, x - x_1 \rangle$$

$H = \{ z = v(x_1) + \langle p, x - x_1 \rangle \}$ is the supporting hyperplane at $(x_1, v(x_1))$

$\Longrightarrow$ Also holds at x_0, namely $p \in \partial u(x_0)$.

Namely
$$\boxed{v(x)} \geq v(x_1) + \langle p, x - x_1 \rangle$$
$$= v(x_0) - \langle p, x_0 - x_1 \rangle + \langle p, x - x_1 \rangle$$
$$= \boxed{v(x_0) + \langle p, x - x_0 \rangle}$$

H passing vertex
$$v(x_0) = v(x_1) + \langle p, x_0 - x_1 \rangle$$

$\to$ otherwise the line passing x_1 won't be on the same side of H

$\Longrightarrow$ The claim.

In particular $\partial v(\Omega)$ is convex

③ $\partial v(x_0)$ contains a ball, namely

$\rightarrow B_{\frac{-u(x_0)}{D}}(0)$.

Namely, $\forall p \in B_{\frac{-u(x_0)}{D}}(0)$

$\Longrightarrow p \in \partial v(x_0)$.

In fact,

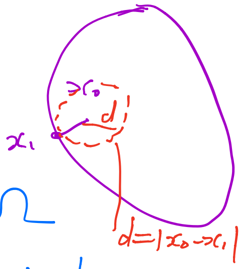
$$\begin{cases} \forall x \in \Omega \\ x = t x_0 + (1-t) x_1 \end{cases}$$

$x_1 \in \partial\Omega$.

$$\left\{ \begin{aligned} &\underset{\overset{\|}{v(x_0)}}{u(x_0)} + \langle \underline{p}, \underline{x - x_0} \rangle \\ &\leq u(x_0) + \boxed{\frac{-u(x_0)}{D}} \, \underbrace{|x_1 - x_0|}_{\leq D} (1-t) \end{aligned} \right.$$
$$\leq u(x_0) - (1-t) u(x_0)$$
$$= t\, u(x_0) = u(x)$$

④ $\partial v(x_0)$ contains a vector

where $x_1 \in \partial\Omega$



$$p_0 = \frac{u(x_0)}{|x_0 - x_1|^2} (x_0 - x_1),$$

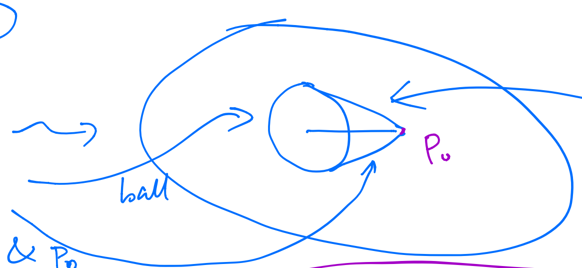
$\text{dist}(x_0, \partial\Omega) = |x_0 - x_1|$

$$|p_0| = \frac{-u(x_0)}{|x_0 - x_1|}$$

Assume ④

$\partial v(x_0)$ $\rightsquigarrow$

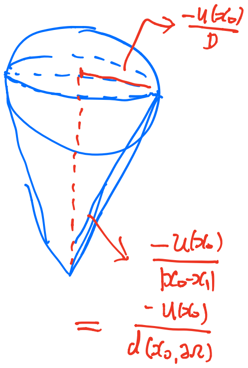
Contains ball & p_0



Volume of the cone

$$|\underline{\partial v(x_0)}| \geq \boxed{C_n \left(\frac{-u(x_0)}{D}\right)^{n-1} \frac{|u(x_0)|}{|x_0-x_1|}}$$

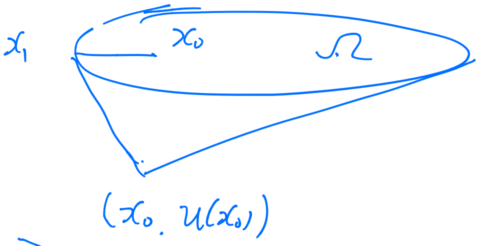
C_n

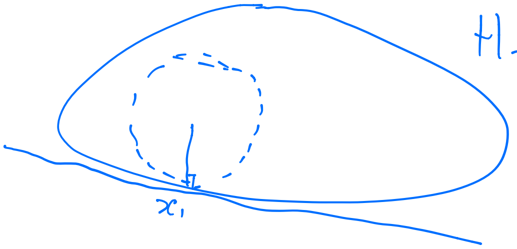


$$\Longrightarrow \quad |u(x_0)|^n \leq C_n D^{n-1} |x_0-x_1| \, |\underline{\partial u(\Omega)}|,$$

since $\partial v(\Omega) \supset \partial u(\Omega)$.

Now prove ④:





H ↘

$$\Longrightarrow \boxed{\ell_1(x) = \langle x-x_1, x_0-x_1 \rangle = 0}$$

$B_{|x_0-x_1|}(x_0) \subset \Omega$

$x_1 \in \partial\Omega$

{ Hyperplane is the unique support plane at x_1 of Ω.

Since it uniquely supports $B_{|x_0-x_1|}(x_0)$ it is unique!

$\ell_2(x,z)$

{ $\underline{z = u(x_0) + \langle p_0, x-x_0 \rangle}$ is the hyper.plane $\subset \mathbb{R}^{n+1}$

span by H & $(x_0, u(x_0))$

$\Longleftrightarrow \quad l_2(x_0, u(x_0)) = 0$

$\& \quad l_2\big|_{z=0} \quad // \, l_1 \qquad (P_0 \, // \, x_0 - x_1)$

$\Rightarrow \exists\, \overset{k}{,}\ k\, l_1(x) = l_2\big|_{z=0} = u(x_0) + \langle P_0, x - x_0\rangle$

$k\langle x - x_1, x_0 - x_1\rangle$

$= k\langle x - x_0, x_0 - x_1\rangle + k|x_0 - x_1|^2 \Rightarrow P_0 = \boxed{k \ \ x_0 - x_1}$

$k|x_0 - x_1|^2 = u(x_0)$

$z = u(x_0) + \langle P_0, x - x_0\rangle$

$$\Longrightarrow \quad P_0 = \frac{u(x_0)}{|x_0 - x_1|^2}(x_0 - x_1)$$

One may check $P_0 \in \partial v(x_0)$ directly as well

$\boxed{\text{Namely } v(x') \geq v(x_0) + \langle P_0, x' - x_0\rangle} \qquad \forall x' \in \Omega$

$\qquad\qquad\qquad\qquad \overset{\|}{\text{RHS}}$

$\text{LHS} = t\, u(x_0)$

$\underbrace{x' = t x_0 + (1-t) x_1'}$

$\boxed{(1-t)(x_1 - x_0) + (1-t)(x_1' - x_1)}$

$\|$

$(1-t)(x_1' - x_0)$

$\|$

$$\text{RHS} = u(x_0) + \frac{u(x_0)}{|x_0 - x_1|^2}\langle \underbrace{x_0 - x_1 \overset{\|}{\ } x' - x_0}\rangle$$

$$= \underbrace{u(x_0) + \frac{u(x_0)}{|x_0 - x_1|^2}\overset{t}{\ }(1-t)|x_0 - x_1|^2}_{\| \; t\,u(x_0)} + \underbrace{\frac{u(x_0)}{|x_0 - x_1|^2}(1-t)\underbrace{\langle x_1' - x_1, x_0 - x_1\rangle}_{\geq 0}}_{\leq 0}$$

since $x_1' \in \overline{\Omega}$

$\boxed{\text{RHS} \leq t\,u(x_0)}$

□

Ⅱ Now we prove $\begin{cases} \det(\nabla^2 u) = 0 \\ u|_{\partial\Omega} = g \end{cases}$ can be solved, by the "Perron" method

Namely taking sup among a class of functions which is $\leq$ the solution (if it exists).

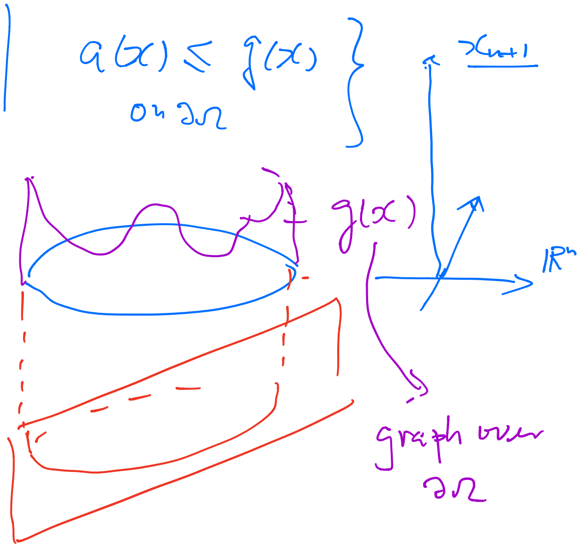
- $\Delta u = f$. — orthogonal transformation is important

- $\det(\nabla^2 u) = f$ Affine transformation is important + & linear functions

$$\langle a, x\rangle + \lambda$$
$$\|$$

$$S := \left\{ a(x) - \text{'linear' function} \;\middle|\; \begin{array}{c} a(x) \leq g(x) \\ \text{on } \partial\Omega \end{array} \right\}$$

Clearly S is NoT empty



$$u(x) := \sup_{a \in S} a(x)$$

Trivially,
$u(x)$ is convex since it is the sup of convex functions
& $u(x)|_{\partial\Omega} \leq g(x)|_{\partial\Omega}$

Claim: (1) $\mathcal{M}(u)(E) = 0$, $\forall$ E Borel.

(2) $u(x) = g(x)$ $\forall x \in \partial\Omega$.

(3) $\lim_{y \to x \in \partial\Omega} u(y) = f(x)$

namely u solves $\begin{cases} \mathcal{M}(u) = 0 \\ u|_{\partial\Omega} = g \end{cases}$

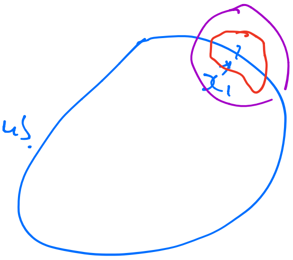
$\exists$ G affine $G|_{\partial\Omega} \geq g|_{\partial\Omega} \Rightarrow G \geq a(x)$ $\forall a \in S$
$\Rightarrow u \leq G \Rightarrow u$ continuous in Ω.

Proof of (2) ∀ $x_1 \in \partial\Omega$, Clearly $u(x_1) \le g(x_1)$

We shall show ∀ $\varepsilon > 0$ $u(x_1) \ge g(x_1) - \varepsilon$

$\exists \delta$, $\forall x \in B_\delta(x_1)$ $x \in \partial\Omega$

$|g(x) - g(x_1)| \le \varepsilon$, by g is continuous



$\Leftrightarrow g(x_1) - \varepsilon \le g(x) \le g(x_1) + \varepsilon$

At x_1 ∃ supporting plane $P(x)$ — linear in x

$P(x_1) = 0$ & $P(x) \ge 0$ $\forall x \in \bar{\Omega}$

Define $B_\eta = \{ P(x) \le \eta \}$

Claim $B_\eta \cap \bar{\Omega} \subset B_\delta(x_1)$ if $\eta << 1$

We use Ω is strictly convex.

Otherwise $\exists x_j \in \bar{\Omega}$ $P(x_j) \le \frac{1}{j}$

But $d(x_1, x_j) \ge \delta$ $\boxed{x_j \to x_\infty \in \bar{\Omega}}$

$$\Rightarrow \begin{cases} P(x_\infty) \le 0 \\ x_\infty \in \bar{\Omega} \end{cases} \Rightarrow P(x_\infty) = 0$$

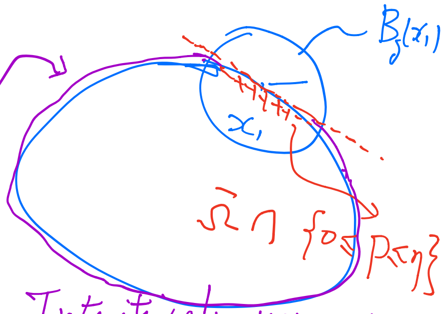
(Since $\forall x \in \bar{\Omega}$, $P(x) \ge 0$)

$\Rightarrow$ x_∞ is on the supporting plane

$\Rightarrow \overline{x_1 x_\infty} \subset \{P = 0\}$ - plane. This forces $\overline{x_1 x_\infty}$ all on $\partial\Omega$

This is contradiction! with strictly convexity

Now we use the similar arguement as the Laplace equation case



Let $b \doteq \min \left\{ g(x) \,\middle|\, \begin{array}{l} \text{for } x \text{ with} \\ P(x) \geq \eta \end{array} \right\}$ $x \in \partial\Omega$

$$a(x) := g(x_1) - \varepsilon - \underline{A P(x)}$$

a barrier similar to Perron method in $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = \varphi \end{cases}$

$$A = \max\left\{ \frac{g(x_1) - \varepsilon - b}{\eta}, 0 \right\}$$

$A \geq 0$

Intuitively very easy

$a(x) \leq g(x)$ in $B_\delta(x_1)$

$a(x) \leq g(x)$ by A big outside the ball.

$\leq b \leq$

Claim $g(x) \geq a(x)$ on $\partial\Omega$

Case 1. $\boxed{0 \le P(x) \leq \eta} \Longrightarrow g(x) \geq g(x_1) - \varepsilon$

$\Rightarrow x \in B_\delta(x_1)$ $\nearrow$

$\geq g(x_1) - \varepsilon - AP(x) = a(x)$

$AP(x) \geq 0$ on $\partial\Omega$

Case 2 $P(x) \geq \eta$

$$a(x) = g(x_1) - \varepsilon - AP(x) \leq g(x_1) - \varepsilon - A\eta$$

$$= \begin{cases} g(x_1) - \varepsilon \leq b \\ g(x_1) - \varepsilon - (g(x_1) - \varepsilon - b) = b \end{cases}$$

$A = 0 \Longleftrightarrow g(x_1) - \varepsilon - b \leq 0$

$g(x_1) - \varepsilon \leq b$

$\leq b \leq g(x)$

$a \in S_-$

$\Longrightarrow \boxed{u(x) \geq a(x)} \Longrightarrow u(x_1) \geq a(x_1) = g(x_1) - \varepsilon$

proof of (1) $\forall\, p \in \partial u(\Omega) \quad \exists\, x_0 \in \Omega$

(*) $u(x) \geq \boxed{u(x_0) + \langle p, x - x_0 \rangle \doteq \ell(x)}$ ← in Ω — linear function

We shall show $\exists\, x_1 \neq x_0 \quad p \in \partial u(x_1)$ (This shows $|\partial u(\Omega)| = 0$)

1st. $\boxed{g(x)\big|_{\partial\Omega} = u(x)\big|_{\partial\Omega} \geq \ell(x)} \quad \forall\, x \in \partial\Omega.$ [Assume (3), (*) $\Rightarrow \ell(x)$, by taking limit]

2nd

But if $g(x) - \ell(x) > 0$ on $\partial\Omega \Rightarrow$

$g(x) \geq \boxed{\ell(x) + \varepsilon}$ since $\in S$ $\Rightarrow$ $u(x) \geq \ell(x) + \varepsilon \quad \forall\, x \in \Omega$

$\Downarrow$

$u(x_0) = \ell(x_0) \geq \ell(x_0) + \varepsilon.$

$\Rightarrow (\Rightarrow\Leftarrow)$

Hence $\exists\, x_1 \in \partial\Omega$ & $g(x_1) = \ell(x_1)$

Thirdly $\Rightarrow \forall\, t \in [0, 1]$

Convexity

$$u(\underbrace{t x_0 + (1-t)}_{}x_1) \leq \boxed{t\, u(x_0) + (1-t)\, u(x_1)}$$

$x = t x_0 + (1-t) x_1$

$u(x_0) = \ell(x_0)$, $u(x_1) = g(x_1) = \ell(x_1)$

$\mathbin{\rotatebox[origin=c]{90}{$\leq$}} \leftarrow p \in \partial u(x_0)$

$u(x_0) + \langle p, t(x_0) + (1-t)x_1 - x_0 \rangle$

$\|$

$u(x_0) + (1-t)\langle p, x_1 - x_0 \rangle$

$= t\,\ell(x_0) + (1-t)\,\ell(x_1)$

$\|$

$t(u(x_0) + \langle p, x_0 - x_0 \rangle)$

$+ (1-t)(u(x_0) + \langle p, x_1 - x_0 \rangle)$

$= \boxed{u(x_0) + (1-t) \cdot \langle p, x_1 - x_0 \rangle}$

⇒ All equality holds

$$\boxed{u(tx_0+(1-t)x_1) = u(x_0)+(1-t)\langle p, x_1-x_0\rangle}$$

Then: $\forall x\in\Omega$

using

$$\boxed{u(x)} \geq \boxed{u(x_0) + \langle p, x-x_0\rangle}$$

by $p\in\partial u(x_0)$

using above

$$= \underbrace{u(tx_0+(1-t)x_1)}_{x_t} - \underline{(1-t)\langle p, x_1-x_0\rangle + \langle p, x-x_0\rangle}$$

$$= \underline{u(\overset{x_t}{\overset{\|}{t x_0 + (1-t)x_1}})} + \langle p, x - [\overset{x_t}{\overset{\|}{(1-t)x_1+tx_0}}]\rangle$$

$$\Rightarrow p \in \partial u(\underbrace{\overset{\in\Omega}{tx_0+(1-t)x_1}}_{\| x_t}). \quad 1>t>0$$

⇒ $|\partial u(\Omega)|=0$. Namely $M(u)=0$

If H-(ℓ graph) support epi(u) & touches a boundary point ⇒ a line segment $\in H$.

The proof of (3) is similar to (2)

∃ a paper of mine (2014) on Amer. Math. Monthly on Perron method!

The $M(u)=\nu$ case is similar.

$$S(\nu, g) := \left\{ u\in C(\bar{\Omega}) \;\middle|\; \begin{matrix} u|_{\partial\Omega}=g \\ M(u)\geq\nu \end{matrix} \right\}$$

Then $U(x) := \sup\limits_{u\in S} u(x)$ — solves $\begin{cases} M(u)=\nu \\ u|_{\partial\Omega}=g \end{cases}$

Compactness & the 'Perron lifting' used for non-homogenous case.

Also A- maximum principle.

Regularity:

① $u\in C^{\frac{1}{n}}(\Omega)$

Higher ones see. Pogorelov, Caffarelli, Caffarelli-Nirenberg-Spruck

Ⅲ ABP – maximum principle — Ω – bounded convex set

C^2 – Version & Continuous version

① $u \in C^2(\Omega) \cap C(\bar{\Omega})$

$$\max_{\Omega} u \leq \max_{\partial\Omega} u + \frac{D}{\omega_n^{\frac{1}{n}}} \left(\int_{C_*(-u)} |\det(\nabla^2 u)|(x)\, dx \right)^{\frac{1}{n}}$$

$\max_{\partial\Omega} u \overset{\text{(W.L.G}}{=} 0)$

(Convex | Contact set) $\uparrow C_*(-u)$

$\downarrow$

② $$\max_{x\in\Omega} u(x) \leq \frac{D}{\omega_n^{\frac{1}{n}}} \Big| \underbrace{\partial((-u)_*)}_{\mathcal{M}((-u)_*)} (C_*(-u)) \Big|^{\frac{1}{n}}$$

$\mathcal{M}((-u)_*)(C_*(-u))$

$\underline{\qquad\qquad} = \square$

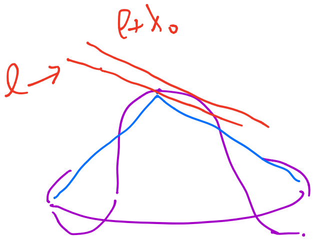
Remark: $\square = \partial(-u)(\Omega)$

Namely $\max u$ can be bounded by the concavity contact set's MA-measure.

$C_*(v)$, v_* are convex function & set - contructed to capture the "Convexity" / " Concave " points of u.

Pf We show ②. Namely

$$(*) \quad \boxed{\max_{x\in\Omega} u \leq \frac{D}{\omega_n^{\frac{1}{n}}} \Big| \underset{\mathcal{M}(-u)(\Omega)}{\partial(-u)(\Omega)} \Big|^{\frac{1}{n}}}$$

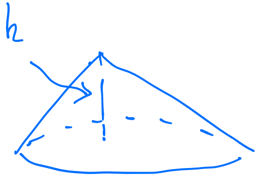


Our discussion before is the mirror case. That's $-u$ is used

$D = \text{diam}(\Omega)$

$\omega_n = |B_1(0)|$

Observation!:



$$\partial(-v)(x_0) \supset B_{\frac{h}{D}}(0)$$

Since $-v(x) \geq -h + \langle p\ x - x_0 \rangle \quad \forall p \in B_{\frac{h}{D}}(0)$

$x = tx_0 + (1-t)x_1$

$-th \geq -h + \langle p, x - x_0 \rangle$

$\Longrightarrow -v(x) = -th$

$\Leftrightarrow (1-t)h \geq + \langle p, (1-t)(x_1 - x_0) \rangle$

$\langle p, x_1 - x_0 \rangle \leq \frac{h}{D} \cdot D = h$ $\Uparrow$ always holds.

Observation 2 : $\partial(-v)(x_0) \subset \partial(-u)(\Omega)$.

$$\underline{-v(x)} \geq \underline{-v(x_0)} + \langle p, x-x_0\rangle$$

$$\Longleftrightarrow \quad v(x) \leq \underbrace{v(x_0) - \langle p, x-x_0\rangle}$$

[In particular $v(x_0) - \langle p, x-x_0\rangle \geq 0$ on $\bar{\Omega}$]

$v(x_0) - \langle p, x-x_0\rangle + \lambda$ — linear function

$F(x)$ $\boxed{v(x_0) - \langle p, x-x_0\rangle + \lambda - u(x)}$

$\exists \lambda$ this is ≥ 0
$F(x)$
& $\exists \lambda_0$ such that

this holds $\forall x \in \bar{\Omega}$ & λ_0 is the smallest.

This argument was used in Lecture (-1)

$$\Longrightarrow \quad v(x_0) - \langle p, x-x_0\rangle + \lambda_0 \geq u(x)$$

But $\quad v(x_0) - \langle p, x-x_0\rangle - \lambda_0 = u(x)$ somewhere say at x_1

At x_0 $\quad \underset{\overset{\|}{h}}{v(x_0)} + \lambda_0 \geq \underset{\overset{\|}{h}}{u(x_0)} \Longrightarrow \lambda_0 \geq 0$

If $\lambda_0 = 0 \Longrightarrow \quad u(x) \leq u(x_0) - \langle p, x-x_0\rangle$

$\Longleftrightarrow \quad -u(x) \geq -u(x_0) + \langle p, x-x_0\rangle \Rightarrow p \in \partial(-u)$

Otherwise $\lambda_0 > 0$

Now use $\quad v(x_0) - \langle p, x-x_0\rangle \geq 0$

$$\Rightarrow \begin{cases} v(x_0) - \langle p, x-x_0\rangle + \lambda_0 > 0 & \text{on } \partial\Omega \\ \& \ u(x) \leq 0 & \text{on } \partial\Omega \end{cases} \Longrightarrow \quad x_1 \bar{\in} \partial\Omega$$

Then $\quad v(x_0) - \langle p, x_1-x_0\rangle + \lambda_0 = u(x_1)$

But $\quad v(x_0) - \langle p, x-x_0\rangle + \lambda_0 \geq u(x) \quad \forall x \in \Omega$

$$\Longleftrightarrow \quad u(x) \leq u(x_1) - \langle p, x-x_1\rangle$$

This $\Rightarrow p \in \partial(-u)(\Omega)$.

Combining ob① & ob② $\Rightarrow$ $\left(\frac{h}{D}\right)^n \omega_n \leq (\partial(-u))(\Omega)$ $\quad \square$

Next We shall prove a Liouville-type theorem. on $\det(\nabla^2 u) = 1$

Namely prove: $\forall u$ $\nabla^2 u > 0$, $\det(\nabla^2 u) = 1$, $\Rightarrow$ u is a quadratic function polynomial.

If u is convex function on $\mathbb{R}^n$ $\Longrightarrow$ $\exists$ $\langle x \ p\rangle - \lambda$

Such that $u(x) \geq \langle x. \ p\rangle - \lambda$ [Rockaffellar P. 103, Coro 12.1.2]

$$u' = u - \langle x, p\rangle + \lambda \qquad \nabla^2 u' = \nabla^2 u, \qquad u' \geq 0$$

Hence we may assume W. L. G $u \geq 0$

Even better, $u(x) = u(0) + \langle \nabla u(0), x\rangle + \int_0^1 (1-t)\langle \left(\nabla^2 u\big|_{tx}\right)(x), x\rangle dt$

If $\tilde{u} = u(x) - u(0) - \langle \nabla u(0). x\rangle$

$$\Rightarrow \quad \tilde{u}(x) = \int_0^1 (1-t)\langle \nabla^2 \tilde{u}\big|_{tx}(x), x\rangle dt > 0$$

$$\Rightarrow \quad \tilde{u}(0) = 0 \quad \nabla\tilde{u}(0) = 0$$

Namely we focus on the case:
$$\begin{cases} \det(\nabla^2 u) = 0 \\ u \geq 0 \\ u(0) = 0, \nabla u(0) = 0 \end{cases}$$

Added: $x_n \to x_1 \in \partial\Omega$, $x_n \in \Omega$ $\quad \lim_{n\to\infty} u(x_n) = g(x_1) = u(x_1)$

The Continuity of u on Ω is ensured by the Convexity.

Hence we only need to worry about boundary points.

ⓘ $\underline{a(x) = g(x_1) - \varepsilon - A\,p(x)}$ $\quad \underline{a(x) \leq g(x)}$

$\Rightarrow$ $\underline{a(x) \leq u(x)}$ $\Longrightarrow$ $\underline{\lim_{n\to\infty} a(x_n) \leq \varliminf_{n\to\infty} u(x_n)}$ $\Longrightarrow$ $\boxed{\varliminf_{n\to\infty} u(x_n) \geq g(x_1) - \varepsilon}$

(ii) Let $h(x)$ be $\begin{cases} \Delta h = 0 \\ h|_{\partial\Omega} = g \end{cases}$ $\Rightarrow$ $\underline{h(x) - l(x) \text{ harmonic}}$ $\Rightarrow$ $h(x) - l(x) \geq 0$ in Ω $\forall \varepsilon$

$\forall l \in S_-$ & $h(x) - l(x)\big|_{\partial\Omega} \geq 0$ $\Rightarrow$ $\underline{h(x) \geq u(x)}$

$\Longrightarrow$ $g(x_1) = \lim_{x\to x_1} h(x) \geq \varlimsup_{x_n \to x_1} u(x_n)$