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# A generalized complex Hopf Lemma and its applications to CR mappings

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## 0 Introduction

In this paper we study local geometric properties of smooth CR mappings between hypersurfaces in $\mathbf{C}^{n+1}$. In particular, one of our main results (Theorem 4 below) is that any smooth CR self-map of a hypersurface of D-finite type fixing a point is either constant or a local diffeomorphism. The novel approach here is the use of techniques of analytic discs as in Tumanov [22] for the extension of CR functions as well as the use of a new notion, that of minimal convexity (see below), which is crucial for the proofs. We also apply other results of the authors given in [7] and [11].

Let $M$ be a smooth real hypersurface in $\mathbf{C}^{n+1}$. For $p \in M$, we denote by $T_pM$ the real tangent space of $M$ at $p$ and by $\mathbf{C}T_pM$ its complexification. We denote by $\mathscr{V}_pM$ the complex subspace of $\mathbf{C}T_pM$ consisting of all antiholomorphic vectors tangent to $M$ at $p$, and by $T_p^cM = \operatorname{Re}\mathscr{V}_pM$ the complex tangent space of $M$ at $p$ considered as a real subspace of $T_pM$. Recall that a smooth (germ of a) map $H$ (at $p_0$) from $M$ to another smooth real hypersurface $M'$ is called CR if for every $p \in M$, $H'(\mathscr{V}_pM) \subset \mathscr{V}_{H(p)}M'$, where $H'$ denotes the differential of $H$. Note that if $H = (H_1, \ldots, H_{n+1})$, where $H_j: M \to \mathbf{C}$, then the $H_j$ are smooth CR functions defined on $M$.

Recall also that an *analytic disc* in $\mathbf{C}^N$ is a continuous mapping $A: \bar{\Delta} \to \mathbf{C}^N$ which is holomorphic in $\Delta$, where $\Delta$ is the open unit disc in the plane and $\bar{\Delta} = \Delta \cup S^1$, where $S^1$ is the unit circle. We say that $A$ is *attached* to $M$ if $A(S^1) \subset M$. We shall always assume all analytic discs to be of Hölder class at least $C^{1,\alpha}(\bar{\Delta})$ for some fixed $\alpha \in (0, 1)$, passing through a point $p_0 \in M$, which we fix throughout, i.e. $A(1) = p_0$. The norm of $A$ is always taken to be in this Hölder class.

Following Tumanov [22] (see also Trépreau [21]), we say that $M$ is *minimal* at $p_0$ if there is no germ of a complex holomorphic hypersurface contained in $M$ and passing through $p_0$. Note that if $M$ is of finite type (in the sense of Bloom and

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Graham [16]) at $p_0$ then $M$ is minimal. It is proved in [22] that $M$ is minimal at $p_0$ if and only if in any neighborhood $V$ of $p_0$ in $M$ there is an analytic disc $A(\zeta)$ attached to $V$ of sufficiently small norm such that its real derivative $\frac{\partial}{\partial \xi}[A(\xi)]_{|\xi=1}$ does not lie in $T_{p_0}M$. Here we have used the notation $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbf{R}$ and $\zeta \in \Delta$. Since $T_{p_0}M$ can be regarded as a (real) hyperplane passing through $p_0$, it separates $\mathbf{C}^{n+1}$ into two real half-spaces. We introduce the following.

**Definition.** *Let $M$ be a smooth hypersurface minimal at $p_0$. We shall say that $M$ is minimally convex at $p_0$ if there is a neighborhood $U$ of $p_0$ in $M$ and a side of the hyperplane $T_{p_0}M$ in $\mathbf{C}^{n+1}$ such that the real derivatives $\frac{\partial}{\partial \xi}[A(\xi)]_{|\xi=1}$ lie on that side or in $T_{p_0}M$ for all analytic discs $A$ of sufficiently small norm attached to $U$ and passing through $p_0$.*

The notion of minimal convexity generalizes that of pseudoconvexity and is studied in more detail in Sect. 1 and Sect. 7.

Trépreau [21] proved that every CR function defined in a neighborhood of $p_0 \in M$ extends holomorphically to at least one side of $M$ if and only if $M$ is minimal at $p_0$. The following, the proof of which is essentially contained in the work of Tumanov [22], will be used to reduce the study of a self CR mapping of $M$ to either the case where the mapping extends holomorphically or $M$ is minimally convex.

**Theorem 1** *Let $M$ be a smooth, real hypersurface in $\mathbf{C}^{n+1}$, and assume that $M$ is minimal at $p_0$. Then one of the following two conditions holds.*

(i) *$M$ is minimally convex at $p_0$.*

(ii) *Every CR function in a neighborhood of $p_0$ in $M$ extends holomorphically to a full neighborhood of $p_0$ in $\mathbf{C}^{n+1}$.*

We shall prove the following generalization of the "Hopf Lemma" concerning CR mappings into minimally convex hypersurfaces, extending the known result for pseudoconvex hypersurfaces of finite type. (See Diederich and Fornaess [17], Fornaess [18], Bell [14].)

If $H$ is a CR mapping between two hypersurfaces $M$ and $M'$ in $\mathbf{C}^{n+1}$ we denote by $\operatorname{Jac} H$ the Jacobian determinant of $H$ considered as a mapping from the real manifold $M$ to the real manifold $M'$. Note that $\operatorname{Jac} H$ is independent of the choice of local coordinates on $M$ and $M'$ up to multiplication by a nonvanishing smooth function. In all of the following results, $M$ and $M'$ are smooth hypersurfaces in $\mathbf{C}^{n+1}$ and $H$ a germ of a smooth CR map at $p_0$ mapping $M$ into $M'$. We will call $H$ a *self-map* of $M$ if $M' = M$ and $H(p_0) = p_0$.

The main technical result of this paper is the nonvanishing of the transversal derivative of the mapping when the target hypersurface is minimally convex. This result, which is contained in the following theorem, may be regarded as a generalization of the classical Hopf lemma for harmonic functions.

**Theorem 2** *Suppose that $M$ is minimal at $p_0$ and $\operatorname{Jac} H \not\equiv 0$. If $M'$ is minimally convex at $H(p_0)$, then the differential of $H$ at 0 is nonzero. More precisely,*

$$H'(T_{p_0}M) \not\subset T^c_{H(p_0)}M'. \tag{0.1}$$

*In addition, $M$ is also minimally convex at $p_0$.*

Note that (0.1) is equivalent to the nonvanishing of the derivative (in the transversal direction) of the transversal component of $H$ at $p_0$. (See (5.1) and the remarks above.)

Following D'Angelo [2] we shall say that a hypersurface $M$ is of *D-finite type at* $p_0$ if there is a positive integer $N$ such that there is no nontrivial germ of a complex analytic variety through $p_0$ with order of contact $\geqq N$ with $M$ at $p_0$. In particular, if $M$ is of D-finite type, then there is no complex analytic variety through $p_0$ with infinite order of contact with $M$. (In $\mathbf{C}^2$, the notion of D-finite type is the same as that of finite type, as introduced first by Kohn [19].) Theorem 2 as well as other results of the authors [11] will be used to prove the following.

**Corollary 0.2** *If $M$ and $M'$ are of D-finite type at $p_0$ and $H(p_0)$ respectively and $M'$ is minimally convex at $p_0$, then either $H$ is constant or* (0.1) *holds.*

For the case of self-maps, we may make use of Theorems 1 and 2, as well as previous results of the authors [7] to obtain a sharper result in the case of essentially finite hypersurfaces (See Sect. 6 for precise definitions).

**Theorem 3** *Suppose that $M$ is essentially finite at $p_0$ and $H$ is a self-map of $M$. Then one of the following holds.*

(i) *$H$ is constant.*
(ii) *$H$ is nonconstant and* $\operatorname{Jac} H \equiv 0$.
(iii) *$H$ is a local diffeomorphism at $p_0$.*

For the case of self-maps when $M$ is of D-finite type, we may combine Theorem 3, as well as a result in [11] to obtain the following.

**Theorem 4** *If $M$ is of* D-*finite type at $p_0$ and $H$ is a self-map of $M$, then either $H$ is constant or $H$ is a local diffeomorphism.*

A more general result is the following.

**Theorem 5** *Suppose $M$ is essentially finite at $p_0$ and does not contain a germ of a nontrivial complex manifold through any point near $p_0$. If $H$ is a self-map of $M$, then either $H$ is constant or $H$ is a local diffeomorphism.*

The conclusion of Theorem 5 may fail if $M$ is essentially finite but contains a complex manifold as shown by example in Sect. 7.

When the hypersurfaces $M$ and $M'$ are real analytic, the results above, in conjuction with previously known results on holomorphic extendability of CR mappings (see [6], [7]) yield the following.

**Theorem 6** *Let $M$, $M'$ be real analytic hypersurfaces essentially finite at $p_0$ and $H(p_0)$ respectively. Suppose, in addition, that $M'$ is minimally convex at $H(p_0)$ and* $\operatorname{Jac} H \not\equiv 0$. *Then $H$ extends holomorphically in a full neighborhood of $p_0$ in $\mathbf{C}^{n+1}$.*

**Theorem 7** *If $M$ is a real analytic hypersurface of* D-*finite type at $p_0$ and $H$ is a self-map of $M$, then either $H$ is constant or $H$ extends as a local biholomorphism in a neighborhood of $p_0$.*

The applications of Theorems 1 and 2 for mappings may be regarded as results in unique continuation for CR mappings. Indeed, under the hypothesis of essential finiteness, it is shown that a CR self mapping $H$ with $\operatorname{Jac} H \not\equiv 0$ cannot have any component flat at a point (i.e. all derivatives vanishing there). To our knowledge, these are the first general local results on mappings of nonpseudoconvex domains which do not rely on the Taylor series of the mapping at a point. Similarly, previously known results on holomorphic extendability of CR mappings of real analytic hypersurfaces [5, 4, 6, 17, 7, 8] assume some nondegeneracy of the Taylor series of the mapping at the point of extendability. It should be noted that in our Theorems 6 and 7 above, no such condition is assumed at $p_0$.

For simplicity we have assumed all hypersurfaces and mappings to be of class $C^\infty$; however, some results, in particular Theorems 1 and 2, can be formulated for $C^k$ regularity with $k \geqq 3$.

## 1 Analytic discs, minimality, and minimal convexity; proof of Theorem 1

Let $M$ be a smooth hypersurface in $\mathbf{C}^{n+1}$ and $p_0 \in M$. Without loss of generality, we may assume $p_0 = 0$ and choose local holomorphic coordinates $(z, w) \in \mathbf{C}^{n+1}$ so that $M$ is given by

$$\text{(1.1)} \qquad \operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w),$$

with $\phi$ a smooth, real valued function defined in a neighborhood of 0 in $\mathbf{R}^{2n+1}$ and satisfying $\phi(0) = 0$ and $d\phi(0) = 0$. We write $s$ for $\operatorname{Re} w$. Then $M$ is parametrized by $(z, \bar{z}, s)$ in a neighborhood of 0 in $\mathbf{R}^{2n+1}$, and in these coordinates $T_0^c M$ consists of all vectors of the form $\sum_1^n a_j \frac{\partial}{\partial z_j} + \bar{a}_j \frac{\partial}{\partial \bar{z}_j}$, where the $a_j$ are complex numbers.

Using the coordinates above, any analytic disc $A(\zeta)$ attached to a small neighborhood of 0 in $M$ may be written $A(\zeta) = (z(\zeta), w(\zeta))$, where $z(\zeta)$ is an analytic disc valued in $\mathbf{C}^n$ of class $C^{1,\alpha}(\bar{\Delta})$, with $z(1) = 0$, and $w(\zeta) = s(\zeta) + i\phi(z(\zeta), \overline{z(\zeta)}, s(\zeta))$ with $s(\zeta)$ satisfying the Bishop equation (see [15])

$$\text{(1.2)} \qquad s(e^{i\theta}) = -T_1(\phi(z(\cdot), \overline{z(\cdot)}, s(\cdot)))(e^{i\theta}),$$

where $T_1 u$ is the Hilbert transform of a function $u$ defined on $S^1$ normalized by the condition $T_1 u(1) = 0$. More precisely we have

$$T_1 u(e^{i\theta}) = Tu(e^{i\theta}) - Tu(1),$$

where

$$Tu(e^{i\theta}) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\varepsilon}^{2\pi - \varepsilon} u(e^{i(t-\theta)}) \cot(\theta/2)\, d\theta.$$

Note that by the continuity of the Hilbert transform in Hölder spaces we have $s(e^{i\theta}) \in C^{1,\alpha}(S^1)$.

The proof of the following lemma is elementary, and is left to the reader.

**Lemma 1.3** *Let $u \in C^{1,\alpha}(S^1)$ satisfy $u(1) = u'(1) = 0$. Then*

$$\text{(1.4)} \qquad \frac{d}{dt} T_1 u(e^{it})_{|t=0} = -1/\pi \int_0^{2\pi} \frac{u(e^{i\theta})}{|e^{i\theta} - 1|^2}\, d\theta.$$

Following [22], it will be convenient to introduce the following notation. If $u$ is as in Lemma (1.3) we write

$$\mathscr{J}u = -1/\pi \int_0^{2\pi} \frac{u(e^{i\theta})}{|e^{i\theta}-1|^2}\,d\theta = \frac{1}{i\pi}\int_{S^1} \frac{u(\zeta)}{(\zeta-1)^2}\,d\zeta. \tag{1.5}$$

We have the following characterization of minimality and minimal convexity, which follows closely the work of Tumanov [22].

**Theorem 8.** *The hypersurface $M$ is minimal at 0 if and only if for every neighborhood $U$ of 0 in $M$ there exists an analytic disc attached to $U$, $A(\zeta) = (z(\zeta), w(\zeta))$ of sufficiently small norm, with* Re $w(\zeta) = s(\zeta)$ *such that $\mathscr{J}(\phi(z(\zeta), \overline{z(\zeta)}, s(\zeta))) \neq 0$. Also, $M$ is minimally convex at 0 if, in addition, there exists a neighborhood $V$ of 0 in $M$ such that $\mathscr{J}(\phi(z(\zeta), \overline{z(\zeta)}, s(\zeta)))$ takes only one sign, possibly 0, for all discs $A(\zeta) = (z(\zeta), w(\zeta))$ attached to $V$.*

*Proof.* The characterization of minimality is obtained by inspecting the proof of the main theorem in [22]; see also [23]. (Details are also given in the forthcoming monograph [12] of the authors.) The characterization of minimal convexity is a consequence of Lemma 1.3 and the use of the Cauchy-Riemann equations at $\zeta = 1$ for the $w$ component of analytic discs. □

*Proof of Theorem 1* Assume that $M$ is minimal, but not minimally convex. Then by Theorem 8, in any neighborhood $U$ of the origin in $M$ there are discs $A_j(\zeta) = (z_j(\zeta), w_j(\zeta))$, $j = 1, 2$, attached to $U$, of sufficiently small norm, with $s_j(\zeta) = \operatorname{Re} w_j(\zeta)$ and such that

$$\mathscr{J}(\phi(z_1(\cdot), \overline{z_1(\cdot)}, s_1(\cdot))) > 0, \quad \text{and} \quad \mathscr{J}(\phi(z_2(\cdot), \overline{z_2(\cdot)}, s_2(\cdot))) < 0. \tag{1.6}$$

An inspection of the proof in [22] shows that (1.6) implies that every CR function defined in a neighborhood of 0 in $M$ extends holomorphically to a full neighborhood of 0 in $\mathbf{C}^{n+1}$. This proves Theorem 1. □

Using Theorem 1, the following shows the connection between pseudoconvexity and minimality.

**Proposition 1.7** *Let $M$ be a smooth pseudoconvex hypersurface in $\mathbf{C}^{n+1}$. If $M$ is minimal at $p_0$, then $M$ is minimally convex at $p_0$.*

*Proof.* By Amar [1], there exists a pseudoconvex open set $\Omega \subset \mathbf{C}^{n+1}$ with smooth boundary $\partial\Omega$ such that $\partial\Omega \cap M$ is a neighborhood of $p_0$ in $M$. By the pseudoconvexity of $\Omega$, there exists a function $h$ holomorphic in $\Omega$ which does not have a holomorphic extension in any neighborhood of $p_0$ in $\mathbf{C}^{n+1}$. By a standard argument, this implies that there exists a CR function on $M$ near $p_0$ which does not extend holomorphically to any full neighborhood of $p_0$. The minimal convexity of $M$ at $p_0$ then is a consequence of Theorem 1. □

*Remark 1.8* Assume that $M$ is given by (1.1) with $\phi \geqq 0$ in a neighborhood of 0 in $\mathbf{R}^{2n+1}$. By Theorem 8, if $M$ is minimal at 0, it is minimally convex. Note that $M$ need not be pseudoconvex. Indeed, suppose that $M = \{(x+iy, w)$: $\operatorname{Im} w = \phi(z, \bar{z}) = x^2(x-y)^2\} \subset \mathbf{C}^2$. Then $M$ is not pseudoconvex, since $\Delta\phi$, the Laplacian of $\phi$, is a homogeneous quadratic polynomial which takes both signs. It is easy to check that $M$ is minimal, since $M$ is of finite type.

*Remark 1.9* Ye [24] has given an example of a tubular hypersurface, $M = \{(z, w): \operatorname{Im} w = \phi(x)\} \subset \mathbf{C}^2$, with $\phi$ a real-valued, smooth function on $\mathbf{R}$, flat at 0, taking both positive and negative values in any interval containing 0, such that $M$ has the following property. In any sufficiently small neighborhood of 0 in $M$ there exists a CR function which does not extend holomorphically to a full neighborhood of 0 in $\mathbf{C}^2$. It is easy to see that $M$ is minimal at 0. Hence, by Theorem 1, $M$ is minimally convex at 0. However, $M$ is not pseudoconvex.

*Remark 1.10* It is not known to the authors whether conditions (i) and (ii) of Theorem 1 are mutually exclusive. That is, we do not know of an example of a hypersurface minimally convex at $p_0$ for which all CR functions extend holomorphically to a full neighborhood of $p_0$.

## 2 Mappings and analytic discs

Let $H$ be a (germ at $p_0$ of a) CR mapping from a hypersurface $M \subset \mathbf{C}^{n+1}$ to a hypersurface $M' \subset \mathbf{C}^{n+1}$. We may assume that $M$ is given by (1.1) and similarly, we can choose holomorphic coordinates $(z', w')$ so that $M'$ is given near $H(p_0)$ by

$$\operatorname{Im} w' = \psi(z', \bar{z}', s'), \tag{2.1}$$

with $\psi(0) = 0$, $d\psi(0) = 0$, and $s'$ for $\operatorname{Re} w'$. Using the above coordinates we may write $H = (f_1, \ldots, f_n, g)$, where the functions $f_j, j = 1, \ldots, n$ and $g$ are smooth CR functions defined in a neighborhood of 0 in $M$. Since $H(M) \subset M'$ we have

$$\operatorname{Im} g(z, \bar{z}, s) = \psi(f(z, \bar{z}, s), \overline{f(z, \bar{z}, s)}, \operatorname{Re} g(z, \bar{z}, s)), \tag{2.2}$$

in a neighborhood of 0 in $\mathbf{R}^{2n+1}$.

**Lemma 2.3** *Let $A(\zeta)$ be an analytic disc attached to $U$, a sufficiently small neighborhood of 0 in $M$, and passing through 0. Then $H \circ A$ extends (uniquely) as an analytic disc attached to $M'$ through 0.*

*Proof.* Since $A(S^1) \subset M$ we have $H \circ A(S^1) \subset M'$ and $H \circ A(1) = 0$. It suffices to show that $H \circ A$ extends holomorphically to $\Delta$, i.e., $f_j \circ A$ and $g \circ A$ extend holomorphically to $\Delta$. This follows from the approximation theorem of CR functions by holomorphic polynomials [13] and the maximum principle for holomorphic functions in $\Delta$. □

The following formula connects analytic discs attached to hypersurfaces and their images under CR maps.

**Theorem 9.** *Let $M$ and $M'$, given by (1.1) and (2.1) respectively, be real hypersurfaces in $\mathbf{C}^{n+1}$ and $H = (f_1, \ldots, f_n, g) = (f, g)$ be a CR mapping from $M$ into $M'$ with $H(0) = 0$. Then $\frac{\partial g}{\partial s}(0)$ is real and for any sufficiently small analytic disc $A(\zeta) = (z(\zeta), w(\zeta))$, $\operatorname{Re} w(\zeta) = s(\zeta)$, attached to $M$, we have*

$$-\frac{\partial g}{\partial s}(0)\frac{\partial s}{\partial \theta}(e^{i\theta})|_{\theta=0} = \mathscr{J}(\psi(f(z, \bar{z}, s), \bar{f}(z, \bar{z}, s), \operatorname{Re} g(z, \bar{z}, s))), \tag{2.4}$$

$$\frac{\partial g}{\partial s}(0)\mathscr{J}(\phi(z, \bar{z}, s)) = \mathscr{J}(\psi(f(z, \bar{z}, s), \bar{f}(z, \bar{z}, s), \operatorname{Re} g(z, \bar{z}, s))). \tag{2.5}$$

The following, which will be used in the proof of Theorem 9, is elementary and well known.

**Lemma 2.6** *Let $h(\zeta)$ be holomorphic in $\Delta$ of class $C^1(\bar{\Delta})$ with $h(1) = 0$. Then $\mathscr{I}(h) = h'(1)$, where the integral $\mathscr{I}(h)$ is taken in the sense of principal value.*

*Proof of Theorem 9* We start with (2.2), in which we replace $z$ by $z(\zeta)$ and $s$ by $s(\zeta)$, the components of the analytic disc attached to $M$ and apply the operator $\mathscr{I}$ to both sides of the resulting equation. By Lemma 2.3, the function $\zeta \mapsto g(z(\zeta), \bar{z}(\zeta), s(\zeta)) = h(\zeta)$ extends holomorphically to $\Delta$. Since $h(1) = 0$, we may apply Lemma 2.6 to obtain $\mathscr{I}(h(\cdot)) = h'(1)$. Since the operator $\mathscr{I}$ is real, the result will follow by computing $h'(1)$. We have

$$h'(1) = -ie^{-i\theta}\left[g_z(0)\frac{d}{d\theta}(z(e^{i\theta})) + g_{\bar{z}}(0)\frac{d}{d\theta}\overline{(z(e^{i\theta}))} + g_s(0)\frac{d}{d\theta}(s(e^{i\theta}))\right]_{|\theta=0}.$$

We claim that $g_z(0) = g_{\bar{z}}(0) = 0$ and $g_s(0)$ is real. Indeed, since $g$ is CR and $d\phi(0) = 0$, we conclude that $g_{\bar{z}}(0) = 0$. Applying $\dfrac{\partial}{\partial z}$ to both sides of (2.2) and using the fact that $d\psi(0) = 0$, we conclude that $g_z(0) = 0$ also. Applying $\dfrac{\partial}{\partial s}$ to (2.2) and using a similar argument, gives $\operatorname{Im} g_s(0) = 0$. Now (2.4) follows easily. Using (1.2) and (1.4), we obtain (2.5) from (2.4). The proof of Theorem 9 is complete. □

The following two elementary lemmas concerning minimal hypersurfaces will be used in the proof of Theorem 2.

Recall that if $M$ is of finite type at $p$ in the sense of Bloom–Graham [16], then the *type of $M$* at $p$ is the length of the shortest commutator of sections of $T^c M$ near $p$ which, at $p$, lies outside $T^c_p M$. If $M$ is of infinite type at every point, then $M$ is called *Levi flat*. The following is an immediate consequence of the Frobenius theorem; the proof is left to the reader.

**Lemma 2.7** *If $M$ is minimal at 0 and $U$ is any open neighborhood of 0 in $M$, then $M$ is not Levi flat in $U$, and hence the set $U_F = \{p \in U: M \text{ is of type } 2 \text{ at } p\}$ is open and nonempty.*

Lemma 2.7 will be used in conjunction with the following.

**Lemma 2.8** *Let $M$ and $M'$ be smooth, real hypersurfaces in $\mathbf{C}^{n+1}$ and $H$ a germ of a* CR *map at 0 with $H(M) \subset M'$ and $\operatorname{Jac} H \not\equiv 0$ in any neighborhood of 0 in $M$. If $M$ is minimal at 0, and $V$ is a small neighborhood of 0 in $M$, then the set $V_H = \{p \in V: \operatorname{Jac} H(p) \neq 0\}$ is open and dense in $V$.*

*Proof.* Note first that since $M$ is minimal, by the theorem of Trépreau, the components $f_1, \ldots, f_n$ and $g$ of $H$ extend holomorphically to at least one side of $M$. Let $F_1(z, w), \ldots, F_n(z, w)$, $G(z, w)$ be their holomorphic extensions. Then the complex Jacobian determinant $D(z, w)$ of this holomorphic map restricts to a CR function $d$ on $M$. Since the zeros of $d$ and of $\operatorname{Jac} H$ are the same, as can be easily checked, we conclude that $d \not\equiv 0$ and hence by unique continuation does not vanish on any open set on $M$. This proves the lemma. □

## 3 Consequences of minimal convexity for analytic discs attached to $M$

As in Sect. 1, we write $T_1$ for the normalized Hilbert transform on $S^1$ and assume that $M$ is given by (1.1). Following [22], for an analytic disc attached to $M$, $A^0(\zeta) = (z^0(\zeta), w^0(\zeta))$, $s_0(\zeta) = \operatorname{Re} w^0(\zeta)$, of sufficiently small norm, we define the function $v(e^{i\theta})$ on $S^1$ implicitly by

$$(3.1) \qquad v(e^{i\theta}) = 1 + T_1(v(\cdot)\phi_s(z^0(\cdot), \overline{z^0(\cdot)}, s_0(\cdot)))(e^{i\theta}).$$

Indeed, the existence and uniqueness of $v$ can be proved by showing that for a given function $a(e^{i\theta})$ with small norm the mapping $v \mapsto 1 + T_1(va)$ is a contraction in a closed ball around the constant function 1 in $C^{1,\alpha}(S^1)$. The main result of this section is the following, which is crucial for the proof of Theorem 2.

**Proposition 3.2** *Let $M$ be minimally convex at 0 and $A = (z^0, w^0)$, $s_0 = \operatorname{Re} w^0$, an analytic disc attached to $M$ with sufficiently small norm. If $\mathscr{I}(\phi(z^0(\cdot), \overline{z^0(\cdot)}, s_0(\cdot))) = 0$, then the $1 \times n$ matrix-valued function*

$$(3.3) \qquad e^{i\theta} \mapsto v(e^{i\theta})\phi_z(z^0(e^{i\theta}), \overline{z^0(e^{i\theta})}, s_0(e^{i\theta}))$$

*extends holomorphically to $\Delta$.*

*Proof.* Let $z^1(\zeta)$ be an analytic disc in $\mathbf{C}^n$ with $z^1(1) = 0$. For $\lambda \in \mathbf{C}$ small, we define a family of analytic discs attached to $M$ through 0 given by

$$A_\lambda(\zeta) = (z^0(\zeta) + \lambda z^1(\zeta), w_\lambda(\zeta)),$$

with

$$(3.4) \qquad w_\lambda(\zeta) = s_\lambda(\zeta) + i\phi(z^0(\zeta) + \lambda z^1(\zeta), \overline{z^0(\zeta) + \lambda z^1(\zeta)}, s_\lambda(\zeta)), \quad \zeta \in S^1.$$

Here $s_\lambda$ is given by the Bishop equation (1.2) in which $z$ is replaced by $z_\lambda = z^0 + \lambda z^1$, i.e.

$$(3.5) \qquad s_\lambda(\zeta) = -T_1(\phi(z^0(\cdot) + \lambda z^1(\cdot), \overline{z^0(\cdot) + \lambda z^1(\cdot)}, s_\lambda(\cdot)))(\zeta), \quad \zeta \in S^1.$$

We differentiate the Eq. (3.5) and define $u(\zeta)$ by

$$(3.6) \qquad u(\zeta) = \frac{\partial s_\lambda}{\partial \lambda}(\zeta)_{|\lambda=0},$$

so that $u(\zeta)$ satisfies the equation

$$(3.7) \qquad u = -T_1(au + b),$$

where $a(\zeta) = \phi_s(z^0(\zeta), \overline{z^0(\zeta)}, s_0(\zeta))$ and $b(\zeta) = \phi_z(z^0(\zeta), \overline{z^0(\zeta)}, s_0(\zeta))\, z^1(\zeta)$, for $\zeta \in S^1$. The following is proved in [22] (see also [12]):

**Lemma 3.8.** *Let $a, b \in C^{1,\alpha}(S^1)$ with $a(1) = 0$ and $b(1) = b'(1) = 0$ and $u$, $v$ given by $u = -T_1(au + b)$ and $v = 1 + T_1(va)$. Then*

$$(3.9) \qquad \mathscr{I}(au + b) = \mathscr{I}(vb).$$

*Furthermore,*

$$(3.10) \qquad u = -(1 + a^2)^{-1}[v^{-1}T_1(vb) + ab].$$

We may now complete the proof of Proposition 3.2. Since $M$ is minimally convex at 0, we may assume without loss of generality that $\mathscr{I}(\phi(z, \bar{z}, s)) \geqq 0$ for all choices of small analytic discs $A(\zeta) = (z(\zeta), w(\zeta))$ attached to $M$. Let

$$Q(\lambda, \bar{\lambda}) = \mathscr{T}(\phi(z^0 + \lambda z^1, \overline{z^0 + \lambda z^1}, s_\lambda)).$$

Then $Q(\lambda, \bar{\lambda})$ is a continuously differentiable function in a neighborhood of 0 in **C**. Since by assumption $Q(0, 0) = 0$ and $Q(\lambda, \bar{\lambda}) \geqq 0$, $Q$ has a local minimum at 0 so that $\frac{\partial Q}{\partial \lambda}(0, 0) = 0$. It now follows from (3.6), (3.7) and (3.9) that

$$\mathscr{T}(v(\cdot)\phi_z(z^0(\cdot), \overline{z^0(\cdot)}, s_0(\cdot))\, z^1(\cdot)) = 0, \tag{3.11}$$

for all analytic discs $z^1(\zeta)$ valued in $\mathbf{C}^n$ with $z^1(0) = 0$. For any entire function $k(\zeta)$ valued in $\mathbf{C}^n$, take $z^1 = (\zeta - 1)^2 k(\zeta)$, in (3.11) to conclude that $v(\zeta)\ \phi_z(z^0(\zeta), \overline{z^0(\zeta)}, s_0(\zeta))$ extends holomorphically. This completes the proof of Proposition 3.2. □

## 4 Construction of a family of analytic discs parametrized by $\mathbf{C}^n$

Let $M$ and $M'$ be as in Theorem 2, with holomorphic coordinates $(z, w)$ and $(z', w')$ as in Sect. 2. Here again we write $H = (f_1, \ldots, f_n, g) = (f, g)$ so that (2.2) holds in a neighborhood of 0 in $\mathbf{R}^{2n+1}$.

Before beginning the proof of Theorem 2, more preliminaries are needed.

Let $E$ be the Banach space of functions $z(\zeta)$ valued in $\mathbf{C}^n$, $z(1) = 0$, holomorphic in $\Delta$ and of class $C^{1,\alpha}(\bar{\Delta})$. Denote by $B_\varepsilon$ the open ball in $E$, centered at 0, of radius $\varepsilon$. If $\varepsilon$ is sufficiently small, we define the map $\chi\colon B_\varepsilon \to M$ as follows:

$$\chi\colon z \mapsto A(-1),$$

where $A(\zeta) = (z(\zeta), w(\zeta))$ is the corresponding disc attached to $M$ through 0. It is proved in [22] that since $M$ is minimal, for every positive $\varepsilon$, $\chi(B_\varepsilon)$ is a neighborhood of 0 in $M$. Note that by the definition of $\chi$, $\operatorname{rk}\chi'(z) \geqq 2n$ for all $z \in B_\varepsilon$, where $\chi'(z)$ denotes the Fréchet derivative of $\chi$ at $z$ and rk denotes rank. Using the CR map $H = (f, g)$, for $\varepsilon$ sufficiently small, we define the map $\Theta\colon B_\varepsilon \to \mathbf{C}^n$ by $\Theta = f \circ \chi$, i.e.,

$$\Theta\colon z \mapsto f(z(-1), \overline{z(-1)}, s(-1)), \tag{4.1}$$

where $s(\zeta) = \operatorname{Re} w(\zeta)$.

**Lemma 4.2** *For every positive $\varepsilon$ sufficiently small there exists $\gamma_0 \in \mathbf{C}^n$, and $\mathcal{O}$, an open neighborhood of $\gamma_0$ in $\mathbf{C}^n$ and a map*

$$\mathcal{O} \ni \gamma \mapsto z(\gamma, \bar{\gamma}; \cdot) \in B_\varepsilon \tag{4.3}$$

*of class $C^1$ satisfying*

$$\Theta(z(\gamma, \bar{\gamma}; \cdot)) \equiv \gamma \text{ in } \mathcal{O}. \tag{4.4}$$

*Proof.* We shall first show that there exists $z^0 \in B_\varepsilon$ for which $\operatorname{rk}\Theta'(z_0) = 2n$. Since $\chi(B_\varepsilon)$ is a neighborhood of 0 in $M$, we choose a set $U$ open in $M$ satisfying $0 \in U \subset \chi(B_\varepsilon)$. By Lemmas 2.7 and 2.8 the set $U_F \cap U_H$ consisting of all $p \in U$ for which $M$ is of type 2 at $p$ and $\operatorname{Jac} H(p) \neq 0$ is a nonempty open subset of $U$. Let $p^* \in U_F \cap U_H$ and a disc $z^* \in B_\varepsilon$ such that $\chi(z^*) = p^*$. We need the following.

**Lemma 4.5** *For every positive $\varepsilon$ sufficiently small and every $z^* \in B_\varepsilon$ such that $M$ is minimal at $\chi(z^*)$, any open neighborhood $B^*$ of $z^*$ in $B_\varepsilon$ contains a disc $z^0$ for which* $\operatorname{rk} \chi'(z^0) = 2n + 1$.

*Proof.* We reason by contradiction. If no such $z^0$ exists, then by the above remarks, $\operatorname{rk} \chi'(z) = 2n$ in a full neighborhood $B^*$ of $z^*$ in $B_\varepsilon$. It is also proved in [22] that for every $z \in B_\varepsilon$, $T^c_{\chi(z)} M \subset \chi'(z) E$. Hence by the implicit function theorem $\chi(B^*)$ is a $2n$ dimensional CR manifold passing through $\chi(z^*)$, contradicting the minimality of $M$ at this point. This proves Lemma 4.5. □

We may now complete the proof of Lemma 4.2. Since $M$ is of type 2 at $\chi(z^*) = p^*$, and hence minimal at that point, we may apply Lemma 4.5. We choose $B^*$, a neighborhood of $z^*$ in $B_\varepsilon$ such that $\chi(B^*) \subset U_F \cap U_H$ and choose $z^0 \in B^*$ given by Lemma 4.5 so that $\operatorname{rk} \chi'(z^0) = 2n + 1$. We put $q_0 = \chi(z^0)$ and $q_0' = H(q_0)$. We claim that the rank of $\Theta'(z^0)$ is maximal, i.e. $\operatorname{rk} \Theta'(z^0) = 2n$. Indeed, since $\Theta = f \circ \chi$, we have, by the chain rule, $\Theta'(z^0) = f'(\chi(z^0))\chi'(z^0)$. Since $H$ is a diffeomorphism at $q_0$ (by the choice of $B^*$ above), $f'$ is of rank $2n$ at $\chi(z^0) = q_0$, and the claim follows, since $\operatorname{rk} \chi'(z^0) = 2n + 1$.

Hence there is an open neighborhood $B^0$, with $z^0 \in B^0 \subset B_\varepsilon$ such that $\operatorname{rk} \Theta'(z) = 2n$ for all $z \in B^0$. We put $\gamma_0 = \Theta(z^0)$. Now for $\mathcal{O}$ a sufficiently small neighborhood of $\gamma_0$ in $\mathbb{C}^n$, by the implicit function theorem we can invert the map $\Theta$ near $\gamma_0$ and find a map $\gamma \mapsto z(\gamma, \bar\gamma; \zeta)$ satisfying the conclusions (4.3) and (4.4) of Lemma 4.2. □

From now on we fix $\varepsilon$ sufficiently small so that Lemma 4.2 can be applied. We introduce families of discs on $M$ and $M'$ respectively parametrized by $\gamma \in \mathcal{O}$, where $\mathcal{O}$ is given in Lemma 4.2, as follows. We set $A(\gamma, \bar\gamma; \zeta) = (z(\gamma, \bar\gamma; \zeta), w(\gamma, \bar\gamma; \zeta))$, the disc attached to $M$ passing through 0, with $z(\gamma, \bar\gamma; \zeta)$ given by Lemma 4.2 and $\gamma \in \mathcal{O}$. By Lemma 2.3, the image of a disc attached to $M$ under the CR map $H$ is a disc attached to $M'$. We set

$$A'(\gamma, \bar\gamma; \zeta) = H(A(\gamma, \bar\gamma; \zeta)) = (z'(\gamma, \bar\gamma; \zeta), w'(\gamma, \bar\gamma; \zeta)), \tag{4.6}$$

which is a family of discs attached to $M'$ through 0. As usual, we set $s(\gamma, \bar\gamma; \zeta) = \operatorname{Re} w(\gamma, \bar\gamma; \zeta)$ and $s'(\gamma, \bar\gamma; \zeta) = \operatorname{Re} w'(\gamma, \bar\gamma; \zeta)$. Note that we have by Lemma 4.2 and the definitions,

$$z'(\gamma, \bar\gamma, -1) = \Theta(z(\gamma, \bar\gamma; \cdot)) \equiv \gamma, \quad \gamma \in \mathcal{O}. \tag{4.7}$$

Note also since all the discs pass through the origin we have

$$A(\gamma, \bar\gamma; 1) = 0, \quad A'(\gamma, \bar\gamma; 1) = 0. \tag{4.8}$$

These families of analytic discs will be crucial in the proof of Theorem 2.

## 5 Proof of Theorem 2

We continue with the notation and assumptions of Sect. 4. We take $\left\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar z_j}, \frac{\partial}{\partial s}\right\}$ as a basis of $\mathbb{C}T_0 M$, the complexified tangent space to $M$ at 0, and note that $\left\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\right\}$ form a basis of $\mathbf{C}T_0^c M$. We write a similar basis for $\mathbf{C}T_0 M'$ and $\mathbf{C}T_0^c M'$. Since, by Theorem 9, $\frac{\partial g}{\partial s}(0)$ is real, condition (0.1) is equivalent to

$$\frac{\partial g}{\partial s}(0) \neq 0. \tag{5.1}$$

It is an immediate consequence of Theorem 9 that if (5.1) does not hold, then

$$\mathscr{I}(\psi(f(z,\bar{z},s), \bar{f}(z,\bar{z},s), \operatorname{Re} g(z,\bar{z},s))) = 0, \tag{5.2}$$

for all analytic discs $A(\zeta) = (z(\zeta), w(\zeta))$ attached to $M$. In the course of the proof of Theorem 2 we shall reason by contradiction; hence we assume that (5.2) holds for all analytic discs.

The main step in the proof of Theorem 2 is the following.

**Proposition 5.3** *Let $A'(\gamma, \bar{\gamma}; \zeta)$ be the family of analytic discs attached to $M'$ given by* (4.6). *If* (5.1) *does not hold, i.e. if $\frac{\partial g}{\partial s}(0) = 0$, then*

$$\mathcal{O} \ni \gamma \mapsto A'(\gamma, \bar{\gamma}; -1) \in M' \tag{5.4}$$

*parametrizes a complex holomorphic hypersurface contained in $M'$.*

*Proof.* By (4.6) and (4.7), we have

$$A'(\gamma, \bar{\gamma}; -1) = (\gamma, s'(\gamma, \bar{\gamma}, -1) + i\psi(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}; -1))). \tag{5.5}$$

Hence the proposition will be proved if it is shown that

$$\gamma \mapsto s'(\gamma, \bar{\gamma}, -1) + i\psi(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}; -1))$$

is holomorphic, i.e., for $j = 1, \ldots, n$.

$$\frac{\partial}{\partial \bar{\gamma}_j}[s'(\gamma, \bar{\gamma}, -1) + i\psi(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))] = 0, \quad \gamma \in \mathcal{O}. \tag{5.6}$$

To prove (5.6), we apply the Bishop equation (1.2) to $M'$, to obtain

$$s'(\gamma, \bar{\gamma}, \zeta) = -T_1(\psi(z'(\gamma, \bar{\gamma}; \cdot), \overline{z'(\gamma, \bar{\gamma}; \cdot)}, s'(\gamma, \bar{\gamma}; \cdot)))(\zeta). \tag{5.7}$$

Differentiating (5.7) with respect to $\bar{\gamma}_j$ we obtain the system

$$\frac{\partial s'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta) = -T_1\left(\psi_{z'}\frac{\partial z'}{\partial \bar{\gamma}_j} + \psi_{\bar{z}'}\frac{\partial \bar{z}'}{\partial \bar{\gamma}_j} + \psi_{s'}\frac{\partial s'}{\partial \bar{\gamma}_j}\right)(\zeta), \tag{5.8}$$

where $\psi_{z'}$, $\psi_{\bar{z}'}$, $\psi_{s'}$ inside the Hilbert operator $T_1$ are taken at

$$(z'(\gamma, \bar{\gamma}; \cdot), \overline{z'(\gamma, \bar{\gamma}; \cdot)}, s'(\gamma, \bar{\gamma}; \cdot))$$

and $\frac{\partial z'}{\partial \bar{\gamma}_j}, \frac{\partial \bar{z}'}{\partial \bar{\gamma}_j}, \frac{\partial s'}{\partial \bar{\gamma}_j}$ at $(\gamma, \bar{\gamma}; \cdot)$.

We shall compute $\frac{\partial s'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta)$ for $\zeta \in S^1$. For this, we apply Lemma 3.8 with

$$u(\zeta) = \frac{\partial s'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta), \quad a(\zeta) = \psi_{s'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta)), \tag{5.9}$$

and

$$b(\zeta) = \psi_{z'}(z'(\gamma, \bar{\gamma}; \zeta). \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta)) \frac{\partial z'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta) + \psi_{\bar{z}'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta)) \frac{\partial \bar{z}'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta). \tag{5.10}$$

Note that the assumptions of Lemma 3.8 are satisfied; in particular, $a(1) = 0$, $b(1) = b'(1) = 0$, by (4.8) and the vanishing of $d\psi$ at 0.

As in Lemma 3.8 we denote by $v(\zeta)$ the unique solution of the implicit equation $v = 1 + T_1(va)$. We note that since $a$ is real valued, so is $v$. An explicit solution for $u$ is then given by (3.10). In order to calculate $T_1(vb)$ explicitly, we shall use, for the first time, assumption (5.2) and Proposition 3.2, applied to $M'$ instead of $M$, i.e., $\psi$ instead of $\phi$, $z'(\gamma, \bar{\gamma}; \zeta)$ instead of $z^0(\zeta)$, and $s'(\gamma, \bar{\gamma}, \zeta)$ instead of $s_0(\zeta)$, in (3.1) and (3.3). We conclude that $v(\zeta)\psi_{z'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta))$ extends holomorphically to the unit disc $\Delta$. Since $v$ is real, $v(\zeta)\psi_{\bar{z}'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta))$ extends antiholomorphically to $\Delta$. We make use of the following elementary observation: if $c(e^{i\theta})$ is defined on $S^1$ with $c(1) = 0$, then $T_1(c) = -ic$ if $c$ extends holomorphically to $\Delta$, and $T_1(c) = ic$ if $c$ extends antiholomorphically. Since

$$v(\zeta)\psi_{z'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta)) \frac{\partial z'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta)$$

extends holomorphically, while

$$v(\zeta)\psi_{\bar{z}'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta)) \frac{\partial \bar{z}'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta)$$

extends antiholomorphically, we have

$$T_1(vb)(\zeta) = -iv(\zeta)\psi_{z'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}; \zeta)) \frac{\partial z'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta) + iv(\zeta)\psi_{\bar{z}'}(z'(\gamma, \bar{\gamma}; \zeta), \overline{z'(\gamma, \bar{\gamma}; \zeta)}, s'(\gamma, \bar{\gamma}, \zeta)) \frac{\partial \bar{z}'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; \zeta). \tag{5.11}$$

We shall use (3.10) to evaluate $u(-1)$. Note first by using (4.7),

$$\frac{\partial z'_k}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; -1) = 0, \quad \text{and} \quad \frac{\partial \bar{z}'_k}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; -1) = \delta_{jk}, \quad 1 \leq j, k \leq n. \tag{5.12}$$

Using (3.10), (5.7), (5.9), (5.10), (5.11), (5.12), we obtain

$$u(-1) = \frac{\partial s'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; -1) = \frac{-i\psi_{\bar{z}_j}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))[1 - i\psi_{s'}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))]}{1 + \psi_{s'}^2(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))}$$

$$= \frac{-i\psi_{\bar{z}_j}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))}{1 + i\psi_{s'}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))}. \tag{5.13}$$

To complete the proof of (5.6), we use (4.7) and (5.13) to obtain

$$\frac{\partial}{\partial \bar{\gamma}_j}(i\psi(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))) = i\psi_{\bar{z}_j}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1)) + i\psi_{s'}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))\frac{\partial s'}{\partial \bar{\gamma}_j}(\gamma, \bar{\gamma}; -1)$$

$$= \frac{i\psi_{\bar{z}_j}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))}{1 + i\psi_{s'}(\gamma, \bar{\gamma}, s'(\gamma, \bar{\gamma}, -1))}. \tag{5.14}$$

This completes the proof of (5.6) and hence of Proposition 5.3. □

*End of proof of Theorem 2* Suppose that $\frac{\partial g}{\partial s}(0) = 0$. Applying Proposition 5.3, we obtain a complex hypersurface through $q'_0 = H(q_0) = A'(\gamma_0, \overline{\gamma_0}, -1)$, which would imply that $M'$ is not minimal at $q'_0$. However, by the choice of $q_0$ in Sect. 4, $M$ is of type 2 at $q_0$ and $H$ is a local diffeomorphism at $q_0$. Hence $M'$ is of type 2 at $q'_0$ and therefore minimal at $q'_0$. This contradicts the existence of such a complex hypersurface through $q'_0$ and completes the proof of Theorem 2. □

*Proof of Corollary 0.2* We let $M$, $M'$ and $H$ satisfy the hypotheses of Corollary 0.2. It is proved in [11, Corollary 0.4] that if $H$ is a germ of a CR map between two hypersurfaces $M$ and $M'$ of D-finite type, then either $H$ is constant or $\mathrm{Jac}\, H \not\equiv 0$. Hence, since D-finite type implies finite type, which implies minimal, we may apply Theorem 2 to obtain the result. □

## 6 Proofs of Theorems 3 to 7

*Proof of Theorem 3* Assume that $M$ and $H$ satisfy the hypotheses of Theorem 3, and assume that $p_0 = 0$. We may assume that neither (i) nor (ii) holds, and show that (iii) holds, i.e., we may assume that $\mathrm{Jac}\, H \not\equiv 0$ and we must show that $H$ is a local diffeomorphism at 0. Since $M$ is essentially finite at 0 (see precise definition below), it is of finite type at 0 and hence minimal there. By Theorem 1, $M$ is either minimally convex at 0 or every CR function extends to a full neighborhood of 0. We shall first assume $M$ is minimally convex at 0. By Theorem 2, since $\mathrm{Jac}\, H \not\equiv 0$, (0.1) holds. We shall now use some results from previous work of the authors [7]. For this, we introduce the notion of formal transversal coordinates and formal transversal components.

Let $M$ be a (germ of a) smooth hypersurface at 0 in $\mathbf{C}^{n+1}$ given by $\rho(Z, \bar{Z}) = 0$, where $\rho$ is a smooth, real-valued function defined in a neighborhood of 0 in $\mathbf{C}^{n+1}$ with $\rho(0) = 0$ and $d\rho(0) \neq 0$. By a *formal holomorphic change of variables* we shall mean a choice of $n+1$ formal holomorphic power series, with no constant terms, in $n+1$ indeterminates $\tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_{n+1})$ denoted $(Z_1(\tilde{Z}), \ldots, Z_{n+1}(\tilde{Z}))$ and satisfying $\det\left(\frac{\partial Z_j}{\partial \tilde{Z}_k}(0)\right) \neq 0$. One can show (by using the formal implicit function theorem) that if $\rho(Z, \bar{Z})$ denotes also the Taylor series of $\rho$ at the origin in $2n+2$ indeterminates $Z_1, \ldots, Z_{n+1}, \bar{Z}_1, \ldots, \bar{Z}_{n+1}$, there exists a formal holomorphic change of variables $Z(\tilde{Z})$ such that

$$\rho(Z(\tilde{Z}), 0) \sim \alpha(\tilde{Z})\tilde{Z}_{n+1}, \quad \alpha(0) \neq 0, \tag{6.1}$$

where $\alpha(\tilde{Z})$ is a formal holomorphic series. We write $\tilde{Z}_{n+1} = w$ and $(\tilde{Z}_1, \ldots, \tilde{Z}_n) = z$. Then $w$ is called a *transversal formal coordinate* for $M$. We also define the formal power series $\tilde{\rho}(z, w, \zeta, \tau) \sim \rho(Z(z, w), \bar{Z}(\zeta, \tau))$. We write $\rho(z, 0, \zeta, 0) \sim \sum a_\alpha(z)\zeta^\alpha$. Recall that the hypersurface $M$ is called *essentially finite* at 0 if

$$\dim_{\mathbf{C}} \mathbf{C}[[z]]/(a_\alpha(z)) = \operatorname{ess\,type}(M) < \infty, \tag{6.2}$$

where $(a_\alpha(z))$ denotes the ideal generated by the $a_\alpha(z)$ in the ring of formal power series in $n$ indeterminates $\mathbf{C}[[z]]$. Note that the notion of essential finiteness and the essential type of $M$ defined in (6.2) are independent of the choice of formal holomorphic coordinates.

Note that if $u$ is a CR function defined on $M$ near 0, and $(z, w)$ are formal coordinates as above, there is a (unique) formal holomorphic power series $\mathscr{U}(z, w)$ associated to the Taylor series of $u$ in an obvious way. Now let $H: M \to M'$ be a CR map between two hypersurfaces in $\mathbf{C}^{n+1}$ and assume that $(z, w)$ and $(z', w')$ are formal coordinates for $M$ and $M'$ respectively satisfying (6.1). In these coordinates we associate to $H$ $(n+1)$ formal holomorphic power series $\mathscr{H}(z, w) = (\mathscr{F}_1(z, w), \ldots, \mathscr{F}_n(z, w), \mathscr{G}(z, w))$. Then $\mathscr{G}(z, w)$ is called a *formal transversal component* of $H$. Recall that $H$ is of *finite multiplicity* at 0 if

$$\dim_{\mathbf{C}} \mathbf{C}[[z]]/(\mathscr{F}_1(z, 0), \ldots, \mathscr{F}_n(z, 0)) = \operatorname{mult}(H) < \infty, \tag{6.3}$$

where $(\mathscr{F}_1(z, 0), \ldots, \mathscr{F}_n(z, 0))$ denotes the ideal generated by the formal series $\mathscr{F}_j(z, 0)$. In this case, $\operatorname{mult}(H)$ is called the *multiplicity of* $H$ and is independent of the choices of formal coordinates for $M$ and $M'$. By [7, Theorem 5] if $M$ is essentially finite, and a transversal component $\mathscr{G}$ of $H$ does not vanish identically, then $M'$ is also essentially finite, $H$ is of finite multiplicity and the following multiplication formula holds:

$$\operatorname{ess\,type}(M) = (\operatorname{mult}(H))(\operatorname{ess\,type}(M')). \tag{6.4}$$

We may now proceed with the proof of Theorem 3 in the case where $M$ is minimally convex and (0.1) holds. As mentioned in Sect. 5, (0.1) is equivalent to (5.1), and hence $\frac{\partial \mathscr{G}}{\partial w}(0) \neq 0$. Here we have used the coordinates and notation introduced above, i.e., $\mathscr{G}$ is a formal transversal component of $H$. We may use (6.4) above with $M = M'$ to conclude that the multiplicity of $H$ is one. This is equivalent to the invertiblity of the matrix $\left(\frac{\partial F_j}{\partial z_k}(0)\right)$, which, together with $\frac{\partial \mathscr{G}}{\partial w}(0) \neq 0$ implies that $\operatorname{Jac} H(0) \neq 0$. Hence $H$ is a local diffeomorphism.

Assume now that $M$ is not minimally convex at 0, so that the CR mapping $H$ extends holomorphically to a full neighborhood of 0 in $\mathbf{C}^{n+1}$ by Theorem 1. Since $\operatorname{Jac} H \not\equiv 0$, by the holomorphy of $H$, it follows that the holomorphic function $\det \frac{\partial H}{\partial Z}(Z) \neq 0$, $(Z \in \mathbf{C}^{n+1})$, and hence its Taylor series at 0 vanishes at most of finite order. Since this nonvanishing property is clearly invariant by holomorphic and formal holomorphic changes of coordinates, we conclude that no formal transversal component of the CR map $H$ at 0 vanishes identically. By [7, Theorem 4], we conclude the stronger condition that $\frac{\partial \mathscr{G}}{\partial w}(0) \neq 0$, where we have used the same notation as above. The rest of the proof is now the same as that of the minimally convex case. □

*Proof of Theorem 5* Assume that $M$ is essentially finite at the origin with no germ of a complex manifold contained in $M$ through any point in some neighborhood of the origin, and that $H$ is a self CR map of $M$ with $H(0) = 0$. From the conclusion of Theorem 3, to complete the proof of Theorem 5, it suffices to show that (ii) of Theorem 3 does not hold, i.e., if $H$ is nonconstant, then $\operatorname{Jac} H \not\equiv 0$. By [11, Theorem 3], a CR mapping from an essentially finite hypersurface to another essentially finite hypersurface which does not contain germs of holomorphic manifolds, as above, is either constant or has Jacobian not identically zero. Hence Theorem 5 follows. □

*Proof of Theorem 4* Theorem 4 is an immediate consequence of Theorem 5. Indeed, D-finite type implies essentially finite (see [3]), and by the openness property proved in [2], a manifold of D-finite type cannot contain a germ of a complex manifold through any point. Hence all the hypotheses of Theorem 5 are satisfied. □

Theorem 5 is more general than Theorem 4. Indeed, an example of a hypersurface $M$ satisfying the assumptions of Theorem 5 but not of D-finite type is given in Sect. 7.

*Proof of Theorems 6 and 7* We recall first the following result proved in [6] concerning the holomorphic extendability of a CR map: If $M$ and $M'$ are real analytic hypersurfaces in $\mathbf{C}^{n+1}$ essentially finite at 0, and $H$ a smooth CR map from $M$ into $M'$, $H(0) = 0$, such that (0.1) holds, then $H$ extends holomorphically to a full neighborhood of 0 in $\mathbf{C}^{n+1}$. It is clear from Theorems 2 and 5 that these conditions are satisfied under the hypotheses of both Theorems 6 and 7 (unless $H$ is constant in Theorem 7). Hence Theorems 6 and 7 are proved. □

## 7 Remarks and examples

Let $M$ be the flat hypersurface in $\mathbf{C}^2$, i.e. the hypersurface given by $\{(z, w) \in \mathbf{C}^2 : \operatorname{Im} w = 0\}$. The mapping $H = (z, w^3)$ is a CR self map which is a bijection of $M$ onto itself, but which is not a local diffeomorphism at 0 although $\operatorname{Jac} H \not\equiv 0$ in any open set in $M$. Note that (0.1) fails for this mapping.

It follows from Theorem 4 that if $H$ is a CR self mapping from a hypersurface $M \subset \mathbf{C}^2$ into itself, then if $M$ is of finite type, $H$ is either constant or a local diffeomorphism. Indeed, it should be noted that in $\mathbf{C}^2$ the conditions of finite type, D-finiteness, and essential finiteness coincide. For a real analytic hypersurface, these notions also are the same as minimality, but a smooth hypersurface in $\mathbf{C}^2$ can be minimal without being of finite type. The following example shows that the condition of finite type cannot be replaced by that of minimal in Theorem 4.

*Example 7.1* Let $\phi(y)$ be a smooth real valued function defined on $\mathbf{R}$ with $\phi(y) > 0$ for $y < 0$ and $\phi(y) \equiv 0$ for $y > 0$. Consider the hypersurface in $\mathbf{C}^2$ given by $\{(z, w)$: $\operatorname{Im} w = \phi(y)\}$ where $z = x + iy$. It is easy to check that $M$ is minimal at 0, but not of finite type there. Since $\phi(\phi(y)) \equiv 0$ for $y \in \mathbf{R}$, the holomorphic mapping $\mathscr{H}(z, w) = (w, 0)$ restricted to $M$ is a CR self map of $M$, which is neither constant nor a diffeomorphism. Note also that $M$ is minimally convex, by Remark 1.8, but Theorem 2 does not apply since $\operatorname{Jac} H \equiv 0$.

For a real analytic hypersurface $M \subset \mathbf{C}^2$ which is of infinite type at 0, i.e. not minimal, but not flat, there is always a nontrivial CR self-map $H$ with $\operatorname{Jac} H \equiv 0$. Indeed, since $M$ is of infinite type, one can find local holomorphic coordinates $(z, w)$ such that $M$ is given near 0 by (1.1) with $\phi(z, \bar{z}, 0) \equiv 0$. It suffices then to take the self mapping to be $H = (z, 0)$. The following example shows that in the infinite type case the mapping need not be a diffeomorphism even if we restrict attention to the case where $\operatorname{Jac} H \not\equiv 0$.

*Example 7.2* Let $M$ be the hypersurface in $\mathbf{C}^2$ given by $\bar{w} = we^{i|z|^2}$. It is easy to check that this equation does define a real analytic real hypersurface which is neither of finite type nor flat at 0. The mapping $(z, w) \mapsto (\sqrt{2}z, w^2)$ restricts to a CR self map of $M$ with $\operatorname{Jac} H \not\equiv 0$ but for which (0.1) fails.

For $\mathbf{C}^{n+1}$ with $n > 1$ the conclusion of Theorem 5 fails to hold if $M$ is assumed to be only essentially finite (and hence (ii) of Theorem 3 can occur) as is shown by the following example.

*Example 7.3* Let $M \subset \mathbf{C}^2$ be given by $\operatorname{Im} w = |z_1|^2 - |z_2|^2$ and $H$ the CR self mapping given by $(f_1, f_2, g)$, with $f_1 = f_2 = f$ and $g \equiv 0$, where $f$ is any CR function defined on $M$. Note here that $M$ contains the 1-dimensional complex manifold defined by $w = 0$ and $z_1 = z_2$. It is, however, essentially finite with nondegenerate Levi form.

*Remark 7.4* In Example 7.3 $H(M)$ is contained in a submanifold of lower dimension in $M'$ and is therefore of measure. More generally, by using Sard's Theorem, we may replace the hypothesis $\operatorname{Jac} H \not\equiv 0$ in Theorems 2 and 6 by the assumption that there is a neighborhood $U$ of $p_0$ in $M$ such that $H(U)$ has positive measure in $M'$.

To show that Theorem 5 is more general than Theorem 4, as mentioned in the introduction, we give an example of a hypersurface satisfying the hypotheses of Theorem 5 but not of Theorem 4.

*Example 7.5.* Let $\theta(x)$ be a real-valued smooth function defined on $\mathbf{R}$ with the property that $\theta$ is flat at the origin (i.e. $\theta^{(j)}(0) = 0$ for all $j = 0, 1, \ldots$ ), and $\theta$ not real analytic at any $x \in \mathbf{R}$. (Such a function exists by the Baire Category Theorem.) Let $M$ be the hypersurface in $\mathbf{C}^3$ defined by

$$\operatorname{Im} w = x_1^2 - x_2^2 + \theta(x_1), \tag{7.6}$$

where the variables in $\mathbf{C}^3$ are $(z_1, z_2, w)$ and $z_j = x_j + iy_j$. $M$ is essentially finite in a neighborhood of 0, since it has a nondegenerate Levi form. However, it is not of D-finite type at 0, since the holomorphic manifold defined by $w = 0$, $z_1 = z_2$ is tangent of infinite order to $M$ at 0. We claim that there is no germ of a complex manifold contained in $M$ at any point. For this, we reason by contradiction. Assume that $\zeta \mapsto \gamma(\zeta) = (z_1(\zeta), z_2(\zeta), w(\zeta))$ is a holomorphic curve contained in $M$ defined near $\zeta = 0$ in $\mathbf{C}$, with $\gamma'(0) \neq 0$. We write $\zeta = \zeta + i\eta$ and $z_j(\zeta) = x_j(\xi, \eta) + iy_j(\xi, \eta)$ and $w(\zeta) = s(\xi, \eta) + it(\xi, \eta)$. Then we have, by (7.6),

$$t(\xi, \eta) = (x_1(\xi, \eta))^2 - (x_2(\xi, \eta))^2 + \theta(x_1(\xi, \eta)) \tag{7.7}$$

for $\zeta$ in a neighborhood of 0 in $\mathbf{C}$. Since it is easy to see that $x_1$ cannot be constant, since $\gamma'(0) \neq 0$, we obtain from (7.7), using the implicit function theorem, that $\theta$ must be real analytic at any point $x_1(\xi, \eta))$ at which a derivative of $x_1$ is nonzero.

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