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A Mean Value Formula and a Liouville Theorem for the Complex Monge–Ampère Equation

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In this paper, we prove a mean value formula for bounded subharmonic Hermitian matrix valued function on a complete Riemannian manifold with nonnegative Ricci curvature. As its application, we obtain a Liouville type theorem for the complex Monge–Ampère equation on product manifolds.

1. Introduction

Understanding various spaces of harmonic functions on complete noncompact Riemannian manifolds is one of the central questions in geometric analysis. During the last 40 years, there has been much significant progress in this question (see e.g., [7, 9, 17–20, 32, 34, 35], $\cdots$). More importantly, the techniques developed in this field are extremely useful when applied to other problems in geometric analysis. In [18], Peter Li proved the following theorem:

Theorem 1.1. (Theorem 2, [18]) Let (M^n, ω) be a complete Kähler manifold with nonnegative Ricci curvature and $\mathcal{H}^1(M)$ be the space of linear growth harmonic functions

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on (M^n, ω). Then $dim\mathcal{H}^1(M) \leq 2n+1$. Moreover, if $dim\mathcal{H}^1(M) = 2n+1$ then M must be isometric to $\mathbb{C}^n$ with the standard flat metric.

In Peter Li's proof of Theorem 1.1, the following mean value theorem for bounded subharmonic functions plays an important role.

Theorem 1.2. (Lemma B, [18]) Let (M, g) be a complete manifold with nonnegative Ricci curvature. Suppose f is a bounded subharmonic function defined on (M, g), then for any $p \in M$

$$\lim_{r\to\infty} \fint_{B_r(p)} f \ \mathrm{d}V_g = \sup_M f. \tag{1}$$

Besides its application in [18], Theorem 1.2 has some more applications in Riemannian geometry (see e.g., [9]). It is a useful tool in the study of linearly growth harmonic functions on complete Riemannian manifolds with nonnegative Ricci curvature.

In this paper, we study a class of Hermitian matrix valued functions and establish a mean value theorem for them. For convenience, we denote the set of all m-order Hermitian matrices by $\mathbf{Hm}(m)$, and equip it with the metric induced by the inner product

$$\langle A, B\rangle = \mathrm{tr}\, A\overline{B}^T. \tag{2}$$

Definition 1.3. A map $A = (A_{ij})$ from a Riemannian manifold to $\mathbf{Hm}(m)$ is said to be subharmonic, if for any vector $\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{C}^m$, $\xi A \xi^* = A_{ij}\xi_i\overline{\xi_j}$ is a subharmonic function.

By the definition, it is easy to check that a $\mathcal{C}^2$ Hermitian matrix valued function $A = (A_{ij})$ on a Riemannian manifold is subharmonic if and only if $\Delta A = (\Delta A_{ij})$ is semi-positive-definite everywhere. We obtain the following mean value formula to subharmonic Hermitian matrix valued functions.

Theorem 1.4. Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature, and $A = (a_{ij})$ be a bounded subharmonic Hermitian matrix valued function on (M, g). Then there exists a Hermitian matrix A_0, such that

$$A \leq A_0 \tag{3}$$

on M, and for any $p \in M$

$$\lim_{r\to\infty} \fint_{B_r(p)} A \, \mathrm{d}V_g = \lim_{r\to\infty} \left(\fint_{B_r(p)} a_{ij} \, \mathrm{d}V_g \right) = A_0. \tag{4}$$

The complex Monge–Ampère equation has significant applications in complex analysis and complex geometry, and much remarkable progress of complex Monge–Ampère equation was made by many people (see e.g., [1–4, 6, 8, 10–13, 15, 16, 21, 23–30, 33, 36–38], etc.). In this paper, we concentrate on Liouville theorems for the complex Monge–Ampère equation. In [22], Riebesehl and Schulz proved a Liouville theorem for the complex Monge–Ampère equation on $\mathbb{C}^n$, which can be expressed by Kähler forms as follows.

Theorem 1.5. ([22]) Let ω be a Kähler form on $\mathbb{C}^n$ satisfying $C^{-1}\omega_0 \leq \omega \leq C\omega_0$ and $\omega^n = \omega_0^n$, where $\omega_0 = \frac{\sqrt{-1}}{2}\sum_{i=0}^{n} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i$ and C is a positive constant. Then $\nabla_{\omega_0}\omega = 0$, or equivalently

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} A_{ij} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \tag{5}$$

for some constant Hermitian matrix (A_{ij}).

The key of the proof of Theorem 1.5 is a local Calabi $\mathcal{C}^3$ estimate, that is, an estimate on $|\nabla_{\omega_0}\omega|^2_W$. To study the analogous Liouville type theorems on Kähler manifolds with nontrivial Riemannian curvatures should be meaningful (see e.g., [31]). However, in these cases, the Calabi $\mathcal{C}^3$ estimate seems not to work. Recently, Hein ([14]) proved a Liouville theorem for the complex Monge–Ampère equation on product manifolds, which can be restated in short as below.

Theorem 1.6. (Theorem A, [14]) Let (Y, ω_{Y_0}) be a compact Ricci-flat Kähler manifold, and ω be a Ricci-flat Kähler form on $\mathbb{C}^m \times Y$. Assume $C^{-1}(\omega_{\mathbb{C}^m} + \omega_{Y_0}) \leq \omega \leq C(\omega_{\mathbb{C}^m} + \omega_{Y_0})$ for some $C > 1$, where $\omega_{\mathbb{C}^m}$ is the standard flat Kähler form on $\mathbb{C}^m$. Then we can find some Kähler form ω_Y on Y, $T_l \in Auto(\mathbb{C}^m \times Y)$ and complex linear map $S \in Auto(\mathbb{C}^m)$ such that

$$T_l^*\omega = \omega_Y + S^*\omega_{C^m}.$$

In Hein's proof of Theorem 1.6, one key step is to study the convergence property of a sequence of subharmonic functions u_t with respect to Kähler metrics ω_t which are constructed from ω. In this paper, we consider the case that $Ric(\omega_{Y_0}) \geq 0$ and establish the following Liouville theorem.

Theorem 1.7. Let (Y^n, ω_{Y_0}) be an n dimensional compact Kähler manifold with non-negative Ricci curvature, and ω be a Kähler form ω on $\mathbb{C}^m \times Y$ with properties

1) $C^{-1}(\omega_{\mathbb{C}^m} + \omega_{Y_0}) \leq \omega \leq C(\omega_{\mathbb{C}^m} + \omega_{Y_0})$, for some positive constant C;
2) $\omega^{n+m} = (\omega_{\mathbb{C}^m} + \omega_{Y_0})^{m+n}$,

where $\omega_{\mathbb{C}^m}$ is the standard Kähler form on $\mathbb{C}^m$. Then there exists a Kähler form ω_Y on Y with $Ric(\omega_Y) = Ric(\omega_{Y_0})$ such that $\nabla_{\omega_{\mathbb{C}^m}+\omega_Y}\omega = 0$. Furthermore, we have the following representation of ω

$$\omega = \hat{\omega}_{\mathbb{C}^n} + \omega_Y + \frac{1}{2}\sum_{i=1}^{m}\left(\mathrm{d}z^i \wedge \eta^i + \mathrm{d}\bar{z}^i \wedge \overline{\eta^i}\right), \tag{6}$$

where $\hat{\omega}_{\mathbb{C}^n} = \frac{1}{2}\sum\limits_{i,j=1}^{m} u_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$ with the constant Hermitian matrix $\left(u_{i\bar{j}}\right)$, and every η^i is a ω_Y-parallel (0,1)-form.

Taking the construction of ω_Y, T_l, and S in Theorem 1.6 ([14]) in consideration, Theorem 1.7 can be seen as a generalization of Theorem 1.6. Our proof relies on the above mean value formula (i.e., Theorem 1.4) and is very different with Hein's. Theorem 1.7 also can be seen as an application of the mean value formula (4). We hope the mean value formula (4) has more applications in the study of Kähler geometry.

2. A Mean Value Formula for Bounded Subharmonic Hermitian Matrix Valued Function

In this section, we first give a proof of Theorem 1.4 and then give a new proof to Theorem 1.5 by using Theorem 1.4 instead of the Calabi C^3 estimate.

A proof of Theorem 1.4. For any vector $\xi \in \mathbb{C}^m$, define

$$||\xi||_A^2 = \xi A \xi^*. \tag{1}$$

By this definition and the condition on A, for any fixed $\xi \in \mathbb{C}^m$, $||\xi||_A^2$ is a bounded subharmonic function, then Theorem 1.2 implies

$$\lim_{r\to\infty} \fint_{B_r(p)} ||\xi||_A^2 = \sup_M ||\xi||_A^2. \tag{2}$$

For $i = 1, 2, \cdots, n$, let e_i be the i-th direction vector in $\mathbb{C}^n$. We have

$$A_{ij} = \frac{||e_i + e_j||_A^2 - ||e_i - e_j||_A^2}{4} - \sqrt{-1}\frac{||e_i + \sqrt{-1}e_j||_A^2 - ||e_i - \sqrt{-1}e_j||_A^2}{4}. \tag{3}$$

Together with (2), we assert that $\lim_{r\to\infty} ⨍_{B_r(p)} A$ exists. Let

$$A_0 = \lim_{r\to\infty} ⨍_{B_r(p)} A. \tag{4}$$

Then we know that for any $\xi \in \mathbb{C}^m$, it holds that

$$\xi A \xi^* = ||\xi||_A^2 \le \lim_{r\to\infty} ⨍_{B_r(p)} ||\xi||_A^2 = \xi A_0 \xi^*. \tag{5}$$

This shows $A \le A_0$. ■

We obtain the following simple corollary.

Corollary 2.1. Let $A : M \to \mathbf{Hm}(m)$ satisfy the same condition of Theorem 1.4 and $A_0 = \lim_{r\to\infty} ⨍_{B_r(p)} A$. Let F be a bounded function on some neighborhood of the closure of $A(M)$ and continuous at A_0, then we have

$$\lim_{r\to\infty} ⨍_{B_r(p)} F(A) = F(A_0). \tag{6}$$

Proof. From the condition on F, we see that there exists a positive constant C such that

$$F(A) \le C \tag{7}$$

on M. And for any $\varepsilon > 0$, we can find some $\delta > 0$ such that for any $q \in M$ satisfying $|A(q) - A_0| \le \delta$, there holds

$$|F(A(q)) - F(A_0)| \le \varepsilon. \tag{8}$$

For the mentioned ε and δ, we have

$$\begin{aligned}\int_{B_r(p)} |F(A) - F(A_0)| &= \int_{B_r(p)\cap\{|A-A_0|\le\delta\}} |F(A) - F(A_0)| + \int_{B_r(p)\cap\{|A-A_0|>\delta\}} |F(A) - F(A_0)| \\ &\le \varepsilon \mathrm{Vol}(B_r(p) \cap \{|A - A_0| \le \delta\}) + C\mathrm{Vol}(B_r(p) \cap \{|A - A_0| > \delta\}) \\ &\le \varepsilon \mathrm{Vol}(B_r(p)) + C\mathrm{Vol}(B_r(p) \cap \{|A - A_0| > \delta\}).\end{aligned} \tag{9}$$

By Theorem 1.4, $A \leq A_0$, so $A_0 = \lim_{r\to\infty} ⨍_{B_r(p)} A$ implies

$$\lim_{r\to\infty} ⨍_{B_r(p)} |A - A_0| = 0. \tag{10}$$

Together with

$$\mathrm{Vol}(B_r(p) \cap \{|A - A_0| > \delta\}) \leq \delta^{-1} \int_{B_r(p)} |A - A_0|, \tag{11}$$

we derive

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p) \cap \{|A - A_0| > \delta\})}{\mathrm{Vol}(B_r(p))} = 0. \tag{12}$$

Combining (9) and (12) yields

$$\limsup_{r\to\infty} ⨍_{B_r(p)} |F(A) - F(A_0)| \leq \varepsilon. \tag{13}$$

Let $\varepsilon \to 0$, then we get

$$\lim_{r\to\infty} ⨍_{B_r(p)} |F(A) - F(A_0)| = 0. \tag{14}$$

This concludes the proof. ∎

By Therorem 1.4 and Corollary 2.1 we can give a new proof to Theorem 1.5.

A new proof of Theorem 1.5. We can write ω as

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} u_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j, \tag{15}$$

where $(u_{i\bar{j}})$ is a function valued in $\mathbf{Hm}(n)$. Denote $\left(u^{i\bar{j}}\right) = \left(u_{i\bar{j}}\right)^{-1}$, $u_{i\bar{j}k} = \frac{\partial}{\partial z^k} u_{i\bar{j}}$, $u_{i\bar{j}k\bar{l}} = \frac{\partial}{\partial \bar{z}^l} u_{i\bar{j}k}$, etc. Since ω is closed, we have

$$u_{i\bar{j}k} = u_{k\bar{j}i}, \qquad u_{i\bar{j}\bar{k}} = u_{i\bar{k}\bar{j}}. \tag{16}$$

According to the equation ω satisfied, we deduce

$$\det(u_{i\bar{j}}) = 1. \tag{17}$$

A direct computation shows

$$\Delta_\omega u_{i\bar{j}} = u^{k\bar{l}} u^{p\bar{q}} u_{k\bar{q}i} u_{\bar{l}p\bar{j}}. \tag{18}$$

For any $\xi = (\xi^1, \xi^2, \cdots, \xi^n) \in \mathbb{C}^n$, consider the Hermitian quadratic form $F : \mathbb{C}^{3n} \times \mathbb{C}^{3n} \to \mathbb{R}$ defined by

$$(A, B) \mapsto u^{i\bar{\alpha}} u^{\beta\bar{j}} \xi^k \overline{\xi^\gamma} A_{ijk} \overline{B_{\alpha\beta\gamma}}. \tag{19}$$

By choosing a proper frame on $\mathbb{C}^n$, one can easily check that F is semi-positive-definite. So

$$\xi^i (\Delta_\omega u_{i\bar{j}}) \overline{\xi^j} = u^{i\bar{\alpha}} u^{\beta\bar{j}} \xi^k \overline{\xi^\gamma} u_{i\bar{j}k} \overline{u_{\alpha\bar{\beta}\gamma}} \geq 0. \tag{20}$$

This means that $(u_{i\bar{j}})$ is subharmonic. The condition on ω implies that $(u_{i\bar{j}})$ is bounded and $(\mathbb{C}^n, \omega)$ is a complete Ricci flat Kähler manifold. Based on Theorem 1.4 and Corollary 2.1, we can find a constant Hermitian matrix A such that

$$(u_{i\bar{j}}) \leq A \tag{21}$$

on M and

$$\lim_{r\to\infty} \fint_{B_r^\omega(O)} \det(u_{i\bar{j}}) \omega^n = \det A. \tag{22}$$

From (17) and the previous equality, it holds that

$$\det A = \det(u_{i\bar{j}}) = 1. \tag{23}$$

Since $(u_{i\bar{j}})$ is positive-definite, the previous equality and (21) imply $(u_{i\bar{j}})$ is the constant function A. This concludes the proof. ■

Remark: 1) To prove $\left(u_{i\bar{j}}\right)$ is subharmonic, besides direct computations, we can also use the following argument: for any $\xi \in \mathbb{C}^m$, let $X_\xi = \xi_i \frac{\partial}{\partial z^i}$, then

$$\xi_i u_{i\bar{j}} \overline{\xi_j} = 2|X_\xi|^2_\omega. \tag{24}$$

Using the Bochner formula for holomorphic fields and the fact that $Ric(\omega) = 0$, one can easily check that $\xi_i u_{i\bar{j}} \overline{\xi_j}$ is subharmonic.

2) To prove Theorem 1.5, one can also consider $\left(u^{i\bar{j}}\right) = \left(u_{i\bar{j}}\right)^{-1}$. For any $\xi \in \mathbb{C}^m$, let $f_\xi = Re\left(\xi_i z^i\right)$, then

$$\xi_i u^{i\bar{j}} \overline{\xi_j} = |\mathrm{d}f_\xi|^2_\omega. \tag{25}$$

Clearly f_ξ is a pluri-harmonic function and hence a harmonic function with respect to ω. Using the Bochner formula and the fact that $Ric(\omega) = 0$ one can easily check that $\xi_i u^{i\bar{j}} \overline{\xi_j}$ is subharmonic.

3. A Liouville Theorem for the Complex Monge–Ampère Equation

In this section, we obtain a Liouville theorem for the complex Monge–Ampère equation as an application of the mean value formula (4), that is, we give a proof of Theorem 1.7. First we introduce the following lemma concerning the computation of determinant of a block Hermitian matrix.

Lemma 3.1. Let M be an invertible Hermitian matrix. If

$$M = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} \widetilde{A} & \widetilde{C} \\ \widetilde{C}^* & \widetilde{B} \end{pmatrix},$$

where A is invertible. Then

$$\det M = \det A \det \widetilde{B}^{-1}.$$

Proof. Since A is invertible, we have

$$\begin{pmatrix} I & O \\ -C^*A^{-1} & I \end{pmatrix} \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \begin{pmatrix} I & -A^{-1}C \\ O & I \end{pmatrix} = \begin{pmatrix} A & O \\ O & B - C^*A^{-1}C \end{pmatrix}.$$

This implies

$$\det M = \det A \det(B - C^*A^{-1}C). \tag{1}$$

M is invertible, so $B - C^*A^{-1}C$ is also invertible. At the same time, we get

$$M^{-1} = \begin{pmatrix} I & -A^{-1}C \\ O & I \end{pmatrix} \begin{pmatrix} A & O \\ O & B - C^*A^{-1}C \end{pmatrix}^{-1} \begin{pmatrix} I & O \\ -C^*A^{-1} & I \end{pmatrix},$$

which implies

$$\widetilde{B} = (B - C^*A^{-1}C)^{-1}. \tag{2}$$

The required equality is a combination of (1) and (2). ■

A proof of Theorem 1.7. Let π_Y and $\pi_{\mathbb{C}^m}$ be the two projections:

$$\pi_Y : \mathbb{C}^m \times Y \to Y, \qquad \pi_Y(z, y) = y, \tag{3}$$

$$\pi_{\mathbb{C}^m} : \mathbb{C}^m \times Y \to \mathbb{C}^m, \qquad \pi_{\mathbb{C}^m}(z, y) = z. \tag{4}$$

By Künneth's formula (see e.g., Section 5 of [5]) and the result on the de Rham cohomology groups of $\mathbb{C}^m$

$$H^k_{dR}(\mathbb{C}^m) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k \geq 1, \end{cases} \tag{5}$$

there exists a closed real two-form Θ on Y, such that

$$[\omega] = \left[\pi_Y^*\Theta\right], \tag{6}$$

in the sense of de Rham cohomology classes. For any $z \in \mathbb{C}^n$, denote the embedding from Y to $\mathbb{C}^m \times Y$

$$y \mapsto (z, y), \tag{7}$$

by i_z. For distinct $z_1, z_2 \in \mathbb{C}^m$, since $\pi_Y \circ i_{z_1} = id_Y = \pi_Y \circ i_{z_2}$, we have

$$\left[i_{z_1}^* \omega\right] = [\Theta] = \left[i_{z_2}^* \omega\right]. \tag{8}$$

Obviously all $i_z^*\omega$ are Kähler forms on Y, so (8) shows that all $i_z^*\omega$ are in the same Kähler class. By Calabi–Yau theorem (see [36]), there is a unique Kähler form ω_Y in this Kähler class satisfying $Ric(\omega_Y) = Ric(\omega_{Y_0})$ and consequently

$$\omega_Y^n = c\omega_{Y_0}^n \tag{9}$$

for some positive constant c.

Now we can write the conditions on ω as follows.

1) $C^{-1}(\omega_{\mathbb{C}^m} + \omega_Y) \leq \omega \leq C(\omega_{\mathbb{C}^m} + \omega_Y)$, for some positive constant C;
2) $\omega^{n+m} = c^{-1}(\omega_{\mathbb{C}^m} + \omega_Y)^{m+n}$;
3) for any $z \in \mathbb{C}^m$, $i_z^*\omega$ and ω_Y are in the same Kähler class.

Denote by g_0 and g the Riemannian metrics associated with $\omega_{\mathbb{C}^m} + \omega_Y$ and ω, respectively, and let g^{-1} be the metric on $T^*(\mathbb{C}^m \times Y)$ induced by g. Let $\left\{z^i\right\}_{i=1}^m$ be the standard complex coordinate system on $\mathbb{C}^m$ and for $i, j = 1, 2, \cdots, m$, define

$$u^{i\bar{j}} = \frac{1}{2} g^{-1}\left(\mathrm{d}z^i, \mathrm{d}\bar{z}^j\right), \qquad u_{i\bar{j}} = 2g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right). \tag{10}$$

For any point (z, y), we choose a complex normal coordinates system $\{z^\alpha\}_{\alpha=m+1}^{m+n}$ around y with respect to ω_Y. Computing under the coordinate system $\{z^a\}_{a=1}^{m+n}$, we have

$$\Delta_\omega u^{i\bar{j}} = g^{a_1\overline{b_1}} g^{a_2\overline{b_2}} g^{i\overline{b_3}} g^{a_3\bar{i}} g_{a_1\overline{b_3}a_2} g_{\overline{b_1}a_3\overline{b_2}} + g^{i\overline{b_3}} g^{a_3\bar{j}} R_{a_3\overline{b_3}}, \tag{11}$$

$$\Delta_\omega u_{i\bar{j}} = g^{a_1\overline{b_1}} g^{a_2\overline{b_2}} g_{a_1\overline{b_2}i} g_{a_2\overline{b_1}\bar{j}}, \tag{12}$$

where $g^{a\bar{b}}, g_{a\bar{b}}, g_{a\bar{b}c}, R_{a\bar{b}}$ are the coefficients of components of $g^{-1}, g, \nabla_{g_0} g, Ric(g_0)$, respectively, $a, b, a_k, b_k = 1, 2, \cdots, m+n$ ($k = 1, 2, 3$).

It is clear that for $i, j = 1, 2, \cdots, m$, $g^{i\bar{j}} = u^{i\bar{j}}$ and $g_{i\bar{j}} = u_{i\bar{j}}$. Namely

$$(g_{a\bar{b}}) = \begin{pmatrix} u_{i\bar{j}} & g_{\alpha\bar{j}} \\ g_{i\bar{\beta}} & g_{\alpha\bar{\beta}} \end{pmatrix}, \qquad (g^{a\bar{b}}) = \begin{pmatrix} u^{i\bar{j}} & g^{\alpha\bar{j}} \\ g^{i\bar{\beta}} & g^{\alpha\bar{\beta}} \end{pmatrix}.$$

Furthermore, $\left(u^{i\bar{j}}\right)$ and $\left(u_{i\bar{j}}\right)$ are bounded and uniformly positive-definite.

Clearly $Ric(g_0) \geq 0$, so (11) implies that $\left(\Delta_\omega u^{i\bar{j}}\right) \geq 0$, that is, $\left(u^{i\bar{j}}\right)$ is subharmonic with respect to ω. Applying Theorem 1.4 and Corollary 2.1, we can find some constant Hermitian matrix A such that

$$\left(u^{i\bar{j}}\right) \leq A \tag{13}$$

everywhere, and

$$\lim_{r\to\infty} \fint_{B_r^\omega(x_0,z_0)} \det\left(u^{i\bar{j}}\right) \mathrm{d}V_g = \det A. \tag{14}$$

One can see that there exists some sufficiently large constant C such that for $r \gg diam Y$,

$$B_r(z_0) \times Y \subset B^\omega_{Cr}(z_0, y_0) \subset B_{C^2 r}(z_0) \times Y.$$

Hence it holds that

$$0 \leq \int_{B_r(z_0)\times Y} \left(\det A - \det\left(u^{i\bar{j}}\right)\right) \mathrm{d}V_{g_0} \leq c \int_{B^\omega_{Cr}(x_0,z_0)} \left(\det A - \det\left(u^{i\bar{j}}\right)\right) \mathrm{d}V_g,$$

and

$$\mathrm{Vol}_{g_0}(B_r(z_0) \times Y) = C^{-4n}\mathrm{Vol}\left(B_{C^2 r}(z_0) \times Y\right) \geq C^{-4n} c \mathrm{Vol}_\omega\left(B^\omega_{Cr}(z_0, y_0)\right).$$

Then we can obtain

$$0 \leq \fint_{B_r(z_0)\times Y} \left(\det A - \det\left(u^{i\bar{j}}\right)\right) \mathrm{d}V_{g_0} \leq C^{4n}\left(\fint_{B^\omega_{Cr}(z_0,y_0)} \left(\det A - \det\left(u^{i\bar{j}}\right)\right) \mathrm{d}V_g\right),$$

consequently

$$\lim_{r\to\infty} \fint_{B_r(z_0)\times Y} \det\left(u^{i\bar{j}}\right) \mathrm{d}V_{g_0} = \det A. \tag{15}$$

Computing under the local coordinate system $\{z^a\}_{a=1}^{m+n}$ mentioned above and applying Lemma 3.1, we can check that

$$\left(\det\left(u^{i\bar{j}}\right)\right)^{-1} \frac{\left(i_z^*\omega\right)^n}{\omega_Y^n} = \frac{\omega^{n+m}}{(\omega_Y + \omega_{\mathbb{C}^m})^{m+n}} = c^{-1} \tag{16}$$

at any point (z, y). Then we have

$$\fint_{\{z\}\times Y} \det\left(u^{i\bar{j}}\right)\omega_Y^n = \fint_{\{z\}\times Y} c\left(i_z^*\omega\right)^n = c. \tag{17}$$

This tells us

$$\fint_{B_r(z_0)\times Y} \det(u^{i\bar{j}})\, \mathrm{d}V_{g_0} = c, \tag{18}$$

for any z_0 and $r > 0$. Together with (13) and (14), we deduce

$$\det\left(u^{i\bar{j}}\right) \le \det A = c, \tag{19}$$

and then

$$\det\left(u^{i\bar{j}}\right) = \det A = c. \tag{20}$$

Since $(u^{i\bar{j}})$ is positive-definite, using (13) again, we obtain

$$\left(u^{i\bar{j}}\right) \equiv A. \tag{21}$$

By (16), we have

$$\left(i_z^*\omega\right)^n = \omega_Y^n \tag{22}$$

for any $z \in \mathbb{C}^m$. We already know that $i_z^*\omega$ and ω_Y are in the same Kähler class, so

$$i_z^*\omega = \omega_Y. \tag{23}$$

For any $i = 1, 2, \cdots, m$, from (11) and (21), we derive

$$0 = \Delta_\omega u^{i\bar{i}} \ge C^{-1} \sum_{a_1,a_2=1}^{m+n} |g_{a_1\bar{i}a_2}|^2. \tag{24}$$

(24) implies that $u_{i\bar{i}}$ is a constant function, then consequently

$$0 = \Delta_\omega u_{i\bar{i}} \ge g^{a_1\overline{b_1}} g^{a_2\overline{b_2}} g_{a_1\overline{b_2}i} g_{a_2\overline{b_1}i} \ge C^{-1} \sum_{a,b=1}^{m+n} |g_{a\bar{b}i}|^2. \tag{25}$$

That is to say $\left(u_{i\bar{j}}\right)$ is a constant matrix. At the same time, (23) implies

$$\sum_{a=1}^{m+n} \sum_{\alpha,\beta=m+1}^{m+n} |g_{\alpha\bar{\beta}a}|^2 = 0. \tag{26}$$

Combining (24), (25), and (26) shows $\nabla_{g_0} g = 0$.

We define

$$\eta^i = i_z^* \left(\tfrac{\partial}{\partial z^i} \lrcorner \omega\right), \tag{27}$$

where $z \in \mathbb{C}^m$, $i = 1, 2 \cdots, m$. Since $\nabla_{g_0} g = 0$, this definition doesn't depend on the choice of z and every η^i is an ω_Y-parallel (0,1)-form. The expression (6) can be easily checked under a local coordinate system. This concludes the proof of Theorem 1.7. ■

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