Regularity of $\begin{cases} \det(\nabla^2 u) = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

is very tricky. See example in Exercise 3 1.

Interior regularity is related to the boundary value. This is drastically different from the linear case

Regularity is dual to $\begin{cases} \text{Liouville type theorem} \\ \text{Bernstein type theorem} \end{cases}$

Theorem (Jörgens–Calabi–Pogorelov)

$u \in C^\infty(\mathbb{R}^n)$, $\det(\nabla^2 u) = 1$ $\nabla^2 u > 0$

then u is a quadratic polynomial.

I shall present a proof using John's Lemma & Evan's (1982)

interior $C^{2,\alpha}$ estimate. The key is a C^2-estimate

of Pogorelov. or $\boxed{\text{Calabi's } C^3\text{-estimate}}$ (1958) $\left[\begin{array}{l}\text{1972 paper} \\ \text{1978 book}\end{array}\right]$

[Pogorelov's original proof of JCP theorem & used a geometric arguement]

(I) Theorem (Pogorelov). $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$

is a convex solution of $\begin{cases} \det(\nabla^2 u) = \varphi > 0, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$

Then $|\nabla^2 u|(x) \leq \left[C(|u|_0, |\nabla u|_0, |\nabla \ln\varphi|_0, |\nabla^2 \ln\varphi|_0) \Big/ \big(d(x,\partial\Omega) \operatorname{Max}|u|\big) \right]$ (1)

pf: $\forall \alpha$ $|\alpha| = 1$ direction consider $W = (-u) e^{\frac{1}{2} u_\alpha^2} u_{\alpha\alpha}$ $(= 0 \text{ on } \partial\Omega)$.

$\Rightarrow$ It attains its maximum somewhere x_0. (say $x_0 = 0$) We estimate $W(x_0)$

First pick a frame $\underset{\text{at } x_0 = 0}{\underbrace{\quad}}$ $W = (-u) e^{\frac{1}{2} u_1^2} u_{11}$ & $u_{ij} = u_{ii}\delta_{ij}$ at x_0

$0 = \nabla_i \log w = \frac{u_i}{u} + u_{ii} u_1 + \frac{u_{11i}}{u_{11}}$ $\Rightarrow$ $\frac{u_1}{u} + u_{11} u_1 + \frac{u_{111}}{u_{11}} = 0$, $\frac{u_i}{u} + \frac{u_{11i}}{u_{11}} = 0$ $i \geq 2$ (1)

$0 \geq \nabla_i \nabla_i \log w = \frac{u_{ii}}{u} - \underbrace{\left(\frac{u_i}{u}\right)^2}_{\text{Bad}} + u_{iii} u_1 + \underbrace{u_{1i} u_{1i}}_{= u_{11}^2} + \frac{u_{11ii}}{u_{11}} - \underbrace{\frac{u_{11i}^2}{(u_{11})^2}}_{\text{Bad}}$ (2)

$\sum_{i=1}^n \frac{u_{11}}{u_{ii}}$ multiplying $\Rightarrow$ $n \frac{u_{11}}{u} + \sum_i \frac{u_{11ii}}{u_{ii}} - \sum_i \frac{u_{1i}^2}{u_{ii}} \frac{1}{u_{11}} + u_{11}^2 + u_1 u_{11} \sum_i \frac{u_{1ii}}{u_{ii}}$ using (3)

$- \left(\frac{u_1}{u}\right)^2 - \sum_{i \neq 1} \frac{(u_{11i})^2}{u_{ii}} \frac{1}{u_{11}} \leq 0$ using (4)

$$\ln\det(u_{ij}) = \ln\varphi$$

$$\boxed{\sum\left(\frac{u_i}{u}\right)^2\frac{u_{11}}{u_{ii}} = \left(\frac{u_1}{u}\right)^2 + \sum_{i\neq 1}\left(\frac{u_i}{u}\right)^2\frac{u_{11}}{u_{ii}}}$$

Now we use $u^{ij}u_{ij1} = (\ln\varphi)_1$ $\qquad \boxed{\frac{u_{ii1}}{u_{ii}} = (\ln\varphi)_1}$ ③

$$u^{ij}{}_1 u_{ij1} + u^{ij}u_{ij11} = (\ln\varphi)_{11}$$

$$\Rightarrow \quad \frac{u_{ii11}}{u_{ii}} - \frac{(u_{ij1})^2}{u_{ii}u_{jj}} = (\log\varphi)_{11} \quad ④$$

$$\Rightarrow \begin{cases} n\frac{u_{11}}{u} + \underbrace{\sum_{i,j=1}^n \frac{(u_{ij1})^2}{u_{ii}u_{jj}} + (\log\varphi)_{11}}_{④} - \sum_i \frac{u_{11i}^2}{u_{ii}u_{11}} + u_{11}^2 + u_1u_{11}\underbrace{(\log\varphi)_1}_{③} \\ -\left(\frac{u_1}{u}\right)^2 - \sum_{i\neq 1}\frac{(u_{11i})^2}{u_{ii}u_{11}} \leq 0 \end{cases}$$

Bad

Bad

$$\sum\frac{u_{ij1}^2}{u_{ii}u_{jj}} - \sum_i \frac{u_{11i}^2}{u_{ii}u_{11}} - \sum_{\substack{i\neq 1\\ \Updownarrow \\ i=2}}^n \frac{u_{11i}^2}{u_{ii}u_{11}} \geq 0$$

$$\Rightarrow \boxed{u_{11}^2 + n\frac{u_{11}}{u} + u_1u_{11}(\log\varphi)_1 + (\log\varphi)_{11} - \left(\frac{u_1}{u}\right)^2 \leq 0}$$

Multiplying $u^2 e^{u_1^2} \Rightarrow w^2 + Aw + B \leq 0$

A, B depends on $u, u_1, (\log\varphi)_1, (\log\varphi)_{11}$

$$A = n e^{u_1^2/2} + u_1(\log\varphi_1)ue^{u_1^2/2}$$

$$B = u^2(\log\varphi)_{11}e^{u_1^2} - (u_1)^2e^{u_1^2}$$

Hence $\quad w \leq \frac{-A+\sqrt{A^2-4B}}{2}$

Namely at the maximum $\quad w(x_0) \leq C_1$

$$(-u)e^{\frac{1}{2}u_1^2}u_{11}(x_0) \leq C_1$$

$$\Rightarrow \underline{(-u)(x)}\, e^{\frac{1}{2}(\langle Du, x\rangle)^2}\nabla^2u(x,x) \leq C_1 \Rightarrow \underline{(-u)\nabla^2u(x,x) \leq C_1}$$

$$\boxed{\frac{|u(x)|}{d(x,\partial\Omega)} \geq \frac{\max|u|}{D}} \rightsquigarrow \text{simply using the convexity}$$

$$\Rightarrow \boxed{(\nabla^2u)(x,x) \leq \frac{C_1}{d(x,\partial\Omega)\max|u|}}$$

This is similar to — Aubin-Yau's C^2-estimate on a compact manifold, which was proved later. [See my 2019 IJM paper for a ½-page proof]

Thm II Estimates for $u \in C(\bar\Omega) \cap C^4(\Omega)$, $u \geq 0$.

Can always be done by affine transformation via John's Lemma.

Assume $\boxed{B_{\frac{1}{n^{3/2}}} \subset \Omega \subset B_1}$ convex

$$\begin{cases} \det \nabla^2 u = 1 & \text{in } \Omega \\ u|_{\partial\Omega} = h - \text{constant} \end{cases} \Longrightarrow$$

$u(x_0) = 0$ for some $x_0 \in \Omega$

$\underbrace{S_t \doteq \{x \in \Omega \mid u(x) < t\}}$

Then $\exists\, c_1\ c_2 > 0$ $\quad c_i = c_i(n)$

(i) $c_1 \leq h \leq c_2$

(ii) $\exists\, \tau_0(n) \in (0,1)$ $\quad B_{\tau_0}(x_0) \subset \Omega$

$u < \frac{1}{2} h$ in $\boxed{B_{\tau_0}(x_0)}$ $\to B_{\tau_0}(x) \subset S_{\frac{h}{2}}$

Enough to apply Calabi

(iii) $c_1 I \leq \nabla^2 u < c_2 I$ in $S_{h/2}$

& $\exists\, \alpha \in (0,1)$ $\quad |\nabla^2 u|_{C^\alpha(B_{\tau_0}(x_0))} \leq C_1$

From Evans' estimate (17.41) of Gilbarg-Trudinger to be proved in next Lecture.

The above is the key in gradient

Under the assumption of JCP if $C_1 I \leq \nabla^2 u \leq C_2 I$

$\Longrightarrow$ $ds_u^2 = u_{ij}\, dx^i dx^j$ is a complete metric on $\mathbb{R}^n$

Calabi's result [1958 paper] Corollary on p.108] $\Longrightarrow$

u is quadratic.

Or if $[\nabla^2 u]_{\alpha, B_{\tau_0 R}} \leq \dfrac{C}{R^\alpha}$

$\Longrightarrow \forall\, x, y \in B_{\tau_0 R}$

$$\frac{|\nabla^2 u(x) - \nabla^2 u(y)|}{|x-y|^\alpha} \leq \frac{C}{R^\alpha}$$

$$\Longrightarrow |\nabla^2 u(x) - \nabla^2 u(y)| \leq \frac{C\,|x-y|^\alpha}{R^\alpha} \longrightarrow 0 \quad \text{as } R \to \infty$$

$\Longrightarrow$ $\nabla^2 u$ is constant

We shall first show how to use Thm to JCP situation.

Then we prove Thm.

By translation, rotation, & unimodular affine transformation, we assume $u(0)=0$, $\nabla u(0)=0$

$\nabla^2 u(0) = I$, $u \geq 0$

Now we consider $\Omega = \{ x \mid u(x) \leq a^2 \}$

which is a convex body in $\mathbb{R}^n$. By John's theorem.

$\exists$ E centered at $x_0 \in \Omega$, x_0 is the center of gravity of Ω,

$$E = \left\{ x \;\middle|\; \frac{(x^i - x_0^i)^2}{b_i^2} \leq 1 \right\}, \quad E_{\frac{1}{n^{3/2}}} \subset \Omega \subset E$$

This is due to that

$\det \nabla^2 u = 1$ does not change under an affine transformation $T: x \to Ax + c$ with $\det A = \pm 1$

Namely if $\tilde{u}(x) = u(Ax+c)$

$$\frac{\partial \tilde{u}}{\partial x^i} = \frac{\partial u}{\partial y^k} a_i^k \qquad \frac{\partial^2 \tilde{u}}{\partial x^i \partial x^j} = \frac{\partial^2 u}{\partial y^k \partial y^l} a_j^l a_i^k$$

$$= (a_i^k) \left(\frac{\partial^2 u}{\partial y^k \partial y^l} \right) a_j^l \Rightarrow \nabla^2 \tilde{u} = (A)^{tr} (\nabla^2 u) A$$

$$\Rightarrow \det(\nabla^2 \tilde{u}) = \det(A^{tr}) \det(A) \det(\nabla^2 u)$$

This has been used when we assume $\nabla^2 u(0) = id$.

Clearly using orthogonal transformation E can be put into the form we claimed, But it does not change $\nabla^2 u|_0 = I$

Now let $y^i = \dfrac{x^i - x_0^i}{b_i}$ $\iff$ $\boxed{x^i = b_i y^i + x_0^i}$

$E \longrightarrow B_1$ — ball

$E_{\frac{1}{h^{3/2}}} \longrightarrow B_{\frac{1}{h^{3/2}}}$ — smaller ball.

$\Omega \longrightarrow \tilde{\Omega}$, $B_1 \supset \tilde{\Omega} \supset B_{\frac{1}{h^{3/2}}}$

$\boxed{v(y)} = \lambda u(By + x_0)$

$\nabla^2 v = B^{tr}(\lambda \cdot \nabla^2 u) B \implies \det(\nabla^2 v) = \left(\prod_{i=1}^n b_i^2\right)\lambda^n$

$\implies \lambda^n = \dfrac{1}{\prod_{i=1}^n (b_i)^2}$

Now $v(\underbrace{-B^{-1}x_0}_{\tilde{x}_0}) = 0$ $\qquad v|_{\partial\Omega} = \underbrace{\boxed{\lambda a^2}}_{h}$

This is the h in Thm. $\implies$ $\underbrace{c_1 \le \lambda a^2 \le c_2}$

$\iff c_1 a^{-2} \le \lambda \le c_2 a^{-2}$

On $S_{h/2} = \{ v \le \frac{h}{2} \} \iff \{ \lambda u \le \frac{h}{2} \}$

$\iff \{ u \le \frac{a^2}{2} \}$

$\underbrace{c_1 I \le \nabla^2 v \le C_2 I}$

$\iff c_1 I \le B^{tr} \lambda \nabla^2 u|_{x=0} B \le c_2 I$

In particular $\implies$ at $x=0$ for u or $\tilde{x}_0$ for v $\quad c_1 I \le \lambda B^{tr} B \le c_2 I$

$\iff c_1 \le \lambda b_i^2 \le c_2$

$\boxed{c_1 \lambda^{-1} \le b_i^2 \le c_2 \lambda^{-1}}$

$c_1 c_2^{-1} a^2 \le b_i^2 \le c_2 c_1^{-1} a^2$

$\implies \boxed{c_1 a^2 \le b_i^2 \le c_2 a^2}$ $\quad (*)$

$$\Rightarrow \quad C_1' I \le \nabla^2 u \le C_2' I \qquad \text{in } \underbrace{\{x \mid u \le \tfrac{a^2}{2}\}}_{\| \atop S^u_{a^2/2}}$$

$S^u_{a^2/2}$ also contains a smaller ball (with radius $\tilde{\tau}_0 a$)

This put us into the situation of applying Calabi's theorem.

<u>Precisely</u>, ($\bigstar$) shows that E contains

$$B_{\sqrt{c_1}a}(x_0) \qquad \sum \frac{|x^i - x^i_0|^2}{b_i^2} \le \sum \frac{|x^i - x^i_0|^2}{c_1 a^2} \le 1$$

<u>Also</u> since $\underline{B_{\tau_0}(\tilde{x}_0) \subset S^v_{h/2}}$

$$\Leftrightarrow \{y \mid |y - \tilde{x}_0| \le \tau_0\} \subset S^v_{\frac{h}{2}}$$

$$\Leftrightarrow \underbrace{\{x \mid |B^{-1}x| \le \tau_0\}}_{y = B^{-1}x + \tilde{x}_0} \subset S^u_{\frac{a^2}{2}}$$

$x = By + x_0$

$\Leftrightarrow \tilde{x}_0 = -B^{-1}x_0$

$$\Rightarrow B_{\tilde{c}\tau_0 a^2}(0) \subset S^u_{a^2/2}$$

$\Bigg\}$ $B_{\eta a^2}(0) \subset S^u_{a^2/2}$ for some $\eta > 0$ independent of a

Namely, Thm provide the normalized situation to obtain the needed estimates for JCP.

Now we focus on the proof of Thm.

III Proof of Thm (u is the one in Thm, not in JCP).

We use ①, the Comparison principle 2, the Alexandrov maximum principle.
↑ Pogorelov estimate; CP2

For (i). Apply CP2 to $\underbrace{\frac{1}{2}(|x|^2-1)+h}_{u_1}$ & u

$\Longrightarrow$ on $\partial\Omega$ $u_1 \le u$ $\Rightarrow$ $\min_{\partial\Omega}(u-u_1) \ge 0$

$\Longrightarrow$ $u-u_1 \ge 0$ in Ω

$\det(D^2 u_1) = 1 = \det(D^2 u)$

$\Longrightarrow$ $u(x) \ge \frac{1}{2}(|x|^2-1)+h$

$\underset{0}{\overset{u(x_0)}{\parallel}} \Rightarrow$ $h \le \frac{1}{2}(1-|x_0|^2) \le \frac{1}{2}$

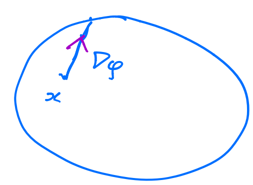
Apply CP2 to $\underbrace{\frac{1}{2}\left(|x|^2 - \frac{1}{n^{3/2}}\right)+h}_{u_2}$ & u

$u_2\big|_{\partial\Omega} \ge u\big|_{\partial\Omega} \Longrightarrow$ $u_2 \ge u$ in Ω

$\Longrightarrow$ $-\frac{1}{2n^{3/2}}+h \ge u(0)$ (≥ 0)

$\Longrightarrow$ $h \ge \frac{1}{2n^{3/2}}$

For (ii). We need the gradient estimate for convex function



φ $\qquad$ $\varphi\big|_{\partial\Omega} = 0$

$\Longrightarrow$ $\varphi(y) \ge \varphi(x) + \langle D\varphi(x), y-x \rangle$

$\Longrightarrow$ If we pick $y = x + \eta \boxed{\frac{D\varphi(x)}{|D\varphi(x)|}}$ $\Longrightarrow$ $|D\varphi(x)| \le \frac{\varphi(y)-\varphi(x)}{\eta}$

As long as such $y \in \Omega$

$\Longrightarrow$ $\boxed{② \rightarrow |D\varphi(x)| \le \frac{-\varphi(x)}{\mathrm{dist}(x, \partial\Omega)}}$ by taking η to the limit of such possibility.

The key is still A- maximum principle:

$$\left. u-h \right|_{\partial\Omega} = 0$$

& $\forall x$

$$(h-u)^n(x) \leq C\, \underline{D}^{n-1}\, d(x, \partial\Omega)\, \underbrace{|\partial u(\Omega)|}_{\Omega = S_h} \leq \omega_n$$

$$\leq \underline{C_n\, d(x, \partial\Omega)} \qquad (D \leq 1)$$

$$\int_{\Omega} \det(\nabla^2 u) = |\Omega| \leq \omega_n$$

Hence $\forall x \in S_{h/2}$

$$\left(\frac{h}{2}\right)^n \leq (h-u)^n \leq C_n\, d(x, \partial S_h)$$

$$\Rightarrow \quad \underbrace{\operatorname{dist}(S_{h/2}, \partial S_h)} \geq C^{-1}\left(\frac{h}{2}\right)^n \geq c(n)$$

Similarly

$$\operatorname{dist}(S_{h/4}, \partial S_{h/2}) \geq c(n)$$

Note that $\boxed{x_0 \in S_{h/4}}$

$$\Rightarrow \quad \boxed{B_{c(n)}(x_0) \subset S_{h/2}} \quad \Rightarrow \quad B_{c(n)}(x_0) \text{ is away from } \partial\Omega.$$

In fact $B_{c(n)/2}(x_0)$ will not touch the boundary of $\partial S_{h/2}$ hence $u(x) < \frac{1}{2}h$.

If want to be sure

$$u(x) - u(x_0) \leq -\langle \nabla u(x), x - x_0 \rangle$$

$$\Rightarrow \quad |u(x)| \leq |\nabla u|\, |x - x_0| \leq \boxed{\frac{h/2}{d(x, \partial\Omega)}} \cdot \delta$$

If $\delta \ll 1$ $\quad \delta = \delta(n)$

$$\Rightarrow \quad |u(x)| < \frac{h}{2}$$

Not really needed

To get (iii), the Hessian estimate, we apply ② to $x \in S_{\frac{3}{4}h}$ & $\underbrace{u-h}_{= \varphi}$

Then, $$|\nabla u|(x) \le \frac{h}{\underset{= \partial S_h}{\operatorname{dist}(x, \partial\Omega)}} \le \frac{h}{\operatorname{dist}(S_{\frac{3}{4}h}, \partial S_h)} \le C$$

$\exists\, C(n) \le \operatorname{dist}(S_{\frac{3}{4}h}, S_h)$

Now Pogorelov Estimate

$$\sup\left[\left(\tfrac{3}{4}h - u\right) u_{\alpha\alpha}\right] \le C$$

$\Rightarrow$ In particular in $B_{C(n)}(x_0) \subset S_{h/2}$

$$\sup\left[u_{\alpha\alpha}\right] \le \frac{C}{\frac{3}{4}h - \frac{1}{2}h} \le C$$

Since $\prod_{i=1}^{n} u_{ii} = 1 \Rightarrow$ the minimum one $u_{\beta\beta} \ge C'$

$$\left[\quad u_{\beta\beta} = \frac{1}{\prod_{j \ne \beta} u_{jj}} \ge \left(\frac{1}{C}\right)^{n-1} \quad\right]$$

This proves the uniform estimate in (iii).

The Hölder estimate now follows from Evans' estimate

G-T 17.4.

Next, we show how to prove Evans' estimate. The key is a Harnack estimate of Krylov-Safanov. [Only need the weaker form.]

<u>Added</u>: Affine transformation to diagonalize (u_{ij})

W.L.G. $x_0 = 0$

$$\left.\left(\nabla^2_{ij} u\right)\right|_{x_0=0} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

$$\Rightarrow \quad B^t \nabla^2_{ij} u\big|_0 B = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_1 & \\ \vdots & & & \end{pmatrix}$$

$\exists \begin{pmatrix} 1 & 0 \\ 0 & O_1 \end{pmatrix} \doteq O$ such that $O_1^{tr} A_1 O_1$ is diagonal.

Now consider $v(x) = u(BOx)$

$$\nabla^2 v\big|_0 = O^{tr} B^{tr} \nabla^2 u \, B \, O\big|_0$$

$$\Rightarrow \quad = \text{diagonal}$$

In fact

$$C = BO = \begin{pmatrix} 1 & \alpha^{tr} \\ 0 & id \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & O_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \alpha^{tr} O_1 \\ 0 & O_1 \end{pmatrix} \Rightarrow \nabla^2_{11} v = \nabla^2_{11} u$$

Also $v_1 = u_1$

This takes care of the issue.