COMM. IN PARTIAL DIFFERENTIAL EQUATIONS, 4(6), 645-699 (1979)

A CRITERION FOR HYPOELLIPTICITY OF OPERATORS CONSTRUCTED FROM VECTOR FIELDS

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0. Introduction. Suppose that $X_1, X_2, \ldots, X_p$ are smooth, real vector fields on a manifold M. Let L be the differential operator

$$L = \sum_{|\alpha| \leq d} a_\alpha(x) X^\alpha,$$

where $X^\alpha = X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_p}$ $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, $|\alpha| = p$ and the $a_\alpha(x)$ are smooth, complex-valued functions. We seek the least restrictive conditions that will guarantee that L is hypoelliptic, i.e. that if $Lf = g$ with g smooth in an open set U, then f is also smooth in U.

Our results will be restricted to the case where $X_1, X_2 \ldots X_p$ satisfy the following two conditions at each point.

(0.1) $\{X_k\}$, together with their commutators $X_\alpha = [X_{\alpha_1}, [X_{\alpha_2}, \ldots, X_{\alpha_s}], \ldots]$ up to some fixed length r span the tangent space at each point,

(0.2) For each $j \leq r$ the dimension of the space spanned by the commutators of length $\leq j$ at each point is constant in a neighborhood. (By convention the X_k themselves will be regarded as commutators of length one.)

Condition (0.1) was introduced by Hörmander [8], who showed that if it was satisfied the sum of squares of the X_k must be hypoelliptic. Metivier [11] imposed the additional condition (0.2) in order to study spectral properties of the sum of squares operator.

By conditions (0.1) and (0.2) one can choose, in a neighborhood of any point, real vector fields $\{X_{jk}\}$, where each X_{jk} is a commutator of length j, such that for any $r' \leq r$ the subspace of the tangent space spanned by X_{jk} $j \leq r'$ is the same as that spanned by all commutators of length $\leq r'$. If the X_j are not already linearly independent, we may by (0.2) replace them by a linearly independent subset which still satisfies (0.1) and (0.2). Hence we may assume $X_{1k} = X_k$. For fixed $x \in M$ the choice of X_{jk} defines a (canonical) coordinate system around x by the mapping Θ_x given by

$$(0.3) \quad \Theta_x(y) = u = (u_{jk}) \quad \text{if} \quad y = \exp(\Sigma\, u_{jk}X_{jk}).x,$$

where exp denotes the exponential map defined in some small neighborhood of x. Thus we identify a neighborhood of $x \in M$ with a neighborhood of 0 in $\mathbb{R}^n$, where $n = \dim M$. We define

$$X_{jk,x} = \Theta_x^*(X_{jk}),$$

where Θ_x^* denotes the differential; i.e. $X_{jk,x}$ is X_{jk} written in local canonical coordinates around x.

On $\mathbb{R}^n$, with coordinates $u = (u_{jk})$ we introduce the family of dilations $\delta_t(u_{jk}) = (t^j u_{jk})$ for $t > 0$. A function $f(u)$ is <u>homogeneous</u> of degree s if $k(\delta_t u) = t^s k(u)$. A differential operator of the form $f(u)\partial/\partial u_{jk}$ is <u>homogeneous</u> of degree $j-s$ if $f(u)$ is homogeneous of degree s. A homogeneous differential operator of degree ℓ reduces homogeneity by ℓ degrees. A differential operator D on $\mathbb{R}^n$ is of <u>local degree</u> $\leq j$ if its Taylor expansion at 0

is a sum of differential operators homogeneous of degrees $\leq j$. (See also Rothschild-Stein [13, Section 7].) Metivier has proved the following.

(0.4) Lemma. *For any x, the local degree of $X_{j,x}$ is ≤ 1. Furthermore, if*

$$X_{k,x} = \hat{X}_{k,x} + R_{k,x},$$

where $\hat{X}_{k,x}$ is homogeneous of degree 1 and $R_{k,x}$ is of local degree ≤ 0, then $\hat{X}_{k,x}$ generate a nilpotent Lie algebra $\mathfrak{g}_x$ of dimension n and rank r. The mapping $x \longrightarrow \hat{X}_{j,x}$ is smooth.

For $x \in M$ let G_x be the simply connected Lie group with Lie algebra $\mathfrak{g}_x$. The main result of this paper is the following.

(0.5) Theorem. *Let $\{X_k\}$ satisfy (0.1) and (0.2), and let $L = \sum_{|\alpha|\leq d} a_\alpha(x) X^\alpha$. For x_0 fixed put $\hat{L}_{x_0} = \sum_{|\alpha|=d} a_\alpha(x_0)\hat{X}^\alpha_{x_0}$ be the corresponding left invariant operator on G_{x_0}. If $\hat{L}_{x_0}$ is hypoelliptic, then L is hypoelliptic in some neighborhood of x_0.*

Combining Theorem 0.5 with the criterion of Helffer-Nourrigat [6] for hypoellipticity of left invariant operators on nilpotent groups we obtain

(0.6) Theorem. *With notation as above, L is hypoelliptic in a neighborhood of $x_0 \in M$ if for every nontrivial irreducible unitary representation π of G_{x_0}, $\pi(L)$ is injective in S_π, the space of C^∞ vectors of π.*

Following Helffer-Nourrigat [7] we call L *maximally hypoelliptic* at x_0 if there is an open neighborhood U of x_0 such that for every sequence α, $|\alpha| \leq d = \deg L$, there exists $C_\alpha > 0$ *such that*

$$\|X^\alpha f\|^2_{L^2(U)} \leq C_\alpha(\|Lf\|^2_{L^2(U)} + \|f\|^2_{L^2(U)})$$

for all $f \in C_0^\infty(U)$. It can be shown [7] that if L is maximally hypoelliptic at x_0, then it is hypoelliptic in a neighborhood of x_0. Helffer and Nourrigat [7] have shown that the criterion of Theorem 0.5 is also necessary for maximal hypoellipticity. Combining their result with Theorems 0.5 and 0.6 we obtain

(0.7) Theorem. *L is maximally hypoelliptic at x_0 if and only if for all non-trivial irreducible unitary representations π of G_{x_0}, $\pi(\hat{L}_{x_0})$ is injective in S_π.*

Theorem 0.7 was conjectured by Helffer and Nourrigat [7], who proved it in the special case where the $\{X_k\}$ are free up to step r, i.e. the Lie algebras $\mathfrak{g}_x$ are all isomorphic to the free nilpotent Lie algebra of step r on p generators. In the general case the $\mathfrak{g}_x$ need not even be isomorphic. The relationship between the injectivity of $\pi(L)$ and the hypoellipticity of L, when L is left invariant and homogeneous, was first conjectured by Rockland [12], inspired by earlier work of Grušin [5].

(0.8) Remark. A sufficient condition for hypoellipticity of an operator $L = \Sigma\, a_\alpha(x)X^\alpha$, where $\{X_k\}$ satisfies just (0.1) may be given as follows. By the methods in [13, Part II], one may associate with X_k a vector field Y_k on a higher dimensional space, such that the Y_k generate a free nilpotent Lie algebra of step r. L is maximally hypoelliptic at x_0 if $\hat{L} = \Sigma\, a_\alpha(x_0)Y_\alpha$ is hypoelliptic. However, if (0.2) is satisfied, this criterion is far from necessary, and in general is much more restrictive than that of Theorem (0.5).

The author wishes to thank Bernard Helffer for bringing this problem to her attention.

1. Smoothing operators and parametrices. We shall prove Theorem 0.5 by modifying the methods of Rothschild-Stein [13] in order to construct a left

parametrix for L. We define the following function spaces as in [15]. L^p_s will denote the classical L^p based Sobolev spaces for $1 < p < \infty$, $s \geq 0$ for $p \neq 2$, $s \in \mathbb{R}$ for $p = 2$. Λ_s, $s > 0$ will denote the classical Lipschitz spaces. The non-isotropic Sobolev spaces are defined as in [13]:

$$S^p_k = \{f \in L^p : X^\alpha f \in L^p,\ |\alpha| \leq k\} \quad \text{with norm}$$

$$\|f\|_{S^p_k} = \sum_{|\alpha| \leq k} \|X^\alpha f\|_{L^p} + \|f\|_{L^p}$$

An operator T, initially defined on $C^\infty_0(M)$ will be said to be <u>smoothing of order</u> λ if T is bounded from

$$L^p_\alpha \longrightarrow L^p_{\alpha+\lambda/r} \quad 1 < p < \infty \text{ and } \alpha \geq 0 \text{ if } p \neq 2$$
$$\alpha \in \mathbb{R} \text{ if } p = 2$$

$$S^p_k \longrightarrow S^p_{k+\lambda} \quad k \geq 0 \text{ if } \lambda \text{ integral,}$$

$$\Lambda_\alpha \longrightarrow \Lambda_{\alpha+\lambda/r} \quad \text{if } \lambda > 0$$

$$L^\infty \longrightarrow \Lambda_{\lambda/r} \quad \text{if } \lambda > 0.$$

In [13], there is no discussion of the space L^2_α, $\alpha < 0$. However, one may easily reduce questions of boundedness for L^2_s, $s < 0$, to that of $L^2_{s'}$, some $s' > 0$, for the class of operators considered.

Theorem 0.3 may be obtained easily from the following.

(1.1) Theorem. *Let* $L = \sum_{|\alpha|\leq d} a_\alpha(x)X^\alpha$ *as in Theorem 0.3 and suppose* $\hat{L}_{x_0} = \sum_{|\alpha|=d} a_\alpha(x_0)\hat{X}^\alpha$ *is hypoelliptic. Then there exist* $\phi \in C_0^\infty(M)$ *with* $\phi \equiv 1$ *in a neighborhood of* x_0 *and operators* K *and* S *smoothing of orders* 2 *and* ∞ *respectively such that*

$$KL = \phi I + S.$$

To construct K we need more information on homogeneous hypoelliptic operators on groups. Condition (0.2) and the results of [11] guarantee that each $\mathfrak{g}_x$ is a vector space direct sum

$$(1.2) \quad \mathfrak{g}_x = \sum \mathfrak{g}_x^i, \qquad [\mathfrak{g}_x^i, \mathfrak{g}_x^j] \subset \mathfrak{g}_x^{i+j}$$

and δ_t defines an automorphism of $\mathfrak{g}_x$ by $\delta_t \mid \mathfrak{g}_x^i = t^i \mathfrak{g}_x^i$, with $\dim \mathfrak{g}_x^i = n_i$, independent of x. An algebra $\mathfrak{g}$ satisfying (1.2) will be called *stratified*. The *homogeneous dimension* of G_x is defined as $Q = \sum_i in_i$. The *homogeneous norm* on G_x, identified with R^n by canonical coordinates, is defined as $\|u\| = \{\sum_{j=1}^{r} |u_{jk}|^{2r!/j}\}^{1/2r!}$. The reader is warned that the homogeneous norm was denoted by $|u|$ in

[9], [4], [3], and [13], while the above notation was used for the usual Euclidean norm. In order to construct a parametrix K we need the following abstract existence theorem.

(1.3) Proposition. [3, Theorem 2.1]. Suppose that D is a self-adjoint left invariant differential operator homogeneous of degree $d < Q$, where Q is the homogeneous dimension of G. If D is hypoelliptic there exists a unique function $k \in C^{\infty}(G-\{0\})$ homogeneous of degree $-Q + d$ such that $Dk = \delta$, the delta function.

Following [7], we shall first consider the case where Proposition 1.3 applies. The general case follows easily (see section 8).

(1.4) Theorem. Suppose L and $\hat{L}_x$ are as in Theorem 0.5. Assume also that $\hat{L}_x$ is self-adjoint and homogeneous of degree $d < Q$. Then if $\hat{L}_{x_0}$ is hypoelliptic, there is a neighborhood U of x_0 such that $\hat{L}_x$ is hypoelliptic for all $x \in U$. Let $k_x \in C^{\infty}(G_x-\{0\})$ be the unique homogeneous fundamental solution for $\hat{L}_x$, and define the operator K_1 by

$$(1.5) \quad K_1 f(x) = \int \phi_1(x) k_y(\Theta(y,x))\phi_2(y) f(y)\,dy$$

for any $\phi_1, \phi_2 \in C_0^{\infty}$ with $\phi_2 \equiv 1$ on $\operatorname{supp} \phi_1$.

Then K_1 *is smoothing of order* 2 *and there exists* S_1 *smoothing of order* 1 *such that*

$$K_1 L = \phi_2 I + S_1.$$

2. *Operators of type* λ. In order to prove that the operators K_1 and S_1 defined in Theorem 1.3 have the right smoothing properties we show, as in [13] that they belong to families of integral operators enjoying these properties. We first recall some definitions from [4] and [13]. A function $h \in C^\infty(\mathbb{R}^n - 0)$ is said to be of *type* λ if h is homogeneous with respect to the dilations of degree $-Q + \lambda$, $0 \leq \lambda < Q$, with the added condition for $\lambda = 0$ that

$$\int_{a \leq \| u \| \leq b} h(u)\,du = 0 \quad \text{for any } a, b \text{ with } 0 < a < b < \infty.$$

Integration against a kernel of type λ is a distribution for $\lambda > 0$. If $\lambda = 0$, a distribution may be defined by taking the integral in the principal value sense; i.e. $f \longrightarrow \lim_{\varepsilon \to 0} \int_{\varepsilon < \| u \| < \infty} f(u)h(u)\,du$ is a distribution if h is a function of type 0. A *distribution of type* $\lambda > 0$ is one obtained from a function of type λ. A *distribution of type* 0 is one of the form $c\delta + \tau$, where c is a constant, and τ is a distribution obtained from a function of type 0. If h is a distribution of type $\lambda \geq 1$, then Dh is a distribution of type $\lambda - 1$ for any D homogeneous of degree 1. (The reader is referred to [3], [9] and [13] for proofs of these and other relevant properties of distribution of type λ.) If h is a function of type $\lambda \geq 0$ we shall identify it with the distribution it defines.

Now suppose the vector fields $\{X_k\}$ satisfy (0.1) and (0.2). A function $K(x,y)$ on $M \times M$ is a <u>kernel of type</u> λ if for all integral $\ell \geq 0$,

$$(2.1)\quad K(x,y) = \sum_{i=1}^{s} a_i(x) k_x^{(i)}(\Theta(y,x)) b_i(y) + E^{\ell}(x,y),$$

where $a_i, b_i \in C_0^{\infty}(M)$, $k_x^{(i)}$ is a kernel of type $\geq \lambda$ with $(x,u) \longrightarrow k_x^{(i)}(u)$ smooth away from $u = 0$, and $E^{\ell} \in C^{\ell}(M\times M)$, the space of functions with ℓ continuous derivatives. An <u>operator of type</u> λ is a mapping T originally defined on $C_0^{\infty}(M)$ given by

$$Tf(x) = \int K(x,y) f(y)\,dy,$$

(2.2) <u>Remark</u>. The definition of operator of type λ given above is exactly the same as that given in [13], for the special case where $\{X_k\}$ is assumed to be free up to step r. The above definitions are given independent of any group structure, and the main properties of the operators may be established without reference to any groups. However, we shall have to use convolutions on the groups G_x for the proof of Theorem 1.4.

The main properties of kernels of type λ are summarized in the following.

(2.3) Theorem. Let T be an operator of type $\lambda \geq 0$.

(i) If $\lambda \geq 1$, the $X_j T$ and TX_j are operators of type $\lambda - 1$.

(ii) For any j, k there exist operators $T_{s\ell}$, $T'_{s\ell}$ of type $\lambda + (s-j)$ and T_0, T'_0 of type λ such that

$$X_{jk}T = \sum_{\substack{s \geq j \\ \ell}} T_{s\ell} X_{s\ell} + T_0$$

$$TX_{jk} = \sum_{\substack{s \geq j \\ \ell}} X_{s\ell} T'_{s\ell} + T'_0$$

(ii) T is smoothing of order λ.

Proof. We shall be very brief, since most of Theorem 2.3 may be proved exactly as for the special case in [13]. In particular, the proof of (i) is the same as that of Theorem 8, Section 14 of [13]. For (ii) we modify the proof of Theorem 9 of [13]. Recall that for any j, k we may write

(2.4) $X_{jk,x} = \hat{X}_{jk,x} + R_{jk,x}$, where $\hat{X}_{jk,x}$ is homogeneous of degree j and $R_{jk,x}$ is of local degree $\leq j-1$. For any vector field X homogeneous of degree j we define X', again homogeneous of degree j, by

$$X'f(-u) = -X(f(-u)). \tag{2.5}$$

Now suppose $\{Y_{jk}\}$ $\{Z_{jk}\}$ are two bases of the tangent space to R^n with Y_{jk}, Z_{jk} homogeneous of degree j, such that

$$Y_{jk} = \frac{\partial}{\partial u_{jk}} + \sum_{\substack{s>j \\ \ell}} g_{s\ell}^{jk}(u) \frac{\partial}{\partial u_{s\ell}}, \quad \text{and}$$

$$Z_{jk} = -\frac{\partial}{\partial u_{jk}} + \sum_{\substack{s>j \\ \ell}} h_{s\ell}^{jk}(u) \frac{\partial}{\partial u_{s\ell}}$$

Then we claim that there exist functions $a_{s\ell}(u)$ and $a'_{s\ell}(u)$ homogeneous of degree $s-j$ such that

$$Y_{jk} = -Z_{jk} + \sum a_{s\ell}(u) Z_{s\ell}, \quad \text{and} \tag{2.6}$$

$$Z_{jk} = -Y_{jk} + \sum a'_{s\ell}(u) Y_{s\ell}. \tag{2.7}$$

Indeed, this amounts to inverting a triangular system; see [13, Theorem 9] for details.

Now notice that we may take $Y_{jk} = \hat{X}_{jk,y}$ and $Z_{jk} = \hat{X}'_{jk,x}$ for any $x, y \in M$. By (2.6) we obtain

$$(2.8)\quad \hat{X}_{jk,y} = -\hat{X}'_{jk,x} + \sum a^{jk}_{s\ell}(x,y,u)X'_{s\ell,y}$$

where the $a_{s\ell}(x,y,u)$ are smooth and homogeneous in u. (Similarly, we may express $X'_{jk,y}$ in terms of $X_{s\ell,x}$, but we shall restrict ourselves to proving the first equality of (ii).)

For $x \in M$ we write X^x for when we want to emphasize that x is the variable. Now $X_{jk}T$ is a sum of operators with kernels

$$X^x_{jk}(a(x)k_y(\Theta(y,x))b(y)).$$

It will suffice to consider those for which k_y is of type λ. Then we are reduced to considering

$$X^x_{jk}k_y(\Theta(y,x)) = (X_{jk,y}k_y)(u),$$

where $u = \Theta(y,x)$. By (2.6)

$$(2.9)\quad \hat{X}_{jk,y}k_y(u) = -\hat{X}'_{jk,x}k_y(u) + \sum a^{jk}_{s\ell}(x,y,u)\hat{X}'_{s\ell,x}k_y(u)$$

Now since by (2.4)

$$(2.10)\quad \hat{X}_{s\ell,y} = X_{s\ell,y} + \text{lower degree, and}$$

$$\hat{X}'_{s\ell,x} = X'_{s\ell,x} + \text{lower degree}$$

we obtain from (2.6),

$$(2.11)\quad X^x_{jk}k_y(\Theta(y,x)) = -X'_{jk,x}k_y(u) + \sum a^{jk}_{s\ell}(x,y,u)X'_{s\ell,x}k_y(u) + \text{higher},$$

where the error is a sum of kernels of homogeneity higher than $\lambda - j$.

For the kernel of TX_{jk}, it suffices to consider, using integration by parts,

$$-X^y_{jk}(a(x)k_y(\Theta(y,x))b(y)).$$

Since $v = \Theta(x,y) = -\Theta(y,x) = -u$,

$$-X^y_{jk}(k_y(\Theta(y,x))) = k'_y(\Theta(y,x)) - X^v_{jk,x}(k_y(-v)),$$

where k'_y is a kernel of type λ obtained by differentiating k_y with respect to y. Also, $X_{s\ell,x}(k_y(-v)) = -(X'_{jk,x}k_y)(-v)$ for any index $s\ell$. Hence

$$(2.12)\quad X'_{s\ell,x}k_y(-v) = X^y_{s\ell}(k_y(\Theta(y,x))) + \text{higher terms}$$

where the error is a kernel of type greater than $\lambda - j$.

Substitution of (2.11) into (2.12) then gives

$$X_{jk}^{x}k_y(\Theta(y,x)) = -X_{jk}^{y}(k_y(\Theta(y,x))) + \sum a_{s\ell}^{jk}(x,y,u)X_{s\ell}^{y}(k_y(\Theta(y,x))) + \text{higher terms.}$$

Now by induction the higher terms can be dealt with similarly, and we then obtain (ii). See [13], Theorem 9] for further details.

Part (iii) can be reduced, by (i) and (ii), to the case where $\lambda = 0$. For this case the same proof as in [13, Theorem 6] will work once one has established the estimates for $\Theta(x,y)$ as in [13, Proposition 12.3]. The proof of Theorem 2.3 will be completed by the following.

(2.13) Lemma. Let $x, y, z \in M$. If $\|\Theta(x,y)\|$ and $\|\Theta(z,y)\|$ are both ≤ 1, then there exists $C > 0$ such that

$$\|\Theta(x,y) - \Theta(z,y)\| \leq C(\|\Theta(x,z)\| + \|\Theta(x,z)\|^{1/r}\|\Theta(x,y)\|^{1-1/r}).$$

Proof. We essentially follow that of [13, Proposition 12.3], except that there are new error terms which must be estimated. Let $v = \Theta(x,y)$, $w = \Theta(x,z)$, and $u = \Theta(z,y)$. Then

$$(2.14)\quad (\exp v\cdot X)(\exp - w\cdot X)z = y,$$

where $v\cdot X = \Sigma\, v_{jk}X_{jk}$ and $w\cdot X = \Sigma\, w_{jk}X_{jk}$. Now fix z and regard u_{jk} as a function of y be setting

$$u_{jk}(y) = (\Theta(z,y))_{jk}.$$

We must estimate $\|v-u\|$. To do this, observe that by (2.14)

$$u_{jk}(y) = e^{v\cdot X}e^{-w\cdot X}u_{jk}(z).$$

Now we claim that formally

$$(2.15)\quad e^{v\cdot X}e^{-w\cdot X} = e^{(v-w)\cdot X + \sum_{\alpha,k} C_{\alpha,k}(y)q_{\alpha,k}(v,w)X_{|\alpha|,k}}$$

where $C_{\alpha,k}(y)$ is smooth,

$$q_{\alpha,k}(v,w) = O\Big(\sum_{0<\ell<|\alpha|} \|v\|^{\ell}\|w\|^{|\alpha|-\ell}\Big)$$

Indeed, by the Baker-Campbell-Hausdorff formula the left-hand side of (2.15) may be expanded so that

$$e^{v\cdot X}e^{-w\cdot X} = e^{(v-w)\cdot X + R},$$

where each term of R is a commutator of length s of $v\cdot X$ and $-w\cdot X$. Hence each term of R may be written as

$$\sum_{\substack{\ell \\ |\alpha|\leqslant s}} q_{\alpha,\ell}(v,w)X_\alpha ,$$

where $X_\alpha = [X_{j_1k_1},[X_{j_2k_2},\dots,X_{j_tk_t}]\dots]$

with $\alpha = ((j_1,k_1),(j_2,k_2)\dots(j_t,k_t))$, $|\alpha| = \sum_{i=1}^{t} j_i$, where

$$|q_{\alpha,\ell}(v,w)| = O(\sum_{0<\ell<|\alpha|} \|v\|^{\ell}\|w\|^{|\alpha|-\ell})$$

By (0.2) each X_α may be written

$$X = \sum_{\substack{k \\ j\leqslant|\alpha|}} c_{\alpha,j,k}(y)X_{jk}$$

with $c_{\alpha,j,k}(y)$ smooth. This proves (2.15). To complete the proof of the lemma, we shall show that

$$(2.16)\quad X_{j_1k_1}X_{j_2k_2}\dots X_{j_tk_t}u_{jk}(z) = 0$$

if $\sum_{i=1}^{t} j_i < j$. Indeed, suppose (2.16) holds. Then one may formally expand the right-hand side of (2.15) to obtain

$$e^{vX}e^{-wX}u_{jk}(z) = u_{jk}(e^{(v-w)\cdot X}) + O(\sum_{0<\ell<j} \|v\|^{\ell}\|w\|^{j-\ell})$$

(since the terms with coefficients which are $O(\|v\|^{\ell}\|w\|^{s-\ell})$ with $s < j$ vanish by (2.16).) As in [13, (12.9)] we use the inequality

$$A^{a_1}B^{1-a_1} \leqslant C(A+A^aB^{1-a})$$

for positive real numbers A, B and $0 < a \leqslant a_1 < 1$, to complete the proof.

Finally, we must prove (2.16). For this, assume by induction that

$$(2.17)\quad X_{j_1}X_{j_2}\ldots X_{j_sk_s}u_{jk}(z) = 0$$

if $\Sigma j_i < j$ and $s < t$. (Note that $X_{j_ik_i}u_{jk}(z) = 0$ if $j_i < j$.) By definition,

$$X_{j_1k_1}\ldots X_{j_tk_t}u_{jk}(z)$$
$$= \frac{\partial}{\partial s_1\ldots s_t}u_{jk}(e^{s_tX_{j_tk_t}}\ldots e^{s_1X_{j_1k_1}}z)\Big|_{s=0}$$

Now

$$e^{s_tX_{j_tk_t}}\ldots e^{s_1X_{j_1k_1}}$$
$$= e^{\Sigma s_\ell X_{j_\ell k_\ell}+\sum_{|\beta|<t} C_\beta(x)X_\beta+O(|s|^2)},$$

where $X_\beta u_{jk}(z) = 0$ if $|\beta| < t$ by (2.17) and

$e^{\Sigma s_\ell X_{j_\ell k_\ell}} u_{jk}(z) = 0$ since $j_\ell < j$, all ℓ. This proves (2.16), and therefore completes the proof of Theorem 2.3.

3. Smoothly varying families of nilpotent groups. In order to prove Theorem 1.4, we must show that there is a neighborhood U of x_0 such that the following hold.

(3.1) If $y \in U$, $\hat{L}_y$ is hypoelliptic and hence has a homogeneous fundamental solution k_y.

(3.2) $(y,u) \longrightarrow k_y(u)$ is smooth on $U \times (\mathbb{R}^n - \{0\})$.

(3.1) will be proved by showing that $\hat{L}_y$ has a fundamental solution k_y which is sufficiently smooth away from 0. For this, we shall follow the outline of the proof of Theorem 3 [13], in which it is shown that a certain family of operators on a free nilpotent group has a "smoothly varying" family of fundamental solutions. Similar arguments will then prove 3.2.[1]

We shall now be precise. A smoothly varying family of nilpotent groups is a collection $\{G_x\}$, $x \in M$, such that each G_x topologically isomorphic to $\mathbb{R}^n$ and satisfies the following.

[1] In fact, the proof of Theorem 3 [13] is not quite correct, since Lemma 6.7 there fails for the case $Q = 4$. The proof of Theorem 3.6 below corrects this gap.

(3.3) $G_x = \exp \mathfrak{g}_x$, $\mathfrak{g}_x = \mathfrak{g}_x^1 + \mathfrak{g}_x^2 + \dots + \mathfrak{g}_x^r$,

linear sum, $[\mathfrak{g}_x^i, \mathfrak{g}_x^j] \subset \mathfrak{g}_x^{i+j}$ and

$\dim \mathfrak{g}_x^k = n_k$, independent of x,

(3.4) There is a smooth mapping

$$x \longrightarrow X_{jk,x}$$

such that for each j, $\{X_{jk,x}\}$ is a basis of $\mathfrak{g}_x^j$.

Metivier [11] has proved the following.

(3.5) Lemma. *If $\{X_j\}$ satisfies (0.1) and (0.2) the groups $G_x = \exp \mathfrak{g}_x$ defined by Lemma 0.4 form a smoothly varying family of groups with $Y_{jk,x} = \hat{X}_{jk,x}$.*

In this context, (3.1) and (3.2) are contained in the following key result, whose proof will occupy most of the remainder of this paper.

(3.6) Theorem. *Suppose $\{G_x\}$ is a smoothly varying family of nilpotent groups. Let $L_x = \Sigma\, a_\alpha(x) X_x^\alpha$, where $a_\alpha(x)$ is smooth, be a family of homogeneous left invariant operators on G_x such that each L_x is self-adjoint and $\deg L_x < Q =$ the homogeneous degree of G_x. If L_y is hypoelliptic for some y, there is a neighborhood U of y such that L_x is* *hypoelliptic for all* $x \in U$. *If* k_x *is the unique homogeneous fundamental solution for* L_x ,

$$(x,u) \longrightarrow k_x(u)$$

is smooth for $u \neq 0$.

Before proving Theorem 3.6 we state some conventions and basic results for convolutions. We shall denote by $f *_x g$ the convolution of functions (or distributions) on the group G_x. In most of what follows, we shall omit the subscript when convenient, and $f*g$ will denote convolution on the varying group G_x. If τ is a distribution and f a function for which $\tau(f)$ is defined, we define the convolution $f*\tau$ by

$$f*\tau(u) = \tau(\tilde{f}^{L_u})$$

where $\tilde{f}(v) = f(v^{-1})$ and $g^{L_u}(v) = g(uv)$. For convenience we now state the fundamental results concerning convolution by distributions of type λ, $0 \leqslant \lambda < Q$.

Theorem A. ([Knapp-Stein [9], Coifman-Weiss [2], Koranyi-Vagi [10] and Folland [3]). *Let* τ *be a homogeneous distribution of type* λ *on* G_x ,

$0 \leqslant \lambda < Q$. *If* $\lambda = 0$, *convolution by* τ *extends to a bounded mapping on* $L^p(G_x)$, $1 < p < \infty$. *If* $\lambda > 0$, *convolution by* τ *extends to a bounded mapping from* L^p *to* L^q, *where* $1/q = 1/p - \lambda/Q$, *if* $1 < p < Q/\lambda$.

Theorem B. (Folland [3, Proposition 1.13]). *Let* k_1, k_2 *be homogeneous distributions of type* λ_1, $\lambda_2 \geqslant 0$ *on* G_x *with* $Q > \lambda_1 + \lambda_2 > 0$. *Then* $k_1 * k_2$ *exists and is a homogeneous distribution of type* $\lambda_1 + \lambda_2$ *satisfying the associative law*

$$(f *_x k_1) *_x k_2 = f * (k_1 *_x k_2)$$

for all $f \in L^p$, $p < Q/(\lambda_1 + \lambda_2)$.

We shall also need the following easy results involving manipulation of derivatives. If X is left invariant on G_x, then

(3.7) $X(f *_x k) = f *_x Xk$,

(3.8) $X'(f *_x k) = X'f *_x k$, and

(3.9) $Xf *_x k = f *_x X'k$,

if $Xf, X'f \in L^p$, $1 < p < Q/\lambda$, k of type $\lambda \geqslant 0$. Indeed, on the formal level (3.7) and (3.8) are simple calculations if $f \in C_0^\infty$, and then one merely checks that the appropriate integrals are absolutely convergent. By Theorem A the result extends to $f \in L^p$ provided $X^{(\cdot)} f \in L^p$.

4. Existence of an inverse operator for L_x. We begin the proof of Theorem 3.6 by formally constructing a fundamental solution k_x for L_x if x is close to y. The existence of $k_y \in C^\infty(G-\{0\})$ satisfying

$$(4.1) \quad L_y f *_y k_y = f *_y L_y k_y = f$$

for $f \in C_0^\infty(G_x)$ is guaranteed by Proposition 1.3. For any $x \in M$ let K_x be the operator

$$(4.2) \quad K_x f = \lim_{n\to\infty} f * \sum_{j=0}^{n} (-1)^j k_x^{(j)},$$

where $k_x^{(j)}$ is the distribution of type d

$$(4.3) \quad \begin{aligned} k_x^{(j)} &= (L_x - L_y)k_y * (L_x - L_y)k_y * \ldots * (L_x - L_y)k_y * k_y \\ k_x^{(0)} &= k_y , \end{aligned}$$

all convolutions being taken over the varying group G_x, $K_x f$ will be defined whenever the sum on the right-hand side of (4.2) converges in L^q norm for some q, $1 < q < \infty$, i.e. if

$$\| f * \sum_{j=0}^{n} (-1)^j k_x^{(j)} \|_{L^q} \leq C_f$$

with C_f independent of n. Note that by the properties stated in Section 2, $(L_x - L_y)k_y$ is a distribution of type 0. Since k_y is a function of type $-Q + d$, the convolution defining (4.3) is associative by Theorem B and is a function of type d. Hence $k_x^{(j)}$ is unambiguously defined. Furthermore, since convolution by a function of type d is a bounded operator from L^p to L^q if $1/q = 1/p - d/Q$, by Theorem A, $\| f * k_x^j \|_{L^q} \leq C \| f \|_{L^p}$ for p, q satisfying above with $1 \leq p, q < \infty$. However, in general the sum will not converge, so that it will be necessary to put extra conditions on x so that $K_x f$ is defined for $f \in C_0^\infty(G_x)$. For this we need some preliminary computations on the bounds of the operators $f \longrightarrow f*(L_x - L_y)k_y$.

(4.4) Lemma. $X_{s,x} - X_{s,y} = \sum_{j,k,i} (x-y)_i g_{jksi}(x,y,u) \frac{\partial}{\partial u_{jk}}$ *where* g_{jksi} *is smooth and homogeneous of degree* $j-1$ *in* u.

Proof. By (3.4) $X_{s,z} = \sum_{j,k} a_{jk}(z,u) \frac{\partial}{\partial u_{jk}}$ with a_{jk} smooth and homogeneous of degree $j-1$ in u. Now apply the mean value theorem.

From Lemma 4.4 we easily obtain

(4.5) Lemma. $X_x^\alpha - X_y^\alpha = \sum_{i,s,\beta} (x-y)_i h_{\alpha,\beta,i}(x,y,u) D_u^{\beta,i}$, where $g_{\alpha,\beta,i}$ is smooth and homogeneous of degree $|\beta| - |\alpha|$, and $D_u^{\beta i}$ is homogeneous of degree $|\beta|$.

By Lemma 4.5 we may write

(4.6) $$L_x - L_y = \sum_{\beta,i,l} (x-y)_i h_{\beta,i}(x,y,u) D_u^{\beta,i}$$

with $h_{\beta i}$ homogeneous of degree $|\beta| - d$ and $D_u^{\beta,i}$ homogeneous of degree $|\beta|$. We need the following to estimate $\|f*k_x^{(j)}\|_{L^p}$.

(4.7) Proposition. The operator

$$f \longrightarrow f *_x (L_x-L_y)k_y$$

is bounded from L^p to itself, $1 < p < \infty$ with bound

(4.8) $$\|f *_x (L_x-L_y)k_y\|_{L^p} \leq |x-y| C_{p,x} \|f\|_{L^p}$$

with $C_{p,x}$ bounded when x varies in a bounded neighborhood of y.

Proof. By (4.6) it suffices to show that there exists for each i, β, $C_{i,\beta,p,x} > 0$ such that

(4.9) $\| f *_x (h_{\beta,i}(x,y,u) D_u^{\beta,i} k_y) \|_{L^p} \leq C_{i,\beta,p,x} \| f \|_{L^p}$

with $C_{i,\beta,p,x}$ is bounded as x varies. First we claim that $h_{\beta,i}(x,y,u) D_u^{\beta,i} k_y = k$ is a distribution of type 0. Clearly it is a distribution, since k_y is a distribution. Also by the homogeneity of h and $D_u^{\beta,i}$, k is homogeneous of degree $\lambda = -Q$ in the sense that $k(f \circ \delta_t) = t^{-Q-\lambda} k(f) = k(f)$. Hence by Knapp-Stein [9, Theorem 2], k is a distribution of type 0. Hence $f \longrightarrow f*k$ extends to a bounded operator on L, $1 < p < \infty$ and it suffices to show that the constant $C_{p,x}$ is bounded as x varies over a compact set. For this, we must check the constants which enter into the proof of L^2 boundedness, and then those of the L^p boundedness (which depend on the L^2 case). By checking the proof in Knapp-Stein [9, Theorem 1], we find that $\| f*k \|_{L^2} \leq C \| f \|_{L^2}$, where C depends on C_i, $i = 1,2,\ldots,6$ as follows.

(4.10) $$\int_{a \leq \| u \| \leq b} |k(u)| du \leq C_1 \log(b/a)$$

If $k = k(x,y,u)$ is as above, with y fixed, then k varies smoothly with x, $u \neq 0$ so that C_1 can be chosen independent of x as x varies in a bounded neighborhood of y. (Note that each $G_x \approx \mathbb{R}^n$, topologically, and that the definition of the norm is independent of x. Hence the group multiplication on G_x is not involved in (4.10).)

(4.11) $\left|\frac{\partial}{\partial u_{jk}} k(u)\right| \leq C_2 \|u\|^{-Q-1}$

C_2 is clearly bounded if x is bounded.

(4.12) $\|u \circ_x v - u\| \leq C_3 |v|^{1/r}$

if $\|w\| \leq 1$, where $\circ_x$ denotes multiplication on G_x. C_3 is bounded since multiplication on G_x is smooth as x varies by (3.4).

(4.13) $|u| \leq C_4 \|u\|, \qquad \|u\| \leq 1.$

Here C_4 is actually independent of x.

(4.14) $\|u \circ_x v\| \leq C_5(\|u\| + \|v\|).$

C_5 is bounded by the smoothness of the group multiplication on G_x.

By the remarks in section 2

(4.15) $f*k = cf + PV(f*k),$

a constant, where PV denotes principal value, taken in the homogeneous sense. The above examination of the proof of [9, Theorem 1], shows that

$$(4.16)\quad \| PV(f*(h_{\beta,i}(x,y,u)D_u^{\beta,i}k_y\|_{L^2} \leq C_{i,\beta,2,x}\|f\|_{L^2},$$

with $C_{i,\beta,2,x}$ bounded if x varies in a bounded subset of y. We must show that the constant c in (4.15) is also bounded as x varies. In fact c is determined by C_6, where

$$(4.17)\quad |k(u)| \leq C_6\|u\|^{-Q}.$$

For, by definition of the principal value operator and integration by part 5,

$$cf(0) = \lim_{\varepsilon\to 0}\int_{\|u\|=\varepsilon} k(u)f(u)du \leq |f(0)|\lim_{\varepsilon\to 0}\int_{\|u\|=\varepsilon} k(u)du.$$

Since $\int_{\|u\|=\varepsilon} k(u)du \leq C_6\|\varepsilon\|^{-Q}\int_{\|u\|=\varepsilon}\|u\|^{-Q}du = CG_6$,

the proof of (4.9) for $p = 2$ is complete.

For arbitrary p, $1 < p < \infty$, $C_{i,\beta,p,x}$ depends on the constants C_7, C_8, C_9 defined in the following inequalities.

$$(4.18)\quad \|u \circ_x v^{-1}\| \leq C_7(\|u\|+\|v\|)$$

C_7 can be chosen independent of x, for x in a bounded set by the continuity of the group multiplication.

(4.19) $|k(w^{-1} \circ_x u) - k(u)| \leq C_8\|w\|\,\|u\|^{-Q-1}$

$$\text{if } \|w\| \leq 1/2\, C_7\|u\|$$

Note that $C_7(\|w^{-1} \circ_x u\| + \|w\|) \geq \|u\|$ by (4.17) and so $\|w^{-1} \circ_x u\| \geq \|u\| - C_7\|w\| \geq \frac{1}{2}\|u\|$. Hence $w^{-1} \circ_x u$ is bounded away from 0 if $\|u\| = 1$, and one may proceed as in [4, Lemma 8.10], checking that the constant C_8 may be chosen when x varies in a bounded set. From (4.19) one obtains

(4.20) $$\int_{\|u\| \geq 2C_7\|w\|} |k(w^{-1} \circ_x u) - k(u)|\,du \leq C_9 ,$$

with C_9 depending only on C_7 and C_8. Then

(4.21) $$\|f*k\|_{L^p} \leq C\|f\|_{L^p} \qquad 1 < p < \infty$$

follows as in [2, Theorème 2.4], with C depending only on C_7 , C_9 and the constant in (4.15). This completes the proof of (4.9) and hence of Proposition 4.7.

(4.22) <u>Corollary</u>. <u>Suppose</u> $\frac{1}{q} = \frac{1}{p} - d/Q$, <u>with</u> $1 < q < \infty$, $1 < p < Q/d$. <u>Let</u> $C_p \geq 1$ <u>be chosen so</u>

that $C_{p,x} \leq C_p$ *for all* x, $|x-y| < 1$. *Then if* $|x-y| < 1/C_p$,

$$\sum_{j=0}^{\infty} \| f*k_x^{(j)} \|_{L^q} \leq C \| f \|_{L^p}$$

for some $C > 0$.

Proof.

$$(4.23)\quad \| f*k_x^{j} \|_{L^q}$$

$$= \| \underbrace{f*(L_x-L_y)k_y*(L_x-L_y)k_y*\ldots(L_x-L_y)k_y}_{j}*k_y \|_{L^q}$$

$$\leq C_{pq} \| f*(L_x-L_y)k_y*(L_x-L_y)k_y*\ldots*(L_x-L_y)k_y \|_{L^p}$$

by Theorem A, since k_y is of type d. By Proposition 4.7 applied j times, the right-hand side of (4.23) is bounded by $C_{pq}|x-y|^j C_p^j \| f \|_{L^p}$. Hence the sum converges.

Finally, we claim that the operator $K_x : L^p \longrightarrow L^q$ is formally a left and right inverse for L_x. Indeed, by (3.7)

$$L_x(f*k_x^{(j)}) = f*(L_x-L_y)k_y*(L_x-L_y)k_y * \ldots * (L_x-L_y)k_y*L_xk_y$$

$$= \underbrace{f*(L_x-L_y)k_y*(L_x-L_y)k_y * \ldots * (L_x-L_y)k_y}_{j+1}$$

$$+ f*(L_x-L_y)k_y*(L_x-L_y)k_y*\ldots*L_yk_y$$

$$= \underbrace{f*(L_x-L_y)k_y * \cdots * (L_x-L_y)k_y}_{j+1}$$

$$+ \underbrace{f*(L_x-L_y)k_y * \cdots * (L_x-L_y)k_y}_{j}$$

Hence $\Sigma L_x(f*k_x^{(j)})$ is a collapsing sum, and we need only check that it is absolutely convergent. However,

$$\| f*(L_x-L_y)k_y * \cdots * (L_x-L_y)k_y\|_{L^p} \leq |x-y|^j C_p^j \| f\|_{L^p}$$

which implies $\sum_{j=0}^{\infty} L_x(f*k_x^{(j)}) = f$ provided $|x-y| < C_p$. Since L_x is self-adjoint, K_x is a left and right inverse for L_x. We summarize these results.

(4.24) <u>Theorem</u>. <u>Let</u> K_x <u>be defined by</u> (4.2). <u>Then for any</u> $p, q, 1 < p, q < \infty, 1/q = 1/p - d/Q$, <u>there exists</u> $\delta = \delta_{p,q} > 0$ <u>such that if</u> $|x-y| < \delta$ <u>and</u> $\delta \in L^p$,

$$\| K_x f\|_{L^q} \leq C\| f\|_{L^p} ,$$

$$L_x K_x f = f, \quad \text{\underline{and}}$$

$$K_x L_x f = f.$$

5. <u>Smoothness of the kernel of</u> K_x. We have now constructed a left and right parametrix K_x for L_x, but we know only that K_x is bounded from L^p to L^q.

We shall show that by taking some $\delta' < \delta$, we have

$k_x = \sum_{j=0}^{\infty} (-1)^j k_x^{(j)} \in C^\infty(\mathbb{R}^n - \{0\})$, provided $|x-y| < \delta'$.

Here we shall follow part of the proof of [13, Theorem 3]. However, in this case we do not have uniform estimates for L_x as in [13, Lemma 6.9], since we do not even know that L_x is hypoelliptic in a neighborhood of y.

As in [13], Theorem 3.6 will be proved by the following. Let k_x be defined by $f*k_x = k_x f$.

(5.1) <u>Main Lemma</u>. <u>There exists</u> $\delta' > 0$ <u>such that for all</u> x, $|x-y| < \delta'$ $D_u^\beta k_x(u)$ <u>exists for all</u> β, <u>all</u> u, $\frac{1}{2} < \|u\| < \frac{3}{2}$, <u>and satisfies</u>

$$\sup_{\frac{1}{2}<\|u\|<\frac{3}{2}} |D_u^\beta k_x(u)| \leq C_\beta$$

<u>independent of</u> x. <u>In particular</u>, $k_x \in C^\infty(G_x - \{0\})$ <u>and hence</u> L_x <u>is hypoelliptic</u>.

Assuming the Main Lemma, let us prove Theorem 3.6 now, essentially as in [13, Theorem 3]. First note that since L_z is hypoelliptic for all $z \in U = \{x : |x-y| < \delta'\}$, we may replace y by any $z \in U$ in the Main Lemma. Thus for any z, there exists δ_z such that if $|x-z| < \delta_z$

(5.2) $k_x(u) = \sum_{j=0}^{\infty} k_{x,z}^{(j)}$, where

$$(5.3)\quad k^{(j)}_{x,z} = (L_x-L_z)k_z*(L_x-L_z)k_z * \cdots * (L_x-L_z)k_z*k_z\ ,$$

$$j \geq 1$$

$$k^{(0)}_{x,z} = k_z.$$

(Since both sides of (5.2) are homogeneous fundamental solutions smooth away from 0, they must be equal.)

Next, we define a countable collection of semi norms $\{\| \ \|_\alpha\}$ $\alpha = (\alpha_1,\ldots,\alpha_n)$

$$\|k\|_\alpha = \sup_{\frac{1}{2}\leq\|u\|\leq\frac{3}{2}} \left|\left(\frac{\partial}{\partial u}\right)^\alpha k(u)\right|$$

for $C^\infty = \{k : k$ is smooth on $\{\frac{1}{2} \leq \|u\| \leq \frac{3}{2}\}$. A mapping f from U to C^∞ will be said to be <u>bounded</u> if the positive functions

$$x \longrightarrow \|f(x)\|_\alpha$$

are bounded for each seminorm α. The Main Lemma implies

(5.4) <u>Lemma</u>. <u>$x \longrightarrow k_x$ of U to C^∞ is bounded</u>.

Next, we show that for fixed u, $x \longrightarrow k_x(u)$ is C^∞, $x \in U$. By (5.2), for any $z \in U$ and any positive integer ℓ, if $|x-z| < \delta_z$

$$(5.5)\quad k_x u = \sum_{|\alpha|\leq\ell} \frac{(x-z)}{|\alpha|!} k_z^\alpha(x,u) + R_\ell(x,z,u)|z-x|^{\ell+1}$$

where $k_z^{(\alpha)}$ is a sum of convolutions with k_z over G_x, of $|\alpha|$ terms of the form $h_\beta(x,z,u)D_u^\beta k_z$, where h_β is smooth and homogeneous of degree $|\beta| - d$ in u. Hence by Lemma 5.4 the kernels $(x,z) \longrightarrow k_z^{(\alpha)}(x,u)$ and the remainder $R_\ell(x,z,u)$ vary over a bounded subset of C^∞ for $x, z \in U$, $|x-z| < \delta_z$. By the converse of Taylor's theorem with remainder, applicable since $k_x(u)$ provides a Taylor expansion with bounded coefficients, $x \longrightarrow k_x(u)$ is C^∞ on U and for each α, we may calculate $(\frac{\partial}{\partial x})^\alpha k_x(u)$ as follows. Expand $h_\beta(x,z,u)$ in a Taylor series around z:

$$h_\beta(x,z,u) = \sum_{|\alpha|\leq\ell} \frac{(x-z)^\alpha}{|\alpha|!} h_\beta^{(\alpha)}(z,z,u)$$

$$+ r_\beta(z,u)|x-z|^{\ell+1},$$

where $h_\beta^{(\alpha)}(z,z,u)$ is the α derivative of $h(x,z,u)$ at $x = z$, and $r_\beta(z,u)$ is the remainder. Then

$$(\frac{\partial}{\partial x})^\alpha k_x(u) = \sum_{|\beta|\leq|\alpha|} h_\beta^{(\alpha-\beta)}(x,xu)D_u^\beta k_x ,$$

which is bounded in absolute value on any compact subset of U by the Main Lemma. Now since $(x,u) \longrightarrow k_x(u)$ is separately smooth in x, u, $u \neq 0$, with derivatives bounded on compact subsets, we may conclude as in [13, Theorem 3] that the mapping is jointly smooth. This proves Theorem 3.6, modulo the Main Lemma.

6. The main estimate. We shall now state an estimate and show that it implies the Main Lemma. We write $S^p_{k,x}$ for the spaces $S^p_k(G_x)$ defined with respect to the left invariant vector fields $X_{j,x}$ on G_x. When there is no ambiguity, we may omit the subscript x and write S^p_k. Theorem 3.6 will be proved by the following.

(6.1) *Lemma. Let* $K = \{u : 0 \leqslant \|u\| \leqslant \frac{1}{2}\}$. *There exists* $\delta' > 0$ *satisfying the following. For every integer* $m \geqslant 0$, *there is a constant* C_m *and* a_m, $b_m \in C_0^\infty(K)$ *such that* $a_m(u) \equiv 1$ *for* $\|u\| \leqslant \frac{1}{4}$ *and* $b_m \equiv 1$ *on* $\operatorname{supp} a_m$ *such that for any* $g \in C_0^\infty(G_x)$ *with* $\operatorname{supp} g \subset \{u : \frac{1}{2} \leqslant \|u\| \leqslant \frac{3}{2}\}$ *and* $f = g * k_x$, $|x-y| < \delta'$ *the estimate*

$$(6.2) \qquad \|a_m f\|_{S^q_{m,x}} \leqslant C_m \|b_m f\|_{L^p}$$

holds for some p, q, $1 < p, q < \infty$, $p > Q/d$.

We shall refer to (6.2) as the main estimate. To prove it we shall start with an easy observation. If

$\ell \geq rm$, m integral, then

$$(6.3) \quad \|a\phi\|_{L^p_m} \leq C_x \|a\phi\|_{S^p_{\ell,x}},$$

for all $\phi \in C_0^\infty$ and all $a \in C_0^\infty(K)$, such that C_x is bounded as x varies in a bounded neighborhood of y. Indeed for fixed x the inclusion is clear by the characterization of L^p_m as the space of L^p functions with up to m derivatives in L^p, given that a has fixed support. The boundedness of C_x follows from (3.4) and the fact that $\operatorname{supp} a \subset K$ is fixed.

Next, if $b \in C_0^\infty(K)$ with $b \equiv 1$ on $\operatorname{supp} a$ and $1/p = 1/t - d/Q$, then by Theorem 4.24,

$$(6.4) \quad \|bf\|_{L^p} = \|b(K_x g)\|_{L^p} \leq C\|g\|_{L^t}$$

for some constant C, if $|x-y| < \delta'$. By the main estimate and (6.3),

$$(6.5) \quad \|a_m f\|_{L^q_m} \leq C'_m \|b_m f\|_{L^p} \leq C''_m \|g\|_{L^t}.$$

By Sobolev's Lemma e.g. [15], for any α, there exists $m > 0$ such that

$$(6.6) \quad |D^\alpha_u f(0)| \leq C_{\alpha,q} \|a_m f\|_{L^q_m} \leq C''_{\alpha,p} \|g\|_{L^t}$$

by (6.4). Now if D_u^α is left invariant on G_x by (3.7),

$$(6.7)\quad D_u^\alpha f(0) = (g*D_u^\alpha k_x)(0) = \int g(u)(D_u^\alpha k_x)(u^{-1})du.$$

Hence from (6.6),

$$\left|\int g(u)D_u^\alpha k_x(u^{-1})du\right| \leq C''_{\alpha,p}\|g\|_{L^s}$$

By the converse of Hölder's inequality,

$$\{\int_{\frac{1}{2}\leq\|u\|\leq\frac{3}{2}} |D_\alpha^u k_x(u^{-1})|^{t'}\ du\}^{1/t'} \leq C''_{\alpha,p}\ ,$$

where t' satisfies $1/t' + 1/t = 1$ since g is arbitrary among smooth functions with support in $\{\frac{1}{2} \leq \|u\| \leq \frac{3}{2}\}$. By another application of Sobolev's inequality $D_u^\beta k_x(u)$ exists for all β and satisfies

$$\sup_{\frac{1}{2}\leq\|u\|\leq\frac{3}{2}} |D_u^\beta k_x(u)| \leq C_\beta.$$

Hence by homogeneity, for $|x-y| < \delta'$, $k_x \in C^\infty(G-\{0\})$, which, as is well known, implies that L_x is hypoelliptic.

7. <u>Estimates for</u> $\|D_u^\alpha f\|_{L^q}$, $f = g*k_x$. We begin the proof of the main estimate (6.2). By induction it suf-

fices to prove that for $a_m \in C_0^\infty(K)$ there exists $a_{m-1} \in C_0^\infty(K)$ such that $a_{m-1} \equiv 1$ on $\operatorname{supp} a_m$ and

$$(7.1)\quad \|a_m f\|_{S_m^q} \leq C_m(\|a_{m-1}f\|_{S_{m-1}^q} + \|a_{m-1}f\|_{L^p}).$$

Since K_x is a left inverse for L_x, if $|x-y| < \delta'$

$$(7.2)\quad a_m f = (L_x a_m f)*k_x = [L_x, a_m]f*k_x + a_m L_x f*k_x ,$$

where all convolutions are taken over G_x. The last term in (7.2) vanishes since $a_m L_x f = a_m g = 0$. Hence

$$a_m f = \sum_{j=0}^{\infty} [L_x, a_m]f*k_x^{(j)}.$$

Thus we must estimate

$$(7.3)\quad \|[L_x, a_m]f*k_x^{(j)}\|_{S_m^q} = \sum_{|\alpha|<m}^{\infty} \|X_x^\alpha [L_x, a_m]f*k_x^j\|_{L^q}$$

This will involve mainly manipulation of derivatives as in [13, Section 14]. The main results needed, besides Theorems A and B, are formulas for moving derivatives across convolutions. We write X_{jk} for $X_{jk,x}$ in the following.

(7.4) <u>Lemma</u>. <u>For a vector field</u> X <u>let</u> X' <u>be defined by</u> $X'(f(-u)) = -(Xf)(-u)$. Then

(7.5) $X_k = -X'_k + \sum_{s>1} b_{ks\ell}(x,u)X'_{s\ell}$

(7.6) $X'_k = -X_k + \sum_{s>1} b'_{ks\ell}(x,u)X_{s\ell}$,

where $b_{ks\ell}$ and $b'_{ks\ell}$ are smooth and homogeneous of degree $j-1$ in u.

Proof. These are both special cases of (2.8) with $x = y$.

Now we estimate (7.3). First,

$$[L_x, a_m] = \sum_{|\alpha_i| \leq d-1} X^{\alpha_i} a_{m,i}(u),$$

a finite sum, with each $a_{m,i}(u)$ satisfying $a_{m,i}g = 0$ since $\text{supp}\, a_{m,i} \subset K$. First, we shall show the following.

(7.7) Lemma. Each $X^{\alpha_i} a_{m,i} f * k_x^{(j)}$ is a sum of $(c^j)^{d-1}$ terms of the form

$$a_{m,i} f * k_j * k_{j-1} * \cdots * k_1 * k_0 ,$$

where c is a constant, independent of j and x, each k_s is a kernel of type 0 of the form

$$k_s = b_{\beta_1}(x,u)X^{\beta_1} b_{\beta_2}(x,u)X^{\beta_2} \cdots b_{\beta_{d-1}}(x,u)X^{\beta_{d-1}}(L_x - L_y)k_y ,$$

$s \geq 1$, <u>where</u> $|\beta_i| \leq r-1$ <u>for each</u> i <u>and</u> $b_{\beta_i}(x,u)$ <u>is smooth, and homogeneous of degree</u> β_i <u>in</u> u, <u>and</u> $k_0 = X^{\alpha}k_y$, $|\alpha| = |\alpha_i|$.

<u>Proof</u>. Suppose $\phi \in C_0^\infty$, $h_1 \ldots, h_j$ kernels of type 0 and k_0 a kernel of type ≥ 1. Then by (3.9),

$$(7.8)\quad X_k\phi * k_j * k_{j-1} * \cdots * k_1 * k_0$$

$$= \phi * (X_k' k_j) * k_{j-1} * \cdots * k_1 * k_0$$

By (7.6)

$$X_k' k_j = -X_k k_j + \sum_{s,\ell} b_{s\ell k}'(x,u) X_{s\ell} k_j$$

$$= -X_k k_j + \sum_{i,t} X_t (b_{\alpha_{i,t}}(x,u) X^{\alpha_i}) k_j$$

with $|\alpha_i| \leq r-1$ and $b_{\alpha_{i,t}}(x,u)$ homogeneous of degree $|\alpha_i|$. (The last equality is obtained by expanding X_s as a sum of products of the X_k. Hence applying (3.9) we obtain c terms of the form

$$\phi * b_{\alpha_i} X^{\alpha_i} k_j * X_i'(k_{j-1} * k_{j-2} * \cdots * k_1 * k_0).$$

Continuing, we express the left-hand side of (7.8) as c^j terms of the form

$$\phi * b_{\alpha_j} X^{\alpha_j} k_j * b_{\alpha_{j-1}} X^{\alpha_{j-s}} k_{j-1} * \ldots * b_{\alpha_1} X^{\alpha_1} k_1 * X_i k_0$$

Now since $X^\alpha = X_{k_1} X_{k_2} \ldots X_{k_s}$, $s \leq d-1$, and k_y is of type d, the lemma follows by induction.

Next, we must estimate

$$\| X_{i_1} X_{i_2} \ldots X_{i_s} (a_{m,i} f * k_j * k_{j-1} \ldots * k_1 * k_0) \|_{L^q}.$$

First, since k_0 is of type t, $1 \leq t \leq d$ with $t \leq s$, then

$$(7.9) \quad X_{i_1} X_{i_2} \ldots X_{i_s} (a_{m,i} f * k_j * k_{j-1} * \ldots * k_1 * k_0)$$

$$X_{i_1} X_{i_2} \ldots X_{i_{s-t}} (a_{m,i} f * k_j * k_{j-1} * \ldots * k_1 * k_0'),$$

where $k_0' = X_{i_{s-t+1}} \ldots X_{i_{s-1}} X_{i_s} k_0$ is of type 0.

(7.10) Lemma. If k_i is of type 0, $i = 1, \ldots, j$, and k_0' also of type 0, then $X_{i_1} \ldots X_{i_\ell} ((af) * k_j * k_{j-1} \ldots k_1 * k_0')$ is a sum of C^j terms of the form

$$(X_{j_1} \ldots X_{j_\ell} af) * k_j' * k_{j-1}' * \ldots * k_1' * k_0'',$$

where

$$(7.11) \quad k_t' = C_{\alpha_1}(x,u) X^{\alpha_1} C_{\alpha_2}(x,u) X^{\alpha_2} \ldots C_{\alpha_\ell}(x,u) X^{\alpha_\ell} k_t,$$

with C_{α_i} homogeneous in u of degree $|\alpha_i|$, and k_0'' of the same form as k_t' with k_t replaced by k_0'.

<u>Proof</u>. This is essentially the same method as for Lemma 7.7. First,

$$X_k(\phi * k_j * k_{j-1} * \cdots * k_1 * k_0')$$
$$= (\phi * k_j * k_{j-1} \cdots * k_1) * X_k k_0'$$

by (3.8). Now express X_k in terms of X_s' using (7.5), and move the derivatives left by (3.9) and repeated use of Lemma 7.4. We omit the details.

Now we are ready to estimate $\| X_{i_1} X_{i_2} \cdots X_{i_s} (a_{m,i} f * k_j * k_{j-1} * \cdots * k_1 * k_0) \|_{L^q}$ for $s \leq m$. First, if $s \geq t =$ type of k_0, then by Lemma 7.10,

$$(7.12) \quad \| X_{i_1} X_{i_2} \cdots X_{i_s} (a_{m,i} f * k_j * k_{j-1} * \cdots * k_1 * k_0) \|_{L^q} \leq$$
$$C^{j\ell} \sup \| (X_{j_1} \cdots X_{j_{s-t-1}} (a_{m,i} f) * k_j' * k_{j-1}' * \cdots * k_1' * k_0'' \|_{L^q} ,$$

where the supremum is taken over the finitely many terms of each type. By Theorem A, the mappings $\phi \longrightarrow \phi * k_\ell'$ and $\phi \longrightarrow \phi * k_0''$ are bounded on L^q. It is important to know the norms of these operators. We have shown that the norms depend on the constants $C_1, C_2, \ldots, C_9$ defined by (4.11) to (4.19). By (7.11), it suffices to assume k_ℓ' is of the form

(7.13) $e_{\alpha_1}(x,u)X^{\alpha_1}e_{\alpha_2}(x,u)X^{\alpha_2}\cdots e_{\alpha_\ell}(x,u)X^{\alpha_\ell}h_\beta(x,y,u)D_u^\beta k_y$,

where $\ell \leq (s-t-1)+(d-1) \leq m+d$, e_{α_i} is homogeneous of degree $|\alpha_i|$, and h_β is homogeneous of degree $|\beta|$. Now as x varies in a bounded neighborhood of y, (7.13) varies in a bounded subset of C^∞. Hence

$$\|\phi * k_t'\|_{L^q} \leq |x-y| C_{m,q} \|\phi\|_{L^q}$$

with $C_{m,q}$ independent of x. Similarly,

$$\|\phi * k_0''\|_{L^q} \leq C'_{m,q} \|\phi\|_{L^q}.$$

Hence

(7.14) $$\|(X_{j_1}\cdots X_{j_\ell} af) * k_j' * k_{j-1}' * \cdots * k_1' * k_0''\|_{L^q} \leq$$
$$C'_{m,q}|x-y|^j C^j_{m,q}\|X_{j_1}\cdots X_{j_\ell} af\|_{L^q} \leq$$
$$C'_{m,q}(|x-y| C_{m,q})^j \|af\|_{S^q_{m-1}}$$

if $j_\ell \leq m-1$. Thus, combining Lemmas 7.7 and 7.10 with (7.12) and (7.14) we obtain

(7.15) $$\|X_{i_1}X_{i_2}\cdots X_{i_s}(a_{m,i}f * k_j * k_{j-1} * \cdots * k_s * k_0)\|_{L^q}$$

$$\subset C^j\ |x-y|^j (C_{m,q})^j \| af\|_{S^q_{m-1}}$$

if $s \geqslant t$. Now if $s < t$, then

$$X_{i_1}X_{i_2}\cdots X_{i_s}(a_{m,i}f*k_j*k_{j-1}\cdots *k_1*k_0) =$$

$$a_{m,i}f*k_j*k_{j-1}*\cdots *k_1*k_0' ,$$

where $k_0' = X_{i_s}X_{i_2}\cdots X_{i_s}k_0$ is of type $t \geqslant 0$. Hence if $t = 0$,

$$\| X_{i_1}X_{i_2}\cdots X_{i_s}(a_{m,i}f*k_j*k_{j-1}*\cdots *k_1*k_0)\|_{L^q} \leqslant$$

$$\subset |x-y|^j \| a_{m,i}f\|_{L^q}$$

by the same argument as for (7.14). Finally, if $0 < t \leqslant d < Q$

(7.16) $$\| X_{i_1}X_{i_2}\cdots X_{i_s}(a_{m,i}f*k_j*k_{j-1}*\cdots *k_1*k_0)\|_{L^q} =$$

$$\| (a_{m,i}f*k_j*k_{j-1}*\cdots *k_1)*k_0'\|_{L^q} \leqslant$$

$$C_t\| a_{m,i}f*k_j*k_{j-1}*\cdots *k_1\|_{L^{p_t}}$$

where $1/q = 1/p_t - t/Q$. Note that C_t is independent of x, for x in a bounded neighborhood of y.

Finally, since convolution by k_ℓ is bounded on L^{p_t} with bound given as above for k'_ℓ, we obtain from (7.16)

$$(7.17)\quad \| X_{i_s} \cdots X_{i_s}(a_{m,i} f * k_j * k_{j-1} * \cdots * k_1 * k_0) \|_{L^q} \leq$$

$$C_t |x-y|^j C^j \| a_{m,i} f \|_{L^q} \leq$$

$$C_t |x-y|^j C^j \| a_0 f \|_{L^{p_t}} ,$$

where $a_0 \in C_0^\infty(K)$ $a_0 \equiv 1$ on supp $a_{m,i}$, all m, i, and C, C_t are independent of x, provided x is in a bounded neighborhood of y. Now t may take only the values $t = 1,2,\ldots,d$, so that there are at most d values p_t. Now choose $p > p_t$, $t = 1,2\ldots d$. By Hölder's inequality since $a \in C_0^\infty(K)$,

$$\| ah \|_{L^{p_t}} \leq C_p \| ah \|_{L^p}$$

all $h \in L^{p_t}_{10c}$, provided $p \geq \sup p_t$. Thus, combining (7.15) and (7.17) with Lemma 7.7 and (7.10) we have

$$\| [L_x, a_m] f * k_x^{(j)} \|_{S^q_m}$$

$$\leq Cd^j |x-y|^j C^j_{m,q,p} (\| a_{m-1} f \|_{S^q_{m-1}} + \| a_0 f \|_{L^p})$$

Now choose $\delta_m' < 1/d\, C_{m,q,p}$. Then

$$\| [L_x, a_m] f * k_x \|_{S_m^q} \leq \sum_{j=0}^{\infty} \| [L_x, a_m] f * k_x^{(j)} \|_{S_m^q} \leq$$

$$C_m (\| a_{m-1} f \|_{S_{m-1}^q} + \| a_0 f \|_{L^p})$$

provided $|x-y| < \delta_m'$. Now if δ_m' is chosen so small that $\delta_m' \leq \delta_j'$ for all $j \leq m$, we have the following.

(7.18) <u>Lemma</u>. <u>Put</u> $q = Q/(Q - d - \frac{1}{2})$. <u>Then</u> $1 < q < \infty$. <u>For every</u> $m > 0$, <u>there exists</u> δ_m' <u>such that if</u> $|x-y| < \delta_m'$,

$$\| a_m \|_{S_m^q} \leq C_m \| b_m f \|_{L^p}$$

<u>with</u> $b_m \equiv 1$ <u>on</u> supp a_m, a_m, $b_m \in C_0^\infty(K)$, $a_m \equiv 1$ <u>on</u> $\{\|u\| \leq 1/4\}$, <u>all</u> $p \geq q$.

<u>Proof</u>. We have shown

$$\| a_m f \|_{S_m^q} \leq \sup_{i=0,1,\dots,d} C_m \| b_m f \|_{L^{p_i}},$$

where $1/q = 1/p_i - i/Q$. The choice of q above guarantees that one can solve for p_i with $p_i > 1$. Now since $1/p_i = 1/q_i + i/Q = (Q-d-1/2)/Q + i/Q$ $p_i = Q/Q-(d-i)-1/2 \leq Q/Q-d-1/2$. Now take $p \geq Q/Q-d-1/2$. By Hölder's inequality

$$\|b_m f\|_{L^{p_i}} \leq \|b_m f\|_{L^p}.$$

Unfortunately, Lemma 7.18 is not quite the same as the main estimate (6.2) because of the dependence of δ'_m on m. We shall show that if $m >> 0$ we may take $\delta' = \delta'_m$ constant. The following is proved as in Section 5.

(7.19) Lemma. *For every* $s > 0$ *there exists* $J = J(s)$ *such that if* $|x-y| < \delta'_J$, *then* $u \longrightarrow k_x(u) \in C^s(G_x - \{0\})$.

8. *A criterion for hypoellipticity on groups*. We want to show that if $k_x \in C^s(G_x-\{0\})$ for s sufficiently large, independent of x, then $k_x \in C^\infty(G_x-\{0\})$ and hence L_x is hypoelliptic. For suppose this is true. Then if $\delta'_{J(s)}$ is chosen as in Lemma 7.19, L_z is hypoelliptic for $z \in U = \{z : |z-y| < \delta'_J\}$. Now one may prove the Lemma 6.1 for any compact subset K' of U using Lemma 7.18 with y replaced by any $z \in K'$.

The proof of Theorem 3.6 will depend on the following, which is of independent interest. (See [13, Theorem 2] and [1] for related results.)

8.1 Theorem. *Let* G *be any nilpotent group with dilations and* L *a left invariant differential operator on* G *homogeneous of degree* d. *Then either*

(i) L *is hypoelliptic, or*

(ii) *there exists* ϕ *continuous on* G *with* $L\phi = 0$, *but* $\phi \notin C^{d+1}(G)$.

Proof. Suppose L is not hypoelliptic. By the theorem of Helffer and Nourrigat [6], there exists an irreducible representation π of G on $L^2(\mathbb{R}^N)$, some N, and a non-zero C^∞ vector $H \in L^2(\mathbb{R}^N)$ such that $\pi(L)H = 0$. From π one constructs a family of irreducible representations π_λ, $\lambda > 0$ with $\pi_\lambda(L)H = 0$ by setting

$$\pi_\lambda(X) = \pi(\delta_\lambda X).$$

Now we proceed as in [13, Theorem 2]. Let

$$\phi(u) = \int_1^\infty \langle \pi_\lambda(u)H,H\rangle \lambda^{-k} d\lambda,$$

where $k \geqslant d+2$ will be chosen later, and $\langle\ ,\ \rangle$ denotes the inner product in $L^2(\mathbb{R}^N)$. Since π_λ is unitary, the integral is absolutely convergent for any $u \in G$. Hence ϕ is continuous. Furthermore, by homogeneity $\pi_\lambda(L) = \lambda^d \pi(L)$, so that

$$L\phi(u) = \int_{-\infty}^\infty L\langle \pi_\lambda(u)H,H\rangle \lambda^{-k} d\lambda$$

$$= \int_1^\infty \langle \pi_\lambda(u)\pi_\lambda(L)H,H\rangle \lambda^{-k} d\lambda.$$

(Since $k \geq d+2$ the integrals are absolutely convergent, which justifies differentiation under the integral.)

We shall now choose k so that $\phi \notin C^{d+1}(G)$. First choose $X \in \mathfrak{g}$ so that $\pi(X) = iI$, I = identity. (This is always possible by defining the quotient $\mathfrak{g}' = \mathfrak{g}/\ker \pi$. The center $\mathfrak{z}'$ of $\mathfrak{g}'$ is one dimensional, and $\pi(z') = i\mathbb{R}\cdot I$.) Then

$$\pi_\lambda(X) = \pi(\delta_\lambda \cdot X) = i\lambda^s I$$

for some s, $1 \leq s \leq r$. Let $q \leq d$ be chosen so that $q \geq d/s$, and put

$$k = 1 + qs + s$$

Now let $\psi(t) = \phi(\exp tX)$, $t \in \mathbb{R}$. We claim $\psi \notin C^{q+1}(\mathbb{R})$, which will prove $\phi \notin C^{d+1}(G)$. Indeed,

$$\psi(t) = \int_1^\infty \langle e^{i\lambda^s t} H, H\rangle \lambda^{-k} d\lambda = c \int_1 e^{i\lambda^s t} \lambda^{-k} d\lambda.$$

Now

$$\frac{\partial^q}{\partial t^q}\psi(t) = \int_1^\infty \lambda^{qs-k} e^{i\lambda^s t} d\lambda = \int_1^\infty \lambda^{-(s+1)} e^{i\lambda^s t} d\lambda,$$

since the integral is absolutely convergent. However,

$$\frac{\partial^{q+1}}{\partial t^{q+1}} \psi(0) = \lim_{t\to 0} \int_1^\infty \lambda^{-(s+1)}(e^{i\lambda^s t}-1)/t \, d\lambda,$$

which blows up as $t \longrightarrow 0$, as is easily seen by making the substitution $\mu = \lambda t^{1/s}$.

(8.2) <u>Corollary</u>. <u>Suppose</u> d, Q <u>are fixed integers with</u> $0 < d < Q$. <u>Then there is an integer</u> s, <u>depending only on</u> d, Q <u>such that the following holds. For any stratified nilpotent group</u> G, <u>of total homogeneity</u> Q', $d < Q' \leq Q$, <u>and any homogeneous, self-adjoint left invariant differential operator</u> L <u>homogeneous of degree</u> d <u>on</u> G, <u>if</u> L <u>has a fundamental solution</u> $k \in C^s(G-\{0\})$, <u>homogeneous of degree</u> $-Q' + d$, <u>then</u> $k \in C^\infty(G-\{0\})$ <u>and</u> L <u>is hypoelliptic</u>.

Corollary 8.2, together with the remarks preceding Theorem 7.1, will complete the proof of Theorem 3.6.

<u>Proof of Corollary 8.2</u>. If L is not hypoelliptic, by Theorem 8.1 there exists $\phi \in L^2_{loc}$ such that $L\phi = 0$, but $\phi \notin C^{d+1}$. By Sobolev's Lemma there is an integer $j > 0$ such that $\phi \in S^2_{j,loc}$ but $\phi \notin S^2_{j+1,loc}$. Choose $a \in C_0^\infty$ with $a\phi \notin S^2_{j+1}$. If X_i is homogeneous of degree 1, then

(8.3) $$\|X_{i_1} \cdots X_{i_{j+1}} a\phi\|_{L^2} = \|X_{i_1} \cdots X_{i_{j+1}} (La\phi * k)\|_{L^2}$$

$$= \| X_{i_1} \cdots X_{i_{j+1}} ([L,a]\phi * k) \|_{L^2}$$

since $L\phi = 0$. Now if $k \in C^s(G-\{0\})$ for s sufficiently large, depending on j, then one may show, by using Lemmas 7.7 and 7.10, that the right-hand side of (8.3) is bounded by $\| b\phi \|_{S_j^2}$, where $b \equiv 1$ on supp a, which is finite by the assumption $\phi \in S^2_{j,loc}$. This contradicts the assumption that $a\phi \notin S^2_{j+1}$.

9. Completion of the proof of Theorem 1.1. We may now complete the proof of Theorem 1.1. By Theorem 1.4, the operator K_1 defined by (1.5) is of type 2 and hence is smoothing of order 2 by Theorem 2.1 (iii). The proof that S is smoothing of order 1 follows exactly as in [4, Proposition 16.2] and [13, Theorem 10].

For the proof of Theorem 1.1, we reduce to the special case $L = L^*$, $\deg \hat{L}_x < Q$, following the arguments in [7]. We replace L by

$$L' = L^*L + \sum_{i=1}^{s} \left(\frac{\partial}{\partial t_i}\right)^{2d},$$

where s is chosen so that $s + Q > 2d$. L' acts on $M \times \mathbb{R}^s$, and the set of vector fields X_k, $k = 1,2,\ldots,p$ and $\frac{\partial}{\partial t_i}$, $i = 1,2,\ldots,s$, again satisfy (0.1) and (0.2). Furthermore, for any

$(x,t) \in M \times \mathbb{R}^s$ the operator $\hat{L}'_{(x,t)}$ is self-adjoint and homogeneous of degree 2d. By the theorem of Helffer-Nourrigat [6], $\hat{L}'_{(y,t)}$ is hypoelliptic for all $t \in \mathbb{R}^s$. Now by Theorem 1.3 there exists K_1, S_1 smoothing of orders 2 and 1, respectively, satisfying

$$K_1 L' = \phi I + S_1 ,$$

where $\phi \equiv 1$ in a neighborhood of $(y,0)$. By the construction of Rothschild-Tartakoff [14] we may improve K_1 so that S_1 becomes infinitely smoothing. By integrating over the extra variables in [13, Theorem 15] we obtain K', S on M smoothing of 2 and ∞ respectively such that

$$K' L^* L = \phi' I + S.$$

Now we may take $K = K' L^*$.

Remark. The technique of Helffer-Nourrigat [7] described above gives new results on estimates for solutions of $Lf = g$ even in the case where L is already a left invariant operator on a stratified group G. In particular, the estimates given in [3, Theorem 6.1] are valid whenever L is hypoelliptic without the added assumptions that deg L is less than Q, the homogeneous degree of G, and L^* is also hypoelliptic.

Acknowledgements

The author is an Alfred P. Sloan Foundation Fellow, and has also received support for this work through a grant from the National Science Foundation. The author would also like to thank the Institute of Mathematics, Hebrew University of Jerusalem, Institute des Hautes Études Scientifiques, and the Massachusetts Institute of Technology for their hospitality while this work was being completed.

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Received October 1978