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A General Reflection Principle in $\mathbf{C}^2$*

M. S. BAOUENDI AND LINDA PREISS ROTHSCHILD

Department of Mathematics, University of California, San Diego, La Jolla, California 92093

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A real analytic hypersurface M through 0 in $\mathbf{C}^n$ is said to have the reflection property if any holomorphic mapping defined on one side of M, not totally degenerate at 0, and mapping M into another real analytic hypersurface in $\mathbf{C}^n$, extends holomorphically to a full neighborhood of 0 in $\mathbf{C}^n$. The main result of this paper is that a real analytic hypersurface in $\mathbf{C}^2$ has the reflection property if and only if it is not Levi flat.

CONTENTS

0. INTRODUCTION

If M is a (germ of a) real analytic hypersurface in $\mathbf{C}^2$ defined by $\rho(Z, \bar{Z}) = 0$, $\rho(0) = 0$, $d\rho(0) \neq 0$, and Ω an open neighborhood of 0 in $\mathbf{C}^2$ we define one side of M as $\Omega^+ = \{Z \in \Omega\colon \rho(Z, \bar{Z}) > 0\}$. If $\mathscr{H}\colon \Omega^+ \to \mathbf{C}^2$ is a holomorphic mapping with $\mathscr{H} \in C^\infty(\overline{\Omega^+})$, we shall say that $\mathscr{H}$ is *not totally degenerate* at 0 if its Jacobian determinant $J(\mathscr{H})(Z) = \det(\partial\mathscr{H}_j/\partial Z_k)(Z)$ is not flat at 0; i.e., its Taylor series at 0 does not vanish identically. We say that a real analytic hypersurface M has the *reflection property* at 0 if any holomorphic mapping defined on one side of M as above and not totally degenerate at 0, mapping M into another real analytic hypersurface M' of $\mathbf{C}^2$, extends holomorphically to a full neighborhood of 0 in $\mathbf{C}^2$.

Our main result is the following.

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THEOREM 1. *A real analytic hypersurface M has the reflection property at 0 if and only if M is not locally biholomorphically equivalent to the hypersurface* $\{\operatorname{Im} Z_2 = 0\}$.

We shall say that M is *flat* if after a holomorphic change of coordinates in $\mathbf{C}^2$, M is defined by $\operatorname{Im} Z_2 = 0$. It is easy to show that a flat hypersurface does not have the reflection property (see Section 4). Also, if M is any hypersurface in $\mathbf{C}^2$ on which there is a smooth CR function f which extends holomorphically only to Ω^+, then the mapping $\mathscr{H} = (f, 0)$ is a totally degenerate mapping from M to $M' = \{\operatorname{Im} Z_2 = 0\}$ which does not extend holomorphically in any neighborhood of 0. Hence Theorem 1 is optimal.

In the complex plane the analogue of Theorem 1 is the Schwarz reflection principle, and there is not condition on the degeneracy of M or H at 0. The first reflection principles in higher dimensions were due to H. Lewy [11] and S. Pincuk [12], who proved independently that if M and M' are strictly pseudoconvex, $\mathscr{H}$ defined on one side of M with $\mathscr{H}$ a diffeomorphism from M onto M' then $\mathscr{H}$ extends holomorphically across M. This result was extended to the case where M and M' are essentially finite (finite type in $\mathbf{C}^2$) when $\mathscr{H}$ is a diffeomorphism by Baouendi, Jacobowitz, and Treves [2] and when $\mathscr{H}$ is not totally degenerate by the authors and S. Bell [1] and the authors [3, 4] in $\mathbf{C}^n$, $n > 2$. Other results on the local reflection principle have been obtained by Bell [5], Diederich and Fornaess [7], Diederich and Webster [8], and others.

We believe that Theorem 1 above is the first result which allows the hypersurfaces M and M' to be of infinite type. Also, the mapping $\mathscr{H}$ need not be of finite multiplicity in the sense of [1, 3]. In fact, in view of [1] it suffices to consider here the case where M is of infinite type. The target hypersurface M' is then necessarily nonflat and may be either of finite or infinite type (see Section 1).

Sections 1, 2, and 3 deal with geometric properties of holomorphic and formal mappings between hypersurfaces M and M', with M of infinite type. Several new numerical biholomorphic invariants associated to the map and the hypersurfaces are introduced and certain relationships between them are proved (Theorems 2 and 3). These are needed for the proof of Theorem 1, but are of interest in their own right. Theorem 1 is proved in Section 4.

1. HYPERSURFACES OF INFINITE TYPE AND MAPPINGS

A real hypersurface in $\mathbf{C}^2$ is a set of the form $M = \{Z \in \mathbf{C}^2 \colon \rho(Z, \bar{Z}) = 0\}$, where ρ is a real valued function with $d\rho \neq 0$ on M. If ρ is real analytic, we say that M is real analytic. If ρ is defined in the neighborhood of a point p_0 we refer to M as the *germ* of a hypersurface at p_0. Without loss of generality, we may assume $p_0 = 0$. If M is real analytic, by the Implicit Function Theorem we can find local coordinates $(z, w) \in \mathbf{C} \times \mathbf{C}$ near 0 so that

$$M = \{(z, w): \operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)\} \tag{1.1}$$

with φ real analytic and $\varphi(z, 0, w) \equiv \varphi(0, \zeta, w) \equiv 0$, (see [2]); we shall call (z, w) *normal coordinates* for M. We shall write $z = x + iy$, $w = s + it$. Solving for $\bar{w}$ in (1.1) we find

$$M = \{(z, w):\ \bar{w} = Q(z, \bar{z}, w)\}. \tag{1.2}$$

Here Q is real analytic and satisfies $Q(z, 0, w) \equiv Q(0, \zeta, w) \equiv w$. Similarly we can find P so that

$$M = \{(z, w): w - \bar{w} = P(z, \bar{z}, w)\} \tag{1.3}$$

with $P(z, 0, w) \equiv P(0, \zeta, w) \equiv 0$.

We will denote by L a nonvanishing antiholomorphic vector field tangent to M and by $\bar{L}$ its complex conjugate. If M is given by $\rho(Z, \bar{Z}) = 0$, we may take $L = (\partial\rho/\partial\bar{Z}_1)(\partial/\partial\bar{Z}_2) - (\partial\rho/\partial\bar{Z}_2)(\partial/\partial\bar{Z}_1)$. Following [9, 10] we say that M is of *finite type* at p_0 if the Lie algebra generated by L and $\bar{L}$ spans the tangent space of M at p_0. When M is real analytic and (z, w) are normal coordinates for M, then M is of finite type at 0 if and only if $\varphi(z, \bar{z}, 0) \not\equiv 0$, where φ is as in (1.1). The latter is equivalent to $Q(z, \bar{z}, 0) \not\equiv 0$ or $P(z, \bar{z}, 0) \not\equiv 0$, where Q and P are in (1.2) and (1.3), respectively.

If M is real analytic M is of *infinite type* (i.e., not of finite type) at p_0 if and only if there exists a (germ of *a*) complex line S contained in M and passing through p_0.

Let T be a real valued, real analytic vector field tangent to M such that L, $\bar{L}$, and T are linearly independent near p_0. The *Levi form* α of M, defined up to multiplication by a nonvanishing real analytic function, is given by

$$\frac{1}{2i}[L, \bar{L}] = \alpha T \qquad \operatorname{mod}(L, \bar{L}),$$

where α is a real analytic function defined on M.

The following is well known:

(1.4) PROPOSITION. *The Levi form α vanishes identically if and only if $\varphi(z, \bar{z}, \operatorname{Re} w) \equiv 0$ in (1.1), for any normal coordinates (z, w) in $\mathbf{C}^2$.*

We say that M is *flat* if one of the equivalent conditions of Proposition (1.4) is satisfied.

If M is of infinite type at p_0 and S is the complex line contained in M passing through p_0 then, since L and $\bar{L}$ are tangent to S, the Levi form α vanishes on S. Denote by $j_0 + 1$ the order of vanishing of α on S ($j_0 \geqslant 0$). We write $j_0 = j_0(M)$ in order to avoid any confusion. We put $j_0 = -1$ if M is of finite type at p_0, and $j_0 = \infty$ if M is falt. The integer j_0 is clearly a *biholomorphic invariant* of M. It can be expressed in terms of coordinates in the following way. Let (z, w) be normal coordinates for M and φ and Q as in (1.1) and (1.2). If M is of infinite type (at 0) we write

$$\varphi(z, \bar{z}, s) = \sum_{j=1}^{\infty} \varphi_j(z, \bar{z}) s^j \tag{1.5}$$

with $\varphi_j(z, 0) \equiv \varphi_j(0, \zeta) \equiv 0$ for all j and

$$Q(z, \bar{z}, w) = w\left(1 + \sum_{j=0}^{\infty} R_j(z, \bar{z}) w_j\right) \tag{1.6}$$

with $R_j(z, 0) \equiv R_j(0, \zeta) \equiv 0$.

(1.7) Proposition. *If M is of infinite type, let $j_1 \geqslant 0$ be minimal such $\varphi_{j_1+1}(z, \bar{z}) \not\equiv 0$, in* (1.5), *and j_2 minimal such that $R_{j_2}(z, \bar{z}) \not\equiv 0$, in* (1.6), *and j_0 defined as above* (*i.e.*, $\alpha = s^{j_0+1}\alpha_1(z, \bar{z}, s)$, $\alpha_1(z, \bar{z}, 0) \not\equiv 0$). *Then $j_0 = j_1 = j_2$ with*

$$R_{j_0}(z, \bar{z}) = -2i\varphi_{j_0+1}(z, \bar{z}) \qquad \text{if} \quad j_0 > 0 \tag{1.8}$$

and

$$\frac{R_0(z, \bar{z})}{1 + R_0(z, \bar{z})/2} = -2i\varphi_1(z, \bar{z}) \qquad \text{if} \quad j_0 = 0. \tag{1.9}$$

Proof. We first show that $j_1 = j_2$. Substituting (1.2), (1.5), and (1.6) into (1.1) we obtain

$$\begin{aligned} w - w&\left(1 + \sum_{j=j_2}^{\infty} R_j(z, \bar{z}) w^j\right) \\ &\equiv 2i \sum_{j=j_1+1}^{\infty} \varphi_j(z, \bar{z}) \left[\frac{w + w(1 + \sum R_k(z, \bar{z}) w^k)}{2}\right]^j. \end{aligned} \tag{1.10}$$

The coefficient of the lowest power of w is $-R_{j_2}(z, \bar{z})$ on the left hand side of (1.10) and is $2i\varphi_{j_1+1}(z, \bar{z})(1 + R_0(z, \bar{z})/2)^{j_1+1}$ on the right hand side, which proves $j_1 = j_2$ as well as (1.8) and (1.9). It remains to show that $j_0 = j_1$. Parametrizing M by $(z, \bar{z}, s)$ we take

$$L = \frac{\partial}{\partial \bar{z}} - i \frac{\varphi_{\bar{z}}}{1 + i\varphi_s} \frac{\partial}{\partial s} \tag{1.11}$$

and the real vector field $T = \partial/\partial s$. We have $(1/2i)[L, \bar{L}] = \alpha(\partial/\partial s)$, and a simple computation shows that

$$\alpha = s^{j_1+1}[\varphi^m_{j_1+1, z\bar{z}}(z, \bar{z}) + O(|z|^{m-1} + |s|)], \tag{1.12}$$

where $\varphi^m_{j_1+1}$ is the lowest nonzero homogeneous polynomial in the Taylor expansion of φ_{j_1+1} at 0, with m its degree. This shows that $j_0 = j_1$, which completes the proof of the proposition.

Remark. It follows from (1.11) that in any normal coordinates (z, w) we may take L as

$$L = \frac{\partial}{\partial \bar{z}} + a(z, \bar{z}, s) s^{j_0+1} \frac{\partial}{\partial s}, \tag{1.13}$$

where $a(0, \zeta, 0) \equiv 0$ and $a_z(z, \bar{z}, 0) \not\equiv 0$.

We shall now consider holomorphic mappings defined on one side of a hypersurface. Let M be a real analytic hypersurface in $\mathbf{C}^2$, $p_0 \in M$, defined by $\rho = 0$, $d\rho \neq 0$ on M. Let Ω be a small neighborhood of p_0 in $\mathbf{C}^2$. We denote by Ω^+ the side of M given by

$$\Omega^+ = \{Z \in \Omega : \rho(Z, \bar{Z}) > 0\}. \tag{1.14}$$

We consider mappings $\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2)$, with $\mathscr{H}_1, \mathscr{H}_2$ holomorphic in Ω^+ and C^∞ in $\overline{\Omega^+}$, with the property that $\mathscr{H}(M) \subset M'$, another real analytic hypersurface of $\mathbf{C}^2$. If (z', w') are normal coordinates for M' then $\mathscr{H}$ may be defined by components $\mathscr{F}, \mathscr{G}$, where $z' = \mathscr{F}(z, w)$, $w' = \mathscr{G}(z, w)$, with (z, w) normal coordinates for M, normalized by $\mathscr{F}(0) = 0$, $\mathscr{G}(0) = 0$. We denote by $F(z, w)$ and $G(z, w)$ the (formal) holomorphic Taylor series of $\mathscr{F}$ and $\mathscr{G}$ at 0 and write f and g for the restrictions of $\mathscr{F}$ and $\mathscr{G}$ to M, and we write $H = (f, g)$. If M is given by (1.1) then the Taylor series of f and g in the coordinates $(z, \bar{z}, s)$ are

$$\begin{aligned} f(z, \bar{z}, s) &\sim F(z, w)|_{w=s+i\varphi(z,\bar{z},s)}, \\ g(z, \bar{z}, s) &\sim G(z, w)|_{w=s+i\varphi\varphi(z,\bar{z},s)}. \end{aligned} \tag{1.15}$$

If M' is given by an equation of the form (1.2) then we have

$$\bar{G}(\bar{z}, \bar{w}) \equiv Q(F(z, w), \bar{F}(\bar{z}, \bar{w}), G(z, w)) \tag{1.16}$$

as an identity of formal power series when $w = s + i\varphi(z, \bar{z}, s)$, $\bar{w} = s - i\varphi(z, \bar{z}, s)$. More generally, if $F(z, w)$, $G(z, w)$ are formal power series in (z, w), we shall say that they define a *formal holomorphic map* between M and M' if (1.16) holds.

We observe from the definition of normal coordinates that we have

$$G(z, w) = wG'(z, w), \tag{1.17}$$

with $G'(z, w)$ another formal holomorphic power series. We write

$$F(z, w) = \sum_{j=0}^{\infty} F_j(z) w^j \tag{1.18}$$

and

$$G(z, w) = \sum_{j=1}^{\infty} G_j(z) w^j. \tag{1.19}$$

We define p and l to be minimal, $0 \leqslant p \leqslant l \leqslant +\infty$, for which

$$F_p(z) \not\equiv 0 \qquad \text{and} \qquad F_l(z) \not\equiv F_l(0). \tag{1.20}$$

If $l < \infty$ we define k_1 to be minimal, $1 \leqslant k_1 < \infty$, so that

$$F_{l, z^{k_1}}(0) \neq 0. \tag{1.21}$$

Finally, we define k_0 to be minimal so that

$$G_{k_0}(z) \not\equiv 0. \tag{1.22}$$

We will show in Section 2 that k_0, k_1, and l are biholomorphic invariants, i.e., independent of the choice of normal coordinates (z, w) and (z', w'), whereas p need not be invariant.

We remark here that if the Jacobian of $\mathscr{H}$ is not flat at p_0 then $G \not\equiv 0$. Indeed, the Taylor series of the Jacobian of $\mathscr{H}$ is given by

$$J(\mathscr{H}) \sim F_z G_w - G_z F_w. \tag{1.23}$$

We close this section with the following.

(1.24) Proposition. *Let $\mathscr{H} = (F, G)$ be a formal holomorphic mapping between two real analytic hypersurfaces M and M' defined as above. If M is not flat and $G \not\equiv 0$, then M' is not flat and $F \not\equiv 0$.*

Proof. Assume by contradiction that M' is flat, i.e., M' is given by $w' = \bar{w}'$, or $F \equiv 0$. In both cases we have

$$G(z, w) = \bar{G}(\bar{z}, \bar{w}), \tag{1.25}$$

for $\bar{w} = Q(z, \bar{z}, w)$, where we assume M is given by (1.2). Substituting for $\bar{w}$ in (1.25) and putting $\bar{z} = 0$ we obtain

$$G(z, w) \equiv \bar{G}(0, w),$$

i.e., G is real and independent of z. Then by (1.25)

$$G(w) = G(Q(z, \bar{z}, w)), \tag{1.26}$$

and by differentiating both sides with respect to z we have

$$0 \equiv G_w(Q(z, \bar{z}, w))\, Q_z(z, \bar{z}, w).$$

Since G and Q are nonzero, it follows that $Q_z(z, \bar{z}, w) \equiv 0$, i.e., $Q(z, \bar{z}, w) \equiv w$, so that M is flat, contradicting, the hypothesis.

2. Invariants Associated to Holomorphic Maps

We shall prove the following results for holomorphic mappings defined on one side of a real analytic hypersurface which will be needed for the proof of Theorem 1.

Theorem 2. *Let M be a nonflat real analytic hypersurface in $\mathbf{C}^2$ and $\mathscr{H}$ a holomorphic mapping defined in Ω^+, given by* (1.14), *with $\mathscr{H}$, C^∞ in $\overline{\Omega^+}$. Assume furthermore that $\mathscr{H}(M) \subset M'$, another real analytic hypersurface. Then k_0 defined by* (1.22) *is independent of the choice of normal coordinates. If k_0 is finite, then k_1 and l, defined by* (1.20) *and* (1.21) *are also finite and are independent of the choices of normal coordinates for M and M'.*

Theorem 3. *Let $\mathscr{H}$ be as in Theorem* 1. *If $G \not\equiv 0$, then for any choice of normal coordinates and for any choice of nonvanishing antiholomorphic vector field L tangent to M the following holds for every $j \geqslant 1$,*

$$\bar{L}^j f(z, \bar{z}, s) = s^l f_j(z, \bar{z}, s) \tag{2.1}$$

for some smooth functions f_j on M, with

$$f_{k_1}(0) \neq 0, \qquad \textit{and} \qquad f_1(0) = \cdots = f_{k_1 - 1}(0) = 0. \tag{2.2}$$

Here k_1 and l are as in Theorem 2, *and we have used the notation z, s, f introduced in Section* 1.

We shall first prove (2.1) and (2.2) of Theorem 3 for a given system of normal coordinates. We shall then prove Theorem 2, which will show that the numbers k_1 and l are independent of the choices. The proof of

Theorem 3 depends heavily on the following technical lemma, which will be proved in Section 3.

(2.3) MAIN LEMMA. *Suppose* $\mathscr{H} = (F, G)$ *is a formal holomorphic mapping between two real analytic hypersurfaces* M, M' *with* $G \not\equiv 0$ *and* M *of infinite type. Let* p, l, k_0 *be defined by* (1.20) *and* (1.22) *for a given set of normal coordinates and* $j_0 = j_0(M)$ *as in Section* 1. *Then the following hold*:

If M' *is of finite type then*

$$j_0(M) \geqslant l - p, \tag{2.4}$$

with $j_0(M) = \infty$ *if* $l = p = \infty$.

If M' *is of infinite type, then*

$$G_{k_0}(z) \equiv G_{k_0}(0) \in \mathbf{R} \setminus \{0\}, \tag{2.5}$$

$$j_0(M) \geqslant 2l. \tag{2.6}$$

Proof of (2.1) *and* (2.2) *for a Given System of Normal Coordinates.* We consider first the case where M is of finite type. By [4, Theorem 4] we have $l = 0$ and M' is of finite type. Also, k_1 is then the multiplicity of the map in the sense of [1] and (2.1) and (2.2) follow.

We may now assume M of infinite type and nonflat. It follows from Proposition (1.24) that $F \not\equiv 0$, i.e., $p > \infty$. Therefore by Main Lemma (2.3) l is finite and hence so is k_1. We consider separately the cases $p = l$ and $1 \leqslant p < l$.

Case 1. $p = l$.

Then

$$f \sim F = F_l(z) w^l + \cdots + F_j(z) w^j + \cdots \tag{2.7}$$

with $w = s + i\varphi(z, \bar{z}, s)$, where

$$\varphi = \sum_{j = j_0 + 1}^{\infty} \varphi_j(z, \bar{z}) s^j.$$

We first show that if $l \geqslant 1$ then

$$f(z, \bar{z}, 0) \equiv \cdots \equiv f_{s^{l-1}}(z, \bar{z}, 0) \equiv 0. \tag{2.8}$$

Since by (1.13)

$$L = \frac{\partial}{\partial \bar{z}} + a(z, \bar{z}, s) s^{j_0 + 1} \frac{\partial}{\partial s} \tag{2.9}$$

we have

$$Lf|_{s=0} = \frac{\partial}{\partial \bar{z}} f(z, \bar{z}, 0) \equiv 0, \tag{2.10}$$

which shows $f(z, \bar{z}, 0)$ is holomorphic in z. From (2.7) it follows that the Taylor series of $f(z, \bar{z}, 0)$ at 0 vanishes identically, proving $f(z, \bar{z}, 0) \equiv 0$. From (2.9) we have for all j

$$0 \equiv (Lf)_{s^j} = \frac{\partial}{\partial \bar{z}} (f_{s^j}) + \sum_{k=0}^{j} a_k f_{s^k} + a s^{j_0+1} f_{s^{j+1}}, \tag{2.11}$$

for some real analytic functions a_k. By induction, assuming $f_{s^j}(z, \bar{z}, 0) \equiv 0$, $j < r$, we obtain from (2.11)

$$\left. \frac{\partial}{\partial \bar{z}} f_{s^r} + a_r f_{s^r} \right|_{s=0} \equiv 0, \tag{2.12}$$

from which it follows that $f_{s^r}(z, \bar{z}, 0)$ is real analytic. If $r < l$, the Taylor series of $f_{s^r}(z, \bar{z}, 0)$ is zero, by (2.7), which proves (2.8). Therefore, we have

$$f(z, \bar{z}, s) = s^l f_0(z, \bar{z}, s) \tag{2.13}$$

with f_0 smooth. Then (2.1) follows immediately from (2.9) and (2.13). Furthermore, we have

$$f_j(0) = \bar{L}^j f_0(0). \tag{2.14}$$

Indeed, we have for all j,

$$s^l f_j \equiv \bar{L}^j s^l f_0 = s^l \bar{L}^j f_0 + \sum_{k=1}^{j} c_k (\bar{L}^k s^l)(\bar{L}^{j-k} f_0).$$

It suffices to note that for all k, $\bar{L}^k s^l / s^l$ is smooth and vanishing at 0, which follows from (2.9) and the fact that $a(0, \bar{z}, 0) \equiv 0$ by (1.13).

If we write

$$w = s(1 + i\varphi_1(z, \bar{z}, s))$$

then it follows from (2.7) and (2.13) that

$$f_0 \sim F_l(z)(1 + i\varphi_1(z, \bar{z}, s))^l + \cdots + F_j(z)\, s^{j-l}(1 + i\varphi_1(z, \bar{z}, s))^j + \cdots. \tag{2.15}$$

From (2.9) and the property of φ in (1.1) we easily obtain for all $j \geqslant 0$ and $q \geqslant l$

$$\bar{L}^j (F_q(z)(1 + i\varphi_1(z, \bar{z}, s))^q s^{q-l})|_0 = \frac{\partial^j}{\partial z^j} (F_q(z)(1 + i\varphi_1(z, \bar{z}, s))^q s^{q-l})|_0.$$

Therefore, since $\varphi_1(z, 0, s) \equiv 0$ we obtain from (2.15) and (2.14)

$$(\bar{L}^j f_0)(0) = \partial_z^j F_l(0), \qquad j = 0, 1, ...,$$

and hence the desired conclusion (2.2).

Case 2. $1 \leqslant p < l$.

Here we have

$$f \sim F_p w^p + \cdots + F_{l-1} w^{l-1} + F_l(z) w^l + \cdots, \tag{2.16}$$

where $F_j \in \mathbf{C}$, $p \leqslant j \leqslant l-1$, and

$$w = s + i\varphi(z, \bar{z}, s) = s + is^{j_0+1}\varphi'(z, \bar{z}, s).$$

By Main Lemma (2.3) we have $j_0 \geqslant l - p$; thus we have

$$w = s(1 + is^{l-p}\tilde{\varphi}(z, \bar{z}, s)) \tag{2.17}$$

and

$$L = \frac{\partial}{\partial \bar{z}} + b(z, \bar{z}, s) s^{l-p+1} \frac{\partial}{\partial s}. \tag{2.18}$$

Consider the CR function f' defined on M by

$$f' = f - \sum_{j=p}^{l-1} F_j w^j,$$

where w is given by (2.17). We have

$$f' \sim F_l(z) w^l + \cdots + F_j(z) w^j + \cdots.$$

By the same arguments used in case 1 ($l = p$) we have for all $j = 0, 1, ...$

$$\bar{L}^j f' = s^l f'_j,$$

with f'_j smoothy and

$$f'_j(0) = \partial_z^j F_l(0).$$

In order to prove the desired conclusion it remains only to show that for $p \leqslant r$, and $j \geqslant 1$

$$\bar{L}^j w^r = s^l \chi_{rj} \tag{2.19}$$

with χ_{rj} smooth and $\chi_{rj}(0)=0$. Since $\bar{b}(z,0,0)\equiv 0$ in (2.18) and $\tilde{\varphi}(z,0,0)\equiv 0$ in (2.17) we have for $r\geqslant p$

$$\bar{L}w^r = s^l\chi_{r1} \tag{2.20}$$

with $\chi_{r1}(z,0,0)\equiv 0$. Now (2.19) follows from applying $\bar{L}$ to (2.20) $j-1$ times.

This proves (2.1) and (2.2) of Theorem 3, modulo Main Lemma (2.3).

Proof of Theorem 2. By [4, Theorem 4], the result holds when M is of finite type, so we may restrict ourselves to the case M of infinite type. We need the following lemma.

(2.21) LEMMA. *Let M be a nonflat hypersurface of infinite type and $S\subset M$ the nontrivial complex hypersurface passing through* 0. *Let (z,w) be any normal coordinates near* 0. *Then $S=\{(z,0)\colon z\in\mathbf{C}\}$. Let L, L' be antiholomorphic vector fields tangent to M and M', respectively. We define $\Lambda(z,\bar{z},s)$ on M by*

$$H'(L_{z,s})=\Lambda(z,\bar{z},s)L'_{H(z,\bar{z},s)} \tag{2.22}$$

and l' maximal so that $\Lambda=s^{l'}\Lambda_1(z,\bar{z},s)$ with Λ_1 smooth. Then l' is independent of the choice of (z,w), L and L', and $l'=l$, where l is defined by (1.20). *In addition, if k_1' is minimal satisfying*

$$L^{k_1'}\Lambda = s^{l'}\Lambda' \tag{2.23}$$

with $\Lambda'(0)\neq 0$, then $k_1=k_1'+1$, with k_1 defined by (1.21).

Proof of Lemma (2.21). The integer l' is the order of vanishing of Λ on S. We observe that l' is independent of the choice of L, L' and the coordinates (z,w). Indeed the complex line S is unique and Λ is defined up to multiplication by a nonvanishing smooth function. Similarly, since L is tengent to S, it is clear that the integer k_1' defined by (2.23) is independent of the choice of L, L' and the coordinates (z,w). Lemma (2.21) is then a corollary of Theorem 3 by observing that if L and L' are of the form (1.11) then $\Lambda=L\bar{f}$.

Proof of Theorem 2. The invariance of l and k_1 is an immediate consequence of Lemma (2.21). In order to prove the invariance of k_0 we show that k_0 is the order of vanishing of any transversal component g of H on the complex hypersurface $S\subset M$. Indeed this order of vanishing is invariant (since g is defined up to multiplication by a nonvanishing function). On the other hand, by definition of k_0 the power series of g at 0 is divisible exactly by s^{k_0}. Using the fact that g is CR and an argument similar to the one in the proof of Theorem 3 we conclude that g is exactly divisible by s^{k_0}, which completes the proof of Theorem 2.

(2.24) *Remark.* We have proved that if $H=(f, g)$ then

$$g=s^{k_0}g_1, \tag{2.25}$$

with $g_1(z, \bar{z}, 0) \not\equiv 0$, and k_0 is minimal satisfying (2.25).

(2.26) *Remark.* Unlike the integer l, the integer p defined in (1.20) is not necessarily invariant under a change of holomorphic coordinates as can be shown by the following example. Let $\mathscr{H}=(\mathscr{F}, \mathscr{G})$ be the holomorphic map in $\mathbf{C}^2$ defined by $z'=\mathscr{F}(z, w)=zw^l$, $w'=\mathscr{G}(z, w)=w$. If M is given by

$$w-\bar{w}=2i\,|z|^2\,|w|^{2l},$$

and M' by

$$w'-\bar{w}'=2i|z'|^2,$$

then clearly $\mathscr{H}$ maps M into M' and the coordinates (z, w) and (z', w') are normal for M and M', respectively. For this choice of coordinates we have $p=l$. Consider now the holomorphic change of coordinates on the target space given by

$$z^*=\frac{z'+w'}{1-2iz'-iw'}, \qquad w^*=\frac{w'}{1-2iz'-iw'}.$$

One can easily check that the local biholomorphism $(z', w')\mapsto(z^*, w^*)$ maps M' onto M' (see [6]). For the coordinates (z, w) and (z^*, w^*) the mapping $\mathscr{H}$ is given by $(\mathscr{F}^*, \mathscr{G}^*)$ with

$$\mathscr{F}^*(z, w)=\frac{zw^l+w}{1-2izw^l-iw}, \qquad \mathscr{G}^*(z, w)=\frac{w}{1-2izw^l-iw}.$$

For these coordinates we have $p=1$, indeed $\mathscr{F}^*(z, w)=w+O(|w|^2)$. Note also that we have here $j_0(M)=2l-1$, so that the inequality (2.4) is strict.

(2.27) *Remark.* If $H\colon M\to M'$ is a formal holomorphic mapping with $G\not\equiv 0$ and M of finite type, then it is shown in [4] that $k_0=1$ and $(\partial G/\partial w)(0)\neq 0$. When M is of infinite type then k_0 could be >1, and by Main Lemma (2.3), if M' is also of infinite type then $G_{k_0}(z)$ is a nonzero constant. However, if M is of infinite type and M' of finite type we may have $k_0>1$ and $G_{k_0}(0)=0$ as is shown by the following example.

(2.28) EXAMPLE. Let $\mathscr{F}(z, w)=(1+z)w$, $\mathscr{G}(z, w)=-z(1+z)w^3$, M' given by $w'-\bar{w}'=z'\bar{z}'^2-\bar{z}'z'^2$, and M given by $\operatorname{Im} w=(\operatorname{Re} w)\,\psi(z, \bar{z})$, with

$\psi(z, 0) \equiv \psi(0, \bar{z}) \equiv 0$ and ψ chosen so that $\mathscr{H} = (\mathscr{F}, \mathscr{G})$ maps M into M'. In fact writing the relation $\mathscr{G} - \bar{\mathscr{G}} = \mathscr{F}\bar{\mathscr{F}}^2 - \bar{\mathscr{F}}\mathscr{F}^2$ which must be valid on M, and using the implicit function theorem it is easy to see that ψ is uniquely determined. Here we have $k_0 = 3$ and $G_{k_0}(0) = 0$. Note that we have $j_0(M) = 0$. Hence equality in (2.4) can hold, since here $l = p = 1$.

(2.29) EXAMPLE. The following shows that equality in (2.6) can hold. Let

$$F(z, w) = zw^l, \qquad G(z, w) = w$$

with M' given by $\mathrm{Im}(w'(1 + i\,|z'|^2)) = 0$ and M given by $\mathrm{Im}(w(1 + i\,|zw|^{2l})) = 0$. Here both M and M' are of infinite type and an easy computation shows that $j_0(M) = 2l$.

(2.30) *Remark.* If $k_0 = \infty$, i.e., $G \equiv 0$, then the integer l as defined inh (1.20) is not a biholomorphic invariant. Indeed, let M, M' be real analytic hypersurfaces with M' of infinite type. Then the map

$$\mathscr{F}(z, w) = w + zw^2$$
$$\mathscr{G}(z, w) = 0$$

maps M into M' with $l = 2$. Taking M to be the ball given by $\mathrm{Im}\, w = |z|^2$, an authomorphism of M as in Remark (2.26) will change l to 1.

3. Proof of Main Lemma

We shall need the following for the proof of Main Lemma (2.3).

(3.1) LEMMA. *Let $H(z, \zeta, s, \mu)$ be a formal power series in four indeterminates. Let $t_1, ..., t_j, ...$ be an infinite sequence of indeterminates. Denote*

$$\mu(s) = t_1 s + t_2 s^2 + \cdots + t_j s^j + \cdots \tag{3.2}$$

and define $C^q_\alpha(z, \zeta)$ by

$$\frac{d^q}{ds^q} H(z, \zeta, s, \mu(s))|_{s=0} = \sum_{|\alpha| \leq q} C^q_\alpha(z, \zeta)\, t^\alpha \tag{3.3}$$

with $t^\alpha = t_1^{\alpha_1} \cdots t_q^{\alpha_q}$, $|\alpha| = \sum_{j=1}^q \alpha_j$. Let $R_j(z, \zeta)$, $j = 1, 2, 3, ...,$ be power series satisfying

$$R_j(z, 0) \equiv R_j(0, \zeta) \equiv 0. \tag{3.4}$$

Assume the following three conditions.

$$H(z, \zeta, s, \mu(s))|_{t_j = R_j(z,\zeta)} \equiv 0 \tag{C1}$$

$$R_j(z, \zeta) \equiv 0, \qquad j = 1, 2, ..., n-1, \qquad R_n(z, \zeta) \not\equiv 0. \tag{C2}$$

$$H_{s^{m_1}\mu^{m_2}}(z, \zeta, 0, 0) = a_{m_1 m_2}(z) + b_{m_1 m_2}(\zeta) \tag{C3}$$

for all m_1, m_2 *with* $m_1 + nm_2 \leqslant q_0$.

Then $H_{s^{m_1}\mu^{m_2}}(z, \zeta, 0, 0) \equiv 0$ *for all* m_1, m_2 *with* $m_1 + nm_2 \leqslant q_0$, *and* $C^q_\alpha(z, \zeta) \equiv 0$, *for all* $q \leqslant q_0$ *and* α *for which* $\alpha_1 = \cdots = \alpha_{n-1} = 0$, $|\alpha| \leqslant q$.

Proof. We introduce the following notation. If $\alpha = (\alpha_1, ..., \alpha_N)$, $\beta = (\beta_1, ..., \beta_N)$ $1 \leqslant \beta_1 < \beta_2 < \cdots < \beta_N$, $\alpha_j \geqslant 1$, then we write $(\mu^{(\beta)})^\alpha = (\mu^{(\beta_1)})^{\alpha_1} \cdots (\mu^{(\beta_N)})^{\alpha_N}$ with $\mu^{(\beta_j)} = (d^{\beta_j}/ds^{\beta_j})\, \mu(s)$, and $|\alpha| = \sum_{j=1}^N \alpha_j$.

We compute

$$\frac{d^q}{ds^q} H(z, \zeta, s, \mu(s)) = \sum c_{m_1 m_2 \alpha\beta} H_{s^{m_1}\mu^{m_2}} (\mu^{(\beta)})^{(\alpha)} \tag{3.5}$$

with

$$m_1 + m_2 + \sum_{j=1}^{N} (\beta_j - 1)\alpha_j = q, \qquad |\alpha| = m_2,$$

and the coefficients being nonnegative integers. For $n \geqslant 1$ fixed we split the right hand side of (3.5) into three sums $\sum_1 + \sum_2 + \sum_3$. In $\sum_1$ we put all the terms with at least one $\beta_j < n$. In $\sum_2$ we put all the terms with $N = 1$, $\beta_1 = n$. In $\sum_3$ we put the remaining terms, i.e., those for which $\beta_j \geqslant n$ for all j and at least for one j, $\beta_j > n$. Note that if $n = 1$ then $\sum_1$ is empty and if $q < n$ then $\sum_2$ and $\sum_3$ are empty. We can rewrite $\sum_2$,

$$\sum\nolimits_2 = \sum_{m_1 + m_2 n = q} c_{m_1 m_2} H_{s^{m_1}\mu^{m_2}} (\mu^{(n)}(s))^{m_2} \tag{3.6}$$

with $c_{m_1 m_2}$ positive integers.

Also note that in $\sum_3$ we necessarily have $m_1 + nm_2 < q$.

Substituting $R_j(z, \zeta)$ for t_j in (3.5) and using the hypothesis (C_1) we conclude that for each $q = 0, 1, ...$ both sides of (3.5) vanish identically.

Assume by induction that

$$H_{s^{m_1}\mu^{m_2}}(z, \zeta, 0, 0) \equiv 0 \qquad \text{for} \quad m_1 + nm_2 < q; \tag{3.7}$$

we shall prove the same for $m_1 + nm_2 = q$. (Note that (C_1) implies the desired conclusion for $q = 0$.) For this we make use of (3.5). If $s = 0$, the terms in $\sum_1$ vanish by assumption (C_2). Since for all terms in $\sum_3$ we have $m_1 + nm_2 < q$, the induction hypothesis implies that these vanish also.

By assumption (C_3) the sum $\sum_2$ evaluated at $s=0$ and $t_j=R_j(z,\zeta)$ is of the form

$$\sum_{m_1+nm_2=q} c_{m_1m_2}(a_{m_1m_2}(z)+b_{m_1m_2}(\zeta))(R_n(z,\zeta))^{m_2}$$

and vanishes identically. Then (3.7) with $m_1+nm_2=q$ will follow from Lemma (3.8) below.

(3.8) LEMMA. *If $R(z,\zeta)$ is a power series for which*

$$R(z,\zeta)\not\equiv 0, \qquad R(z,0)\equiv R(0,\zeta)\equiv 0,$$

and

$$\sum_{j=0}^{N}[a_j(z)+b_j(\zeta)][R(z,\zeta)]^j\equiv 0 \tag{3.9}$$

with some power series $a_j(z)$, $b_j(\zeta)$, then

$$a_j(z)+b_j(\zeta)\equiv 0, \qquad 0\leqslant j\leqslant N.$$

Proof. Take $\zeta=0$; then

$$a_0(z)+b_0(0)\equiv 0 \tag{3.10}$$

so that

$$a_0(z)=a_0(0).$$

Similarly taking $z=0$, we conclude

$$b_0(\zeta)=b_0(0).$$

Therefore, from (3.10) obtain

$$a_0(z)+b_0(\zeta)\equiv 0.$$

Since $R(z,\zeta)\not\equiv 0$, by factoring $R(z,\zeta)$ in (3.9), we continue inductively to prove Lemma (3.8).

We may now complete the proof of Lemma (3.1). We note that if $\alpha=(\alpha_1,...,\alpha_q)$ with $\alpha_1=\cdots=\alpha_{n-1}=0$, then

$$C_\alpha^q(z,\zeta)=\sum_{m_1+nm_2\leqslant q} c_{m_1m_2}^{\alpha q}H_{s^{m_1}\mu^{m_2}}(z,\zeta,0,0)$$

and $c_{m_1m_2}^{\alpha q}\geqslant 0$. This implies the last conclusion of Lemma (3.1).

Proof of Main Lemma (2.3). We deal first with the more difficult case when M' is of finite type. Since (2.4) is vacuous if $l \leqslant 1$, we may assume $l \geqslant 2$. (In fact, if $l = 0$ then M is also of finite type by [1].)

Now assume $l \geqslant 2$. We shall first prove (2.4) for $p = 1$ and $F_j = 0$ for $2 \leqslant j \leqslant l-1$, i.e.,

$$F(z, w) = F_1 w + F_l(z) w^l + \cdots + F_j(z) w^j + \cdots \tag{3.11}$$

with $F_1 \in \mathbf{C} \setminus 0$, $F_l(z) \not\equiv F_l(0)$.

Taking normal coordinates (z', w') for M' we have

$$w' - \bar{w}' = 2i\psi\left(z', \bar{z}', \frac{w' + \bar{w}'}{2}\right), \qquad \psi(z', 0, \zeta') \equiv \psi(0, \bar{z}', \zeta') \equiv 0. \tag{3.12}$$

By the implicit function theorem as in (1.3) we obtain

$$w' - \bar{w}' = P(z', \bar{z}', \bar{w}') = \sum_{j=0}^{\infty} P_j(z', \bar{z}')\, \bar{w}'^j. \tag{3.13}$$

It follows from (3.12) that

$$P_j(z', 0) \equiv P_j(0, \bar{z}') \equiv 0;$$

also if M' is of type m then

$$P_{0,m}(z', \bar{z}', 0) = \psi_m(z', \bar{z}', 0), \tag{3.14}$$

where ψ_m, $P_{0,m}$ are the homogeneous parts of degree m of ψ and P_0, respectively. Substituting $F(z, w)$ for z' and $G(z, w)$ for w' in (3.13) we have on M,

$$G(z, w) - \bar{G}(\bar{z}, \bar{w}) = \sum_{j=0}^{\infty} P_j(F(z, w), \bar{F}(\bar{z}, \bar{w}))\, \bar{G}(\bar{z}, \bar{w})^j. \tag{3.15}$$

Since on M we have (by (1.6))

$$w = \bar{w}\left(1 + \sum_{j=0}^{\infty} R_j(z, \bar{z}) \bar{w}^j\right),$$

substituting in (3.15) and writing s for $\bar{w}$ we obtain the identity

$$G(z, s\lambda) - \bar{G}(\bar{z}, s) = \sum_{j=0}^{\infty} P_j(F(z, s\lambda), \bar{F}(\bar{z}, s))\, \bar{G}(\bar{z}, s)^j \tag{3.16}$$

with

$$\lambda = 1 + \sum_{j=0}^{\infty} R_j(z, \bar{z}) s^j. \tag{3.17}$$

Putting $\bar{z}=0$ in (3.16) and taking complex conjugates, (s being counted as real), we obtain

$$\bar{G}(\bar{z}, s) = G(0, s) + \sum_{j=0}^{\infty} \bar{P}_j(\bar{F}(\bar{z}, s), F(0, s))\, G(0, s)^j. \tag{3.18}$$

Substituting for $\bar{G}(\bar{z}, s)$ in the right hand side of (3.16) yields

$$G(z, s\lambda) - \bar{G}(\bar{z}, s)$$
$$= \sum_{j=0}^{\infty} P_j(F(z, s\lambda), \bar{F}(\bar{z}, s)) \left[G(0, s) + \sum_{q=0}^{\infty} \bar{P}_q(\bar{F}(\bar{z}, s), F(0, s))\, G(0, s)^q \right]^j. \tag{3.19}$$

Since $l \geqslant 2$ we show first that $R_0(z, \zeta) \equiv 0$. We consider separately the cases $k_0 < m$, $k_0 = m$, $k_0 > m$.

(i) $k_0 < m$. Taking the coefficient of the lowest power of s in (3.19), which is k_0, gives

$$G_{k_0}(z)(1 + R_0(z, \bar{z}))^{k_0} - \bar{G}_{k_0}(\bar{z}) \equiv 0, \tag{3.20}$$

and since $G_{k_0}(z) \not\equiv 0$, (3.20) implies $R_0(z, \bar{z}) \equiv 0$.

(ii) $k_0 = m$. Taking the lowest power of s in (3.19) which is again k_0 gives the identity

$$G_{k_0}(z)(1 + R_0(z, \bar{z}))^{k_0} - \bar{G}_{k_0}(\bar{z}) \equiv P_{0,m}(F_1(1 + R_0(z, \bar{z})), \bar{F}_1). \tag{3.21}$$

Put $\bar{z}=0$ in (3.21) which shows $G_{k_0}(z) \equiv G_{k_0}(0) \neq 0$. The left hand side of (3.21) is a polynomial of degree exactly k_0 in $R_0(z, \bar{z})$ while the right hand side is of degree $\leqslant k_0 - 1$, since $P_{0,m}(z, 0) \equiv 0$ and $P_{0,m}$ is homogeneous of degree $m = k_0$. Hence $R_0 \equiv 0$.

(iii) $k_0 > m$. Reasoning as before we obtain

$$P_{0,m}(F_1(1 + R_0(z, \bar{z})), \bar{F}_1) \equiv 0.$$

If R_0 is not identically zero, this implies $P_{0,m}(\mu, \bar{F}_1) \equiv 0$ for all $\mu \in \mathbf{C}$ which contradicts the nonvanishing of $P_{0,m}$.

From now on we can assume $l \geqslant 3$ and $R_0 \equiv 0$; thus

$$\lambda = \lambda(s) = 1 + R_1(z, \bar{z})s + R_2(z, \bar{z})s^2 + \cdots. \tag{3.22}$$

Define

$$\mathscr{F} = \mathscr{F}(s) = \mathscr{F}(z, \bar{z}, s) = s^{-1}F(z, s\lambda),$$

i.e.,

$$\mathscr{F} = F_1\lambda + F_l(z)\lambda^l s^{l-1} + \cdots + F_j(z)\lambda^j s^{j-1} + \cdots. \tag{3.23}$$

Similarly $\mathscr{F}^* = s^{-1}\bar{F}(\bar{z}, s)$, i.e.,

$$\mathscr{F}^* = \bar{F}_1 + \bar{F}_l(\bar{z})s^{l-1} + \cdots + \bar{F}_j(\bar{z})s^{j-1} + \cdots. \tag{3.24}$$

Note that since $R_0 \equiv 0$ we have

$$\mathscr{F}(z, \bar{z}, 0) \equiv F_1, \qquad \mathscr{F}^*(\bar{z}, 0) \equiv \bar{F}_1. \tag{3.25}$$

We can write

$$\begin{aligned} P_0(F(z, s\lambda), \bar{F}(\bar{z}, s)) &= P_0(s\mathscr{F}, s\mathscr{F}^*) \\ &= s^m P_{0,m}(\mathscr{F}, \mathscr{F}^*) + \sum_{j=m+1}^{\infty} P_{0,j}(\mathscr{F}, \mathscr{F}^*)s^j, \end{aligned} \tag{3.26}$$

where $P_{0,j}$ is a homogeneous polynomial of degree j. Similarly, for $q = 0, 1, \ldots$, we have

$$\bar{P}_q(\bar{F}(\bar{z}, s), F(0, s)) = \sum_{j=2}^{\infty} K_j^q(\mathscr{F}^*)s^j, \tag{3.27}$$

where K_j^q is a polynomial.

Using (3.19), (3.26), (3.27) and the minimality of k_0 we obtain with $q_0 = \min(k_0 + 1, m)$

$$G(z, s\lambda) - \bar{G}(\bar{z}, s) = s^m P_{0,m}(\mathscr{F}, \mathscr{F}^*) + s^{q_0} \sum_{j=1}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*)s^j, \tag{3.28}$$

where each E_j is a polynomial (not necessarily homogeneous). Substituting (1.19) in the left hand side of (3.28) yields

$$\begin{aligned} &s^{k_0} \sum_{j=k_0}^{\infty} (G_j(z)\lambda^j s^{j-k_0} - \bar{G}_j(\bar{z})s^{j-k_0}) \\ &\qquad = s^m P_{0,m}(\mathscr{F}, \mathscr{F}^*) + s^{q_0} \sum_{j=1}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*)s^j. \end{aligned} \tag{3.29}$$

We shall consider the cases $k_0 < m$ and $k_0 \geqslant m$ separately.

Case 1. $k_0 < m$.

We claim that we have here

$$R_j(z, \bar{z}) \equiv 0, \qquad 0 \leqslant j < 2l + 1. \tag{3.30}$$

To prove this we divide both sides of (3.29) by s^{k_0} to obtain

$$\sum_{j=k_0}^{\infty} G_j(z)\lambda^j s^{j-k_0} - \bar{G}_j(\bar{z})s^{j-k_0}$$
$$= s^{m-k_0} P_{0,m}(\mathscr{F}, \mathscr{F}^*) + s \sum_{j=1}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*)s^j. \tag{3.31}$$

Differentiating (3.31) with respect to s and putting $s=0$ we obtain

$$k_0 G_{k_0}(z)\, \lambda'(0) + G_{k_0+1}(z) - \bar{G}_{k_0+1}(\bar{z}) = C \tag{3.32}$$

with $C=0$ if $m-k_0>1$, and $C=P_{0,m}(F_1, \bar{F}_1)$ if $m=k_0+1$. Since $R_1(z,0) \equiv R_1(0,\bar{z}) \equiv 0$ and $\lambda_1'(0) = R_1(z,\bar{z})$ it follows from (3.32) that

$$R_1(z,\bar{z}) \equiv 0.$$

Assume by induction that $\lambda^{(j)}(0) \equiv 0$ for $j \leqslant j_0$. Differentiating (3.31) j_0+1 times and putting $s=0$ yields

$$k_0 G_{k_0}(z)\, \lambda^{(j_0+1)}(0) = A(z) + B(\bar{z}) \tag{3.33}$$

provided $j_0+1<2l+1$. Indeed if $m-k_0=1$ the first nonharmonic term would be $[(l-1)!]^2 P_{0,m,1,1}(F_1, \bar{F}_1)\, F_l(zs)\, \bar{F}_l(\bar{z})$ which is obtained by taking $2l+1$ derivatives (and putting $s=0$). If $m-k_0>1$ such a term would occur only after more differentiation. Our claim follows by induction from (3.33).

Case 2. $k_0 \geqslant m$.

Dividing both sides of (3.29) by s^m we obtain

$$s^{k_0-m} \sum_{j=k_0}^{\infty} (G_j(z)\, \lambda^j s^{j-k_0} - \bar{G}(\bar{z})s^{j-k_0})$$
$$= P_{0,m}(\mathscr{F}, \mathscr{F}^*) + \sum_{j=1}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*)s^j = \sum_{j=0}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*)s^j \tag{3.34}$$

with

$$E_0(\mathscr{F}, \mathscr{F}^*) = P_{0,m}(\mathscr{F}, \mathscr{F}^*). \tag{3.35}$$

Assuming $l \geqslant 3$ we shall prove

$$\lambda'(0) \equiv R_1(z,\bar{z}) \equiv 0. \tag{3.36}$$

Indeed, if not, we shall show that

$$E_{0\alpha\beta}(F_1, \bar{F}_1) = 0 \tag{3.37}$$

for all $\alpha, \beta > 0$ with $\alpha + \beta \leqslant m$, contradicting $P_{0,m}(z, \bar{z}) \not\equiv 0$. We have used the notation $E_{0\alpha\beta} = (\partial/\partial z^\alpha)(\partial/\partial \bar{z}^\beta) E_0$.

We reason by contradiction and assume $R_1(z, \bar{z}) \not\equiv 0$. In order to prove (3.37) we prove by induction on q

$$E_{j\alpha\beta}(F_1, \bar{F}_1) = 0 \qquad \text{for all } \alpha, \beta > 0 \text{ and } \alpha + (l-1)\beta + j \leqslant q \tag{3.38}$$

and all $j \geqslant 0$.

Condition (3.38) is empty if $q < 3$. We assume (3.38) for some q and prove if for q replaced by $q + 1$.

We shall use Lemma (3.1) with

$$H(z, \zeta, s, \mu) = s^{k_0 - m} \sum_{j=k_0}^{\infty} (G_j(z) \lambda^j s^{j-k_0} - \bar{G}_j(\zeta) s^{j-k_0}) - \sum_{j=0}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*) s^j, \tag{3.39}$$

where $\lambda = 1 + \mu$, and $\mathscr{F}, \mathscr{F}^*$ are given by (3.23) and (3.24). We show first that hypotheses (C_1), (C_2), and (C_3) are valid with $n = 1$, which corresponds to our assumption $R_1(z, \zeta) \not\equiv 0$ (reasoning by contradiction). Indeed, (C_1) follows from (3.34), while (C_2) is satisfied by assumption. We shall prove (C_3) by using the induction hypothesis (3.38), taking q_0 in Lemma (3.1) to be $q + 1$. Thus we must show that

$$H_{s^{m_1}\mu^{m_2}}(z, \zeta, 0, 0) = a(z) + b(\zeta) \tag{3.40}$$

if $m_1 + m_2 \leqslant q + 1$.

In order to compute $H_{s^{m_1}\mu^{m_2}}(z, \zeta, s, \mu)$ we introduce the following notation. If $f(s, \lambda)$ is a power series in two variables s, λ, $\alpha = (\alpha^1, ..., \alpha^N)$, $u = (u^1, ..., u^N)$, and $v = (v^1, ..., v^N)$ are multi-indices we write

$$(f^\alpha)_{uv} = (f^{\alpha_1})_{u_1 v_1} \cdots (f^{\alpha_N})_{u_N v_N}$$

with the notation

$$(f^{\alpha_1})_{u_1 v_1} = \frac{\partial^{u_1}}{\partial s^{u_1}} \frac{\partial^{v_1}}{\partial \lambda^{v_1}} (f(s, \lambda))^{\alpha_1}.$$

Since it is clear that $(\partial^{m_1}/\partial s^{m_1})(\partial^{m_2}/\partial \lambda^{m_2})$ applied to the first sum in the right hand side of (3.39) has the desired form (as in (3.40)), it suffices to show that for each $j = 0, 1, ...,$

$$\frac{\partial^{m_1}}{\partial s^{m_1}} \frac{\partial^{m_2}}{\partial \lambda^{m_2}} [s^j E_j(\mathscr{F}, \mathscr{F}^*)]_{s=0, \lambda=1} = a_j(z) + b_j(\zeta).$$

By the chain rule and using the notation above we have

$$\frac{\partial^{m_1}}{\partial s^{m_1}} \frac{\partial^{m_2}}{\partial \lambda^{m_2}} [s^j E_j(\mathscr{F}, \mathscr{F}^*)]$$

is a linear combination of terms of the form, $0 \leqslant r \leqslant j$,

$$s^{j-r} E_{j, |\alpha_1| + |\alpha_2|, |\beta|} (\mathscr{F}_s^{\alpha_1})_{u_1 v_1} (\mathscr{F}_\lambda^{\alpha_2})_{u_2 v_2} (\mathscr{F}_s^{*\beta})_{u_3} \tag{3.41}$$

with

$$\begin{aligned} m_1 &= r + |\alpha_1| + |u_1| + |u_2| + |u_3| + |\beta| \\ m_2 &= |\alpha_2| + |v_1| + |v_2|. \end{aligned} \tag{3.42}$$

Note that

$$\begin{aligned} \mathscr{F}_s &= (l-1) F_l(z) \lambda^l s^{l-2} + l F_{l+1}(z) \lambda^{l+1} s^{l-1} + \cdots \\ \mathscr{F}_\lambda &= F_1 + l F_l(z) \lambda^{l-1} s^{l-1} + \cdots \\ \mathscr{F}_s^* &= (l-1) \bar{F}_l(\zeta) s^{l-2} + l \bar{F}_{l+1}(\zeta) s^{l-1} + \cdots. \end{aligned}$$

Put $s = 0$ and $\lambda = 1$ in (3.41) and assume that we obtain a nonzero term. This implies necessarily $r = j$ and

$$|u_1| \geqslant (l-2)\,|\alpha_1|, \qquad |u_3| \geqslant (l-2)\,|\beta|. \tag{3.43}$$

We shall show that

$$j + |\alpha_1| + |\alpha_2| + (l-1)\,|\beta| \leqslant q \tag{3.44}$$

which will allow us to use the inductive hypothesis (3.38). Indeed we have

$$\begin{aligned} Q &= j + |\alpha_1| + |\alpha_2| + (l-1)\,|\beta| \\ &= j + (l-2)\,|\alpha_1| - (l-3)\,|\alpha_1| + |\alpha_2| + (l-2)\,|\beta| + |\beta|. \end{aligned}$$

Since $l \geqslant 3$ and using (3.43) we have $Q \leqslant j + |u_1| + |\alpha_2| + |u_3| + |\beta|$, so that

$$Q \leqslant j + |u_1| + |u_2| + |u_3| + |v_1| + |v_2| + |\beta| + |\alpha_2|. \tag{3.45}$$

We consider separately the case $|\alpha_1| \geqslant 1$ and $|\alpha_1| = 0$.

(a) $|\alpha_1| \geqslant 1$.

It follows from (3.45) that

$$Q \leqslant j + |u_1| + |u_2| + |u_3| + |v_1| + |v_2| + |\beta| + |\alpha_1| + |\alpha_2| - 1 = m_1 + m_2 - 1;$$

the last equality follows from (3.42). We therefore conclude, since $m_1 + m_2 \leqslant q+1$, that (3.44) holds in this case.

(b) $|\alpha_1| = 0$.

To obtain a term of the form $a(z)\, b(\zeta)$ necessarily $|\alpha_2| > 0$, $|\beta| > 0$ and

$$|u_2| \geqslant l-1, \qquad |u_3| \geqslant (l-2)\,|\beta|.$$

Therefore, since $l \geqslant 3$,

$$\begin{aligned} j + |\alpha_2| + (l-1)\,|\beta| &= j + |\alpha_2| + (l-2)\,|\beta| + |\beta| + (l-1) - (l-1) \\ &\leqslant j + |\alpha_2| + |v_1| + |v_2| + |u_1| + |u_2| + |u_3| + |\beta| - 2 \\ &\leqslant q-1, \end{aligned}$$

which shows again that (3.44) holds.

This completes the proof of (3.40) and hence condition (C_3) of Lemma (3.1) with $q_0 = q+1$.

In order to prove (3.38) for q replaced by $q+1$, we use Lemma (3.1) to conclude that for each q_1, $0 \leqslant q_1 \leqslant q+1$, the coefficient of $\lambda'(0)^{q_1}$ in the expansion of $(\partial^{q+1}/\partial s^{q+1})\, H(z, \zeta, s, \mu(s))|_{s=0}$ is zero, i.e., in the notation of Lemma (3.1) $C^{q+1}_{q_1}(z, \zeta) = C^{q+1}_{(q_1, 0, \ldots 0, \ldots)}(z, \zeta) \equiv 0$.

We claim that for $1 \leqslant q_1 \leqslant q+1$,

$$C^{q+1}_{q_1} = \sum d^{q_1}_{jk}\, E_{jq_1k}(F_1, \bar{F}_1)\, \bar{F}_l(\zeta)^k + B_{q_1}(z), \tag{3.46}$$

where the sum is for j, k satisfying

$$q+1 = j + k(l-1) + q_1, \qquad k > 0, \tag{3.47}$$

and for such j, k, $d^{q_1}_{jk} \in \mathbf{C}\backslash 0$.

In order to prove the claim note first that the contribution of the first sum in the right hand side of (3.39) to $C^{q+1}_{q_1}$, $q_1 \geqslant 1$, gives a power series of z alone. Therefore, it suffices to compute

$$\frac{d^{q+1}}{ds^{q+1}} \left(\sum_{j=0}^{\infty} E_j(\mathscr{F}, \mathscr{F}^*)\, s^j \right).$$

We clearly have by the chain rule

$$\begin{aligned} &\frac{d^{q+1}}{ds^{q+1}} \left[E_j(\mathscr{F}, \mathscr{F}^*)\, s^j \right] \\ &\qquad = \sum c_{j\alpha\beta uvr}\, s^{j-r} E_{j|\alpha|\,|\beta|}(\mathscr{F}, \mathscr{F}^*) [(\mathscr{F}_s + \mathscr{F}\lambda')^\alpha]_u (\mathscr{F}^{*\beta}_s)_v, \end{aligned} \tag{3.48}$$

where the sum is over

$$q+1=r+|\alpha|+|u|+|\beta|+|v|, \qquad 0\leqslant r\leqslant j,$$

and $c_{j\alpha\beta uvr}>0$. Putting $s=0$ in (3.48) and using the induction hypotheseis (3.38) we ontain that the only nonzero terms in (3.48) correspond to $|u|=0$ and $v=(l-2)\beta$ and $j=r$. Claim (3.46) follows by observing that $(\mathscr{F}_s+\mathscr{F}_\lambda\lambda')|_{s=0}=\lambda'(0)$, and $[(\mathscr{F}_s^*)^\beta]_{(l-2)\beta}=c\bar{F}_l(\zeta)^{|\beta|}$, with $c>0$.

The rest of the induction to prove (3.38) follows from (3.46). Indeed, since $C_{q_1}^{q+1}\equiv 0$ and $d_{j,k}^{q_1}\neq 0$ in (3.46), it follows that

$$E_{jq_1k}(F_1,\bar{F}_1)=0 \qquad \text{for all } \; j, q_1, k, \text{ with } j+q_1+k(l-1)=q+1$$

and $q_1>0$, $k>0$. This completes the proof of (3.38). Thus (3.36) is proved by contradiction.

We shall now prove more generally

$$\lambda^{(j)}(0)=R_j(z,\zeta)\equiv 0, \qquad 0\leqslant j\leqslant l-2 \tag{3.49}$$

by induction on j. Since (3.49) holds for $j=0$ and $j=1$, we shall assume it for $j=0, 1, 2, ..., n-1$ and prove it for $j=n$ with $n\leqslant l-2$. The proof is very similar to the case $n=1$ and we shall indicate the modifications needed. Indeed by contradiction we assume $R_n(z,\zeta)\not\equiv 0$ and we shall show that (3.37) holds for α , $\beta>0$, $\alpha+\beta\leqslant m$, contradicting $P_{0,m}(z,\bar{z})\not\equiv 0$.

In order to prove (3.37) we prove by induction on q that

$$E_{j\alpha\beta}(F_1,\bar{F}_1)=0 \qquad \text{for all } \alpha, \beta>0 \text{ and } n\alpha+(l-1)\beta+j\leqslant q \tag{3.50}$$

and all $j\geqslant 0$.

As for $n=1$ we use Lemma (3.1) but with $n>1$. Conditions (C_1) and (C_2) of Lemma (3.1) are clearly satisfied.

To check (C_3) we begin as before with (3.41), (3.42) and put $s=0$ and $\lambda=1$ in (3.41). If we obtain a nonzero term then necessarily (3.43) holds and we shall show that

$$j+n(|\alpha_1|+|\alpha_2|)+(l-1)\,|\beta|\leqslant q \tag{3.51}$$

which will allow us to use the induction hypothesis (3.50). We have, since $l\geqslant n+2$,

$$\begin{aligned} Q&=j+n(|\alpha_1|+|\alpha_2|)+(l-1)\,|\beta| \\ &=j+(l-2)\,|\alpha_1|-(l-n-2)\,|\alpha_1|+n\,|\alpha_2|+(l-2)\,|\beta|+|\beta| \\ &\leqslant j+|u_1|+|u_2|+|u_3|+n\,|\alpha_2|+n\,|v_1|+n\,|v_2|+|\beta|. \end{aligned} \tag{3.52}$$

If $|\alpha_1|>0$ then (3.53) implies $Q\leqslant m_1+nm_2-1\leqslant q$ which proves (3.51). If $|\alpha_1|=0$ then the argument is the same as in $n=1$.

We can apply Lemma (3.1) with $n>1$ and $q_0=q+1$ to conclude that for each q_1, $0 \leqslant q_1 \leqslant q+1$ the coefficient of $(\lambda^{(n)}(0))^{q_1}$ in the expansion of $(\partial^{q+1}/\partial s^{q+1})\, H(z, \zeta, s, \mu(s))|_{s=0}$ is zero, i.e., in the notation of Lemma (3.1)

$$C^{q+1}_{(0, \dots 0, q_1, 0, \dots)}(z, \zeta) \equiv 0,$$

where q_1 is at the nth position. For this we follow the argument of the proof in the case $n=1$. In particular putting $s=0$ in (3.48) and using the induction hypothesis (3.50) we find that the only nonzero terms correspond to $u=(n-1)\alpha$, $v=(l-2)\beta$, and $r=j$. Hence, we conclude by arguments similar to those used in the case $n=1$ that (3.50) holds also for $q+1$. This completes the proof of (2.4) in the case M' of finite type, $p=1$, and $F_j=0$, $j=2, \dots, l-1$.

We continue to assume M' is of finite type. We first reduce to the case $p=1$, and later to the case $F_j=0$, $2 \leqslant j \leqslant l-1$.

(1) Reduction to $p=1$.

Estimate (2.4) holds if $p=l$. We shall consider the case $p<l$, and therefore, since $F(0)=0$, we have $1 \leqslant p < l$. We will change the target hypersurface M' and define a new hypersurface M'' and a new map $H=(\tilde{F}, \tilde{G})$ from M to M''. Since

$$F = F_p w^p + \cdots + F_l(z) w^l + \cdots, \qquad F_p \neq 0,$$

we define

$$\tilde{F}(z, w) = w(F_p + \cdots + F_l(z) w^{l-p} + \cdots)^{1/p}, \tag{3.53}$$

where the $1/p$th power is defined by the usual power series, and we take $\tilde{G}(z, w) = G(z, w)$. If M' is defined by $\rho'(z', w', \bar{z}', \bar{w}')=0$, where z', w' are normal coordinates for M', then M'' is defined by

$$\rho'(z'^p, w', \overline{z'^p}, \overline{w'}) = 0.$$

Clearly $(\tilde{F}, \tilde{G})$ maps M into M'' and the corresponding pair of integers (p', l') is $(1, l-p+1)$. Since the R_j's are unchanged it suffices to prove the desired result for the new map.

(2) Reduction to the Case $F_j=0$, $2 \leqslant j \leqslant l-1$.

Assume now $p=1$. We make a holomorphic change of coordinates in the target space. We put

$$z'' = z' + c_2 z'^2 + \cdots + c_l z'^l, \qquad w'' = w',$$

where the complex numbers c_j, $2 \leqslant j \leqslant l-1$, are uniquely determined so that

$$\begin{aligned}\bar{\bar{F}}(z, w) &= F(z, w) + c_2(F(z, w))^2 + \cdots + c_l(F(z, w))^l \\ &= F_1 w + F_l(z) w^l + O(w^{l+1}).\end{aligned}$$

It is clear that the new coordinates (z'', w'') are normal for M' and that the R_j's are unchanged. The proof of (2.4) is now complete.

To prove (2.5) and (2.6) we assume now M' is of infinite type. We rewrite (3.15) (here $P_0 \equiv 0$) in the form

$$G(z, w) = \bar{G}(\bar{z}, \bar{w}) \left(1 + \sum_{j=j_0'}^{\infty} R_j'(F(z, w), \bar{F}(\bar{z}, \bar{w}))\, \bar{G}(\bar{z}, \bar{w})^j\right)$$

with

$$w = \bar{w}\left(1 + \sum_{j=0}^{\infty} R_j(z, \bar{z}) \bar{w}_j\right).$$

Substituting $s\lambda$ for w and s for $\bar{w}$ as in (3.16) we obtain

$$G(z, s\lambda) = \bar{G}(\bar{z}, s)\left(1 + \sum_{j=j_0'}^{\infty} R_j'(F(z, s\lambda), \bar{F}(\bar{z}, s))\, \bar{G}(\bar{z}, s)^j\right), \tag{3.54}$$

where λ is given by (3.17).

Substituting (1.18) and (1.19) in (3.54) we obtain, after dividing by s^{k_0},

$$\begin{aligned}&\sum_{k=k_0}^{\infty} G_k(z) \lambda^k s^{k-k_0} \\ &\quad = \sum_{k=k_0}^{\infty} \bar{G}_k(\bar{z}) s^{k-k_0}\left(1 + \sum_{j=j_0'}^{\infty} R_j'\left(\sum_{k=p}^{\infty} F_k(z) s^k \lambda^k, \sum_{k=p}^{\infty} \bar{F}_k(\bar{z}) s^k\right)\right. \\ &\quad\quad \left.\times\left(\sum_{k=k_0}^{\infty} \bar{G}_k(\bar{z}) s^k\right)^j\right).\end{aligned} \tag{3.55}$$

Putting $s=0$ in (3.55) yields

$$G_{k_0}(z)(1 + R_0(z, \bar{z})) \equiv \bar{G}_{k_0}(\bar{z})(1 + R_0'(F_0(z), \bar{F}_0(\bar{z}))),$$

thus by taking $\bar{z} = 0$ we obtain (2.5), and if $l \geqslant 1$, $R_0(z, \bar{z}) \equiv 0$. Assume now by induction that

$$R_j(z, \bar{z}) \equiv 0, \quad 0 \leqslant j \leqslant j_1, \quad \text{and} \quad G_{k_0+j}(z) \equiv G_{k_0+j}(0), \quad j \leqslant \min\{j_1, l\},$$

so that also $\lambda(0)=1$, and $\lambda^{(j)}(0)=0$, $1 \leqslant j \leqslant j_1$. We shall show that if $j_1 < 2l-1$ then $\lambda^{(j_1+1)}(0)=0$, and $G_{k_0+j_1+1}(z) \equiv G_{k_0+j_1+1}(0)$. We differentiate (3.55) (j_1+1) times with respect to s and put $s=0$; we obtain

$$k_0 G_{k_0} \lambda^{j_1+1}(0) + (j_1+1)!\, G_{k_0+j_1+1}(z) = a(z) + b(\bar{z}),$$

with $a(z)=0$ if $j_1+1 \leqslant l$. Indeed, a term in z on the right hand side of (3.55) could first appear by differentiating R_0' $l+1$ times. For $j<2l$ the derivatives of order j on the right hand side of (3.55) are of the form $a(z)+b(\bar{z})$, by the inductive hypotheses. This completes the proof of (2.6) and hence that of Main Lemma (2.3).

4. Proof of Theorem 1

We first show that if M is flat then M does not satisfy the reflection principle; i.e., there exists a holomorphic map $\mathscr{H}$ defined on one side of M, Ω^+, C^∞ on $\bar{\Omega}^+$, not totally degenerate at 0, mapping M into M', but not extending holomorphically in a neighborhood of 0. For this, assume (by Proposition (1.5)) that M is given by $\operatorname{Im} w=0$, and consider the map given by $\mathscr{F}(z,w)=z+h(w)$, $\mathscr{G}(z,w)=w$, where h is holomorphic for $\operatorname{Im} w>0$, C^∞ for $\operatorname{Im} w \geqslant 0$, $h(0)=0$; but not extendinhg holomorphically in a neighborhood of 0. The mapping $\mathscr{H}=(\mathscr{F},\mathscr{G})$ is holomorphic on the side of M given by $\operatorname{Im} w>0$, smooth up to the boundary, and satisfies $\mathscr{H}(M) \subset M$. In addition, the Jacobian determinant $J(\mathscr{H})$ is 1 at 0.

We shall now assume that M is not flat and show that M has the reflection principle. For this, let $\mathscr{H}=(\mathscr{F},\mathscr{G})$ be defined in Ω^+, given by (1.14), C^∞ in $\bar{\Omega}^+$. As in Section 1, we write f, g for the restrictions of $\mathscr{F}$ and $\mathscr{G}$ to M. We choose normal coordinates (z,w) for M and parametrize M by

$$(x, y, s) \mapsto (z = x+iy,\ w = s + i\varphi(z, \bar{z}, s)), \tag{4.1}$$

where φ is an in (1.1). As in [2, 1] we consider the following map from $\mathbf{R}^4$ to $\mathbf{C}^2$ defined near 0,

$$(x, y, s, t) \to (z = x+iy,\ w = s+it+i\varphi(z, \bar{z}, s+it)). \tag{4.2}$$

Note that the hyperplane $t=0$ is mapped onto the hypersurface M.

Let Λ be the vector field in $\mathbf{R}^4$ defined by

$$\Lambda = \frac{\partial}{\partial \bar{z}} - \frac{i\varphi_{\bar{z}}(z, \bar{z}, s+it)}{1+i\varphi_s(z, \bar{z}, s+it)} \frac{\partial}{\partial s}. \tag{4.3}$$

Note that by (1.11) $\Lambda|_{t=0} = L$. A CR function $h(z, \bar{z}, s)$ on M, i.e., a function satisfying $Lh \equiv 0$, extends holomorphically in a neighborhood of 0 if and only if h extends locally as a function $\tilde{h}(z, \bar{z}, s, t)$ from $\mathbf{R}^3$ to $\mathbf{R}^4$ satisfying

$$\Lambda\tilde{h} = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right)\tilde{h} = 0, \tag{4.4}$$

since the two vector fields in (4.4) are the pullbacks of the complex structure $(\partial/\partial\bar{z})$, $(\partial/\partial\bar{w})$ under the map given in (4.2). Since $\mathscr{F}$ and $\mathscr{G}$ are holomorphic in Ω^+ it follows that f and g extend to the half space $t > 0$ and their extensions satisfy (4.4). It remains to show that they also extend for $t < 0$ with (4.4) satisfied.

We begin with the following lemma in one complex variable.

(4.5) LEMMA. *Let R be the rectangle in $\mathbf{C}$ defined by*

$$R = \{s + it : \ |s| < r \ \text{and} \ 0 < t < r\}$$

and $\bar{R} = R \cup (-r, r)$. Let $a(s + it)$ and $b(s + it)$ be holomorphic in R, C^∞ in $\bar{R}$ and satisfying the following conditions

$$a(s) = b(s)\, c(s) \qquad \text{in } (-r, r), \text{ with } c \in C^\infty(-r, r). \tag{4.6}$$

$$|b(s + it)| \geqslant C|s + it|^l \qquad \text{for some } C > 0, \tag{4.7}$$

and some integer $l \geqslant 0$, $s + it \in R$. Then c extends holomorphically to R and is C^∞ in $\bar{R}$.

(4.8) COROLLARY. *If in Lemma (4.5) the assumption (4.7) is replaced by*

$$b(s) = s^l v(s), \qquad v \in C^\infty(-r, r), \tag{4.9}$$

then v extends holomorphically to R and is C^∞ in $\bar{R}$. Furthermore, if for some $0 < r' \leqslant r$, $|v(s + it)| \geqslant C_1 > 0$ in $\bar{R}'$, where R' is the rectangle defined similarly to R but with r replaced by r', then the conclusion of Lemma (4.5) holds with R replaced by R'.

Proof. For $l = 0$ the conclusion of Lemma (4.5) is clear. We shall prove it by induction on l. Assume the conclusion holds for all $l \leqslant l_0$; we shall prove it for $l_0 + 1$. Then (4.7) is

$$|b(s + it)| \geqslant C\, |s + it|^{l_0 + 1}, \qquad s + it \in R. \tag{4.10}$$

Without loss of generality we may assume $b(0) = 0$. From (4.6) we also have $a(0) = 0$. Therefore, $a(s)/s$ and $b(s)/s$ are both in $C^\infty(-r, r)$, as is

$(a(s)/s)/(b(s)/s) = a(s)/b(s) = c(s)$. We claim that $a(s+it)/(s+it)$ and $b(s+it)/(s+it)$ are both C^∞ in $\bar{R}$. Assuming the claim, we note that the conclusion follows from the inductive hypothesis since (4.10) implies

$$\left|\frac{b(s+it)}{s+it}\right| \geqslant C\,|s+it|^{l_0}.$$

It remains to prove the claim above; we shall do so for $a(s)$, since the proof is the same for $b(s)$. For this we shall show that for all $r' < r$, $r' > 0$, there exists $C_{r'}$ such that

$$\left|\frac{a(s+it)}{s+it} - \frac{a(s)}{s}\right| \leqslant C_{r'}\,t \tag{4.11}$$

for $|s| \leqslant r' < r$, $0 < t < r$. Indeed, (4.11) shows that $a(s+it)/(s+it)$ is locally bounded in $\bar{R}$ and its distribution boundary value on $(-r, r)$ is $a(s)/s$. By a classical argument using the Poisson integral we conclude, since $a(s)/s \in C^\infty$, that $a(s+it)/(s+it) \in C^\infty(\bar{R})$.

To prove (4.11) we write

$$\begin{aligned}\frac{a(s+it)}{s+it} - \frac{a(s)}{s} = {} & \frac{a(s+it) - a(s) - a'(s)\,it}{s+it} \\ & - \frac{a(s)(s+it) - sa(s) - sa'(s)\,it}{s(s+it)}.\end{aligned} \tag{4.12}$$

Since $a_s = -ia_t$, the Taylor expansion of $a(s+it)$ shows that the first term on the right hand side of (4.12) is bounded by $C_{r'}$ for $|s| \leqslant r' < r$. On the other hand, the second term on the right hand side of (4.12) is $(s/(s+ir))(it)(a(s)/s)'$, which gives the same bound. This completes the proof of (4.11) and hence that of Lemma (4.5).

Proof of Corollary (4.8). The fact that $v(s)$ extends holomorphically and is C^∞ in $\bar{R}$ follows from Lemma (4.5) by taking $b(s)$ for $a(s)$, s^l for $b(s)$ and $v(s)$ for $c(s)$. By the condition on $v(s+it)$, (4.7) holds for $b(s+it)$ in R', so that the second conclusion also follows from Lemma (4.5).

We shall now use Theorem 3, (2.5) of Main Lemma (2.3), and Lemma (4.5) to show that the component f of the map H satisfies a certain monic polynomial equation.

(4.13) LEMMA. *If M is nonflat and $G \not\equiv 0$, then there exist $r > 0$ and smooth functions $Y = (Y_1(z, \bar{z}, s), \ldots, Y_N(z, \bar{z}, s))$, defined for $|z| < r$, $|s| < r$, satisfying the following*:

(4.14) *For each* z, $|z|<r$, *the functions* $s \to Y_j(z, \bar{z}, s)$ *extend holomorphically to the rectangle* $R=\{s+it \in \mathbf{C}\colon |s|<r,\ -r<t<0\}$ *and the extension is* C^∞ *in* $\bar{R}=R \cup (-r, r)$.

(4.15) *There exists an integer* p_0 *and holomorphic functions* $a_j(Z_1, ..., Z_N)$, $0 \leqslant j \leqslant p_0-1$, *defined in a neighborhood of* $Y(0)$ *such that*

$$f^{p_0}(z, \bar{z}, s)+\sum_{j=0}^{p_0-1} a_j(Y) f^j(z, \bar{z}, s)=0$$

for $|z|<r$, $|s|<r$.

Proof. Since M is nonflat and $G \not\equiv 0$, it follows from Proposition (1.24) that M' is nonflat also. Let $j_0'=j_0(M')$ as defined in Section 1, so that $-1 \leqslant j_0'<\infty$ and $j_0'=-1$ if and only if M' is of finite type. By Proposition (1.7) the equation for M' us given by

$$\bar{w}'=Q(z', \bar{z}', w')=w+w^{j_0'+1} S(z', \bar{z}', w'), \tag{4.16}$$

with $S(z', 0, w') \equiv S(0, \zeta', w') \equiv 0$, and $S_{z'^{p_0}\zeta'^{q_0}}(0) \neq 0$, with q_0 minimal. We shall also assume that p_0 is minimal in the set of indices for which $S_{z'^{p_0}\zeta'^{q_0}} \neq 0$. Therefore, we have on M

$$\bar{g}=Q(f, \bar{f}, g)=g+g^{j_0'+1} S(f, \bar{f}, g). \tag{4.17}$$

Applying L^{k_1}, with k_1 defined in (1.21), to (4.17) yields

$$L^{k_1} \bar{g}=g^{j_0'+1}\left[S_\zeta(f, \bar{f}, g) L^{k_1} \bar{f}+\sum_{q \geqslant 2, \sum j_1=k_1} c S_{\zeta^q}(f, \bar{f}, g)(L^{j_1} \bar{f}) \cdots L^{j_q} \bar{f})\right], \tag{4.18}$$

where c stands for positive constants varying with $q, j_1, ..., j_q$. We put $h_1=1/L^{k_1} \bar{f}$, so that $h_1^{-1}=s^l \bar{f}_{k_1}$ with $\bar{f}_{k_1}(0) \neq 0$ by (2.2) of Theorem 3. Multiplying (4.18) by h_1 we obtain

$$h_1 L^{k_1} \bar{g}=g^{j_0'+1}\left[S_\zeta+\sum_{q \geqslant 2, \sum j_i=k_1} c S_{\zeta^q}(L^{j_1} \bar{f}) \cdots(L^{j_q} \bar{f}) h_1\right]. \tag{4.19}$$

It follows again by Theorem 3 that the right hand side of (4.19) is a smooth function and hence so is the left hand side. By Corollary (4.8), there exists $r>0$ such that for all $|z|<r$ the functions $s \to (h_1 L^{k_1} \bar{g})(z, \bar{z}, s)$ extend holomorphically to a rectangle of the form $\{s+it\colon |s|<r,\ -r<t<0\}$ and C^∞ in $\bar{R}=R \cup (-r, r)$. The same holds for each of the functions $(L^{j_1} \bar{f}) \cdots (L^{j_q} \bar{f}) h_1$.

To define h_2, we apply L^{k_1} to (4.19) to obtain

$$L^{k_1}(h_1 L^{k_1}\bar{g}) = g^{j_0'+1}\left(\sum_{q \geq 2} S_{\zeta^q} A_q\right), \tag{4.20}$$

and we set $h_2 = A_2^{-1}$. We continue this process, alternating multiplication by h_j and application of L^{k_1} to define h_p recursively. We shall prove inductively the formula

$$\begin{aligned} &L^{k_1}(h_{p-1} L^{k_1}(h_{p-2} L^{k_1} \cdots h_1 L^{k_1}\bar{g}) \cdots) \\ &= g^{j_0'+1}\Bigg[S_{\zeta^p} L^{k_1}\bar{f} + \sum_{q \geq p+1, \sum j_i = k_1} c S_{\zeta^q}(L^{j_1}\bar{f}) \cdots (L^{j_q}\bar{f}) \\ &\quad + \sum_{q \geq p, \sum j_i + r_{p-1} = 2k_1} c S_{\zeta^q}(L^{j_1}\bar{f})(L^{r_{p-1}} h_{p-1}) \\ &\quad + \cdots \\ &\quad + \sum_{q \geq p, \sum j_i + r_1 + \cdots + r_{p-1} = pk_1} c S_{\zeta^q}(L^{j_1}\bar{f}) \cdots (L^{j_q}\bar{f})(L^{r_1} h_1) \cdots (L^{r_{p-1}} h_{p-1}) \Bigg], \end{aligned} \tag{4.21}$$

and show that $h_j = s^{-l} v_j$, $v_j \in C^\infty$, $v_j(0) f_{k_1}(0) > 0$, and $1/h_j$ extends holomorphically in s to a rectangle R_j for all $|z|$ small. We shall assume that (4.21) holds for p and the above properties hold for h_j, $j \leqslant p-1$, and prove (4.21) for $p+1$ and the corresponding properties for h_p. We compute h_p^{-1} by calculating the coefficient of S_{ζ^p} in (4.21). We obtain

$$\begin{aligned} h_p^{-1} = L^{k_1}\bar{f} &+ \sum_{\sum j_i + r_{p-1} = 2k_1} c(L^{j_1}\bar{f}) \cdots (L^{j_p}\bar{f})(L^{r_{p-1}} h_{p-1}) \\ &+ \cdots \\ &+ \sum_{\sum j_i + r_1 + \cdots + r_{p-1} = pk_1} c(L^{j_1}\bar{f}) \cdots (L^{j_p}\bar{f})(L^{r_1} h_1) \cdots (L^{r_{p-1}} h_{p-1}). \end{aligned} \tag{4.22}$$

By Theorem 3 and the fact that for $k < p$, $L^j h_k = s^{-l} u_{jk}$ with $u_{jk} \in C^\infty$, it follows that the right hand side of (4.22) is smooth and is of the form $s^l u$, u smooth. Note that the only contributions to u not vanishing at $s = 0$ come from $L^{k_1}\bar{f}$ and the last sum on the right hand side of (4.22). Hence

$$h_p^{-1} = s^l[f_{k_1} + c f_{k_1}^p v_1 \cdots v_{p-1} + \chi],$$

where $\chi(0) = 0$. Then $v_p = f_{k_1} + c f_{k_1}^p v_1 \cdots v_{p-1} + \chi$ satisfies $v_p(0) f_{k_1}(0) > 0$. It follows from the properties of h_j, $1 \leqslant j \leqslant p-1$, that for each r, $L^r h_j = u_{rj}/s^l$, where u_{rj} extends holomorphically to a rectangle R_j in s for all $|z|$ small. Applying Lemma (4.5) to each term on the right hand side of (4.22) yields the desired extendability result for h_p^{-1}.

We must show that (4.21) holds with p replaced by $p+1$. For this, we multiply (4.21) by h_p, then apply L^{k_1}. We obtain the desired result for $p+1$.

In particular, taking $p = q_0$ in (4.21) and multiplying by h_{q_0} we obtain

$$
\begin{aligned}
&h_{q_0} L^{k_1}(h_{q_0-1} L^{k_1} \cdots h_1 L^{k_1} \bar{g}) \\
&\quad = g^{j_0'+1} \Bigg[S_{\zeta^{q_0}} + \sum_{q \geq q_0+1, \sum j_i = k_1} c S_{\zeta^q}(L^{j_1} \bar{f}) \cdots (L^{j_q} \bar{f}) h_{q_0} \\
&\qquad + \cdots \\
&\qquad + \sum_{q \geq q_0+1, \sum j_i + \sum r_j = q_0 k_1} c S_{\zeta^q}(L^{j_1} \bar{f}) \cdots (L^{j_q} \bar{f})(L^{r_1} h_1) \cdots (L^{r_{q_0-1}} h_{q_0-1}) h_{q_0}.
\end{aligned}
\tag{4.23}
$$

We write from (4.17)

$$g = \bar{g}(1 + \bar{g}^{j_0'} \bar{S}(\bar{f}, f, \bar{g})) \tag{4.24}$$

and substitute (4.24) into the right hand side of (4.23). Then divide both sides by $\bar{g}^{j_0'+1}$ to obtain

$$
\begin{aligned}
&\frac{h_{q_0}}{\bar{g}^{j_0'+1}} L^{k_1}(h_{q_0-1} L^{k_1} \cdots h_1 L^{k_1} \bar{g}) \\
&\quad = (1 + \bar{g}^{j_0'} \bar{S}(\bar{f}, f, \bar{g}))^{j_0'+1} [S_{\zeta^{q_0}}(f, \bar{f}, \bar{g}(1 + \bar{g}^{j_0'} \bar{S}(\bar{f}, f, \bar{g}))) + \cdots].
\end{aligned}
\tag{4.25}
$$

Since the right hand side of (4.25) is smooth, and, if $j_0' \geq 0$, $\bar{g} = s^{k_0}\chi$, $\chi(0) \neq 0$, (by (2.5) and (2.25)), we can apply again Lemma (4.5) to obtain that the left hand side of (4.25) extends holomorphically in s for $t < 0$ in a small rectangle uniformly in $|z|$ small. We can now apply the Weierstrass Preparation Theorem with respect to f, considering the left hand side of (4.25), $\bar{f}$, $\bar{g}$ and all terms of the form $(L^{j_1}\bar{f}) \cdots (L^{j_q}\bar{f})(L^{r_p}h_p) \cdots (L^{r_{q_0-1}}h_{q_0-1})h_{q_0}$ as independent variables Y to obtain (4.15). This completes the proof of Lemma (4.13).

We return now to the proof of Theorem 1. We shall show that the components f and g of H extend holomorphically in $s + it$ for $t < 0$. As in [1, 2] we write for $\lambda \in \mathbf{C}$ small

$$Q(f, \lambda, g) = \sum_{\alpha} \frac{(\lambda - \bar{f})^{\alpha}}{\alpha!} Q_{\zeta^{\alpha}}(f, \bar{f}, g). \tag{4.26}$$

It follows from Lemma (4.13) that if $d_\alpha = Q_{\zeta^\alpha}(f, \bar{f}, g)$, then d_α satisfies a polynomial equation of the form

$$d_\alpha^{N'} + \sum_{j=0}^{N'-1} b_{\alpha j}(Y) d_\alpha^j \equiv 0, \tag{4.27}$$

where $b_{\alpha j}$ are holomorphic and $Y = (Y_1, ..., Y_N)$ are as in Lemma (4.13).

The proof of (4.27) from Lemma (4.13) is similar to the proof of Lemma (8.11) in [1]. We also have estimates (8.13) of [1].

For $r > 0$ sufficiently small, we set

$$B_1 = \{z \in \mathbf{C};\ |z| < r,\ s \mapsto L\bar{f}(z, \bar{z}, s) \text{ not identically zero}\}$$
$$B_2 = \{z \in \mathbf{C};\ |z| < r,\ L\bar{f}(z, \bar{z}, s) \equiv 0,\ L^2\bar{f}(z, \bar{z}, s) \not\equiv 0\}$$
$$\vdots$$
$$B_{k_1} = \{z \in \mathbf{C};\ |z| < r,\ L^j\bar{f}(z, \bar{z}, s) \equiv 0,\ 1 \leqslant j \leqslant k_1 - 1\}.$$

Note that by definition of k_1 we have

$$\{z \in \mathbf{C};\ |z| < r\} = B_1 \cup \cdots \cup B_{k_1}.$$

Also $B_i \cap B_j = \emptyset$, $i \neq j$.

Put

$$R = \{s + it;\ |s| < r,\ -r < t < 0\}$$

and $\bar{R} = R \cup (-r, r)$.

We have the following lemma:

(4.28) Lemma. *For every $\alpha \geqslant 1$, there exist k_1 functions $d_{\alpha j}(z, \bar{z}, s + it)$, $1 \leqslant j \leqslant k_1$, C^∞ in the set $\{z, |z| < r\} \times \bar{R}$, holomorphic in $s + it$ for each fixed z, $|z| < r$, and a positive integer n_α such that each $z \in B_j$, and for $|s| < r$,*

$$(L^j\bar{f})^{n_\alpha}(z, \bar{z}, s)\, d_\alpha(z, \bar{z}, s) = d_{\alpha j}(z, \bar{z}, s). \tag{4.29}$$

Proof. We apply L to both sides of (4.17) to obtain

$$L\bar{g} = Q_\zeta(f, \bar{f}, g)\, L\bar{f}. \tag{4.30}$$

We now take $n_1 = 1$ and $d_{11} = L\bar{g}$. Applying L to (4.30) yields

$$L^2\bar{g} = Q_\zeta L^2\bar{f} + Q_{\zeta^2}(L\bar{f})^2.$$

Since for $z \in B_2$ we have $L\bar{f}(z, \bar{z}, s) \equiv 0$ as a function of s, we can take $d_{12} = L^2\bar{g}$. By repeatedly applying L and using the definition of B_j we have $d_{1j} = L^j\bar{g}$. We now prove Lemma (4.28) by induction on α using

$$L^{n\alpha}\bar{g} = Q_{\zeta^\alpha}(L^n\bar{f})^\alpha + \sum_{p_1 + \cdots + p_\beta = n\alpha,\, \beta \leqslant n\alpha} a_{\beta_{p_1 \cdots p_\beta}} Q_{\zeta^\beta}(L^{p_1}\bar{f}) \cdots (L^{p_\beta}\bar{f})$$
$$\text{with } a_{\beta_{p_1 \cdots p_\beta}} \in \mathbf{Z}^+ \text{ and } a_{\alpha_{n \cdots n}} = 0. \tag{4.31}$$

For $z \in B_n$ we deal with the terms on the right hand side of (4.31) as follows. If $\beta \geqslant \alpha$, then $p_j < n$ for some j and hence the term vanishes by the definition of B_n. If $\beta < \alpha$, then we use the induction hypothesis to conclude the desired result.

We can now apply [1, Lemma (8.15)] since each d_α satisfies a polynomial equation of the form (4.27) and, by Lemma (4.28), for fixed z, d_α is a quotient of boundary values of holomorphic functions in R. We conclude that for each z, $|z| < r$, d_α extends holomorphically to the rectangle $R = \{s + it\colon |s| < r,\ -r < t < 0\}$ and is C^∞ in $\bar{R} = R \cup (-r, r)$. We denote this extension by $d_\alpha(z, \bar{z}, s + it)$.

We claim that $d_\alpha(z, \bar{z}, s + it)$ is measurable. For this we first note that the complement of B_1 in the ball of radius r is of measure 0. Indeed, the function $\bar{f}(z, \bar{z}, s + it)$ is real analytic in $\{|z| < r\} \times R$ since f satisfies the elliptic system

$$\Lambda f(z, \bar{z}, s + it) = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right) f = 0$$

(see (4.4)) for $|z| < r$, $|s| < r$, and $0 < t < r$. Hence, $(L\bar{f})(z, \bar{z}, s + it)$ is real analytic in $\{|z| < r\} \times R$. For $z \notin B_1$ we have $L\bar{f}(z, \bar{z}, s + it) \equiv 0$ as a function of $s + it$. Since $L\bar{f} \not\equiv 0$, the complement of B_1 is a real analytic set strictly contained in $\{z\colon |z| < r\}$ and hence is of measure 0. Finally, by the holomorphy of $L\bar{f}(z, \bar{z}, s + it)$ in R for each fixed z and Fubini's theorem, we have also that $\{(z, s + it)\colon z \in B_1,\ L\bar{f}(z, \bar{z}, s + it) = 0\}$ is also of measure zero. Hence, by (4.29) we have

$$d_\alpha(z, \bar{z}, s + it) = \frac{d_{\alpha_1}(z, \bar{z}, s + it)}{L\bar{f}(z, \bar{z}, s + it)} \tag{4.32}$$

except on a set of measure 0, which proves the measurability of d_α.

From the polynomial equations (4.27) we obtain the bounds for d_α,

$$|d_\alpha(z, \bar{z}, s + it)| \leqslant C^\alpha \alpha!. \tag{4.33}$$

Thus the right hand side of (4.26) extends, for each z, $|z| < r$, as a holomorphic function in $s + it$ in R and is C^∞ in $\bar{R}$. Note that the left hand side extends in the rectangle $-R$, since this is the case for f and g. Let h_λ be the extension of the right and left hand sides of (4.26) to $|z| < r$, $|s| < r$, and $|t| < r$. The measurability of d_α together with (4.33) imply that h_λ is measurable and bounded. The holomorphic extendability of f and g for $t > 0$ implies that $\Lambda h_\lambda \equiv 0$, where Λ is given by (4.3), for $t \geqslant 0$. A simple distribution argument, together with the holomorphy of h_λ (with respect to $s + it$ for z fixed) will imply that $\Lambda h_\lambda = 0$ for $t < 0$ also. Therefore, we conclude (since $(\partial/\partial s + i(\partial/\partial t))h_\lambda = 0$ also) that h is real analytic for $|z| < r$, $|s| < r$, and $|t| < r$.

Taking $\lambda = 0$ in (4.26) we conclude that $h_0(z, \bar{z}, s)$ is real analytic and hence so is $g = Q(f, 0, g)$. To obtain the real analyticity of f suppose first that M' is of infinite type. We use (4.17) to write

$$\frac{Q(f, \lambda, g) - g}{g^{j_0' + 1}} = S(f, \lambda, g). \tag{4.34}$$

By (2.25) and (2.5) we have $g = s^{k_0} g_1$, $g_1(0) \neq 0$. Hence $S(f, \lambda, g)$ is also real analytic. Since $S_{p_0 q_0}(0) \neq 0$ we can use the Weierstrass Preparation Theorem to conclude that f satisfies a monic polynomial with real analytic coefficients, which implies the real analyticity of f by an argument similar to that given in [2]. If M' is of finite type then $Q_{p_0 q_0}(0) \neq 0$ and we reach the desired conclusion also. This completes the proof of Theorem 1.

References

1. M. S. Baouendi, S. Bell, and L. P. Rothschild, Mappings of three-dimensional CR manifolds and their holomorphic extension, *Duke Math. J.* **56** (1988), 503–530.
2. M. S. Baouendi, H. Jacobowitz, and F. Treves, On the analyticity of CR mappings, *Ann. of Math.* **122** (1985), 365–400.
3. M. S., Baouendi and L. P. Rothschild, Germs of CR maps between real analytic hypersurfaces, *Invent. Math.* **93** (1988), 481–500.
4. M. S. Baouendi and L. P. Rothschild, Geometric properties of smooth and holomorphic mappings between hypersurfaces in complex space, *J. Diff. Geom.* **31** (1990), 473–499.
5. S. Bell, Analytic hypoellipticity of the $\bar{\partial}$-Neumann problem and extendability of holomorphic mappings, *Acta Math.* **147** (1981), 109–116.
6. S. S. Chern and J. Moser, Real hypersurfaces in complex manifolds, *Acta Math.* **133** (1974), 219–271.
7. K. Diederich and J. E. Fornaess, Proper holomorphic mappings between real analytic pseudoconvex domains in $\mathbf{C}^n$, *Math. Ann.* **282** (1988), 681–700.
8. K. Diederich and S. M. Webster, A reflection principle for degenerate real hypersurfaces, *Duke Math. J.* **47** (1980), 835–843.
9. L. Hörmander, Hypoelliptic second order differential equations, *Acta Math.* **119** (1967), 147–171.
10. J. J. Kohn, Boundary behaviour of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two, *J. Differential Geom.* **6** (1972), 523–542.
11. H. Lewy, On the boundary behavior of holomorphic mappings, *Acad. Naz. Lincei* **35** (1977), 1–8.
12. S. I. Pinčuk, On proper holomorphic mappings of strictly pseudoconvex domains, *Siberian Math. J.* **15** (1974), 909–917.