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# A LOCAL HOPF LEMMA AND UNIQUE CONTINUATION FOR HARMONIC FUNCTIONS

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**0. Introduction.** Let $B$ be the open unit ball in the Euclidean space $\mathbf{R}^n$ centered at 0 and $S$ its boundary, the unit sphere. Let $\mathcal{O}$ be an open neighborhood in $\mathbf{R}^n$ of a point $x_0$ in $S$. Put $\Omega = \mathcal{O} \cap B$ and $V = \mathcal{O} \cap S$; assume that $\Omega$ is connected. A continuous function $u$ in $\overline{\Omega}$ *vanishes of infinite order at* $x_0$ if, for every positive integer $N$,

$$\lim_{\substack{x \in \Omega \\ x \to x_0}} \frac{u(x)}{|x - x_0|^N} = 0. \tag{0.1}$$

Similarly, $u$ *vanishes of infinite order in the normal direction at* $x_0$ if, for every $N$,

$$\lim_{\substack{0<t<1 \\ t \to 1}} \frac{u(tx_0)}{(1-t)^N} = 0. \tag{0.2}$$

The main result of this paper (Theorem 3) is that, if $u$ is harmonic in $\Omega$, $u$ is continuous in $\overline{\Omega}$, $u(x) \geqslant 0$ for $x \in V$, and $u$ vanishes of infinite order in the normal direction at $x_0$, then $u(x) \equiv 0$ for $x$ in some neighborhood of $x_0$ in $V$. Hence, if, in addition, $u$ vanishes of infinite order at $x_0$, then $u \equiv 0$ in $\Omega$ (Corollary 2.6).

The proof of Theorem 3 follows from other preliminary and somewhat stronger results. We show that if $u \in C^0(\overline{\Omega})$ is harmonic in $\Omega$ and vanishes of infinite order in the tangential direction at $x_0$, then $u$ admits an asymptotic expansion in the normal direction at $x_0$ (Theorem 1). If $u(x) \geqslant 0$ for $x \in V$, then the converse also holds (Theorem 2).

A somewhat weaker unique continuation result was previously obtained by the authors [2] for holomorphic functions in one complex variable at boundary points. (See also related results in the joint work of the authors with Alinhac [1] and in Huang-Krantz [4].) It should be noted that similar results also hold when $B$ is replaced by the open half space $\mathbf{R}^n_+$ and $x_0 \in \mathbf{R}^n$, its boundary. However, we give the proofs only in the case of the ball since the case of the half space is similar and somewhat simpler. We conjecture that similar results hold for more general domains and more general second-order elliptic operators.

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**1. Global results.** Let $w \in C^0(\bar{B})$ be real valued and harmonic in $B$. We write $f = w|_S$ and let $x_0 \in S$. We shall consider two conditions on the function $w$.

(A) For every positive integer $N$, the function $y \mapsto |f(y)||y - x_0|^{-N}$ is integrable on $S$.

(B) There is a sequence of real numbers $a_0, \ldots, a_j, \ldots$ such that, for every positive integer $N$, the following holds:

$$\frac{t^{(n-1)/2}}{1+t} w(tx_0) = \sum_{j=0}^{N} a_j \left(\frac{1-t}{\sqrt{t}}\right)^{2j+1} + O((1-t)^{2N+3}), \qquad 0 < t < 1, t \to 1.$$

In this section we shall prove the following.

PROPOSITION 1.1. *For $w$ as above, (A) implies (B) with*

$$a_j = \frac{(-1)^j M_j}{n\omega_n} \int_S \frac{f(y)}{|y - x_0|^{n+2j}} \, d\sigma(y), \tag{1.2}$$

*where $M_j = (1/j!)(n/2)(n/2 + 1)\cdots(n/2 + j - 1)$, $\omega_n$ is the volume of $B$, and $d\sigma(y)$ is the surface measure of $S$.*

PROPOSITION 1.3. *For $w$ as above, if $f(y) \geqslant 0$ for $y \in S$, then (B) implies (A).*

*Proof of Proposition 1.1.* By the Poisson formula (see, e.g., [3]) we have, for $x \in B$,

$$w(x) = \frac{1 - |x|^2}{n\omega_n} \int_S \frac{f(y)}{|x - y|^n} \, d\sigma(y). \tag{1.3}$$

Replacing $x$ by $tx_0$, $0 < t < 1$, we obtain, from (1.3),

$$w(tx_0) = \frac{1 - t^2}{n\omega_n} \int_S \frac{f(y)}{((1-t)^2 + t|y - x_0|^2)^{n/2}} \, d\sigma(y). \tag{1.4}$$

Put $s = (1 - t)/\sqrt{t}$. Then we have

$$\frac{t^{(n-1)/2}}{1+t} w(tx_0) = \frac{1}{n\omega_n} \int_S \frac{s}{(s^2 + |y - x_0|^2)^{n/2}} f(y) \, d\sigma(y). \tag{1.5}$$

Using the Taylor expansion, we have, for every positive $k$ and every positive integer $N$,

$$\begin{aligned} \frac{1}{(\sigma+1)^k} &= \sum_{j=0}^{N} (-1)^j \frac{k(k+1)\cdots(k+j-1)}{j!} \sigma^j \\ &\quad + (-1)^{N+1} \frac{k(k+1)\cdots(k+N)}{N!} \left( \int_0^1 \frac{(1-\tau)^N}{(\tau\sigma + 1)^{k+N+1}} \, d\tau \right) \sigma^{N+1}. \end{aligned} \tag{1.6}$$

Hence, we obtain from (1.6)

$$(1.7)\quad \frac{s}{(s^2+|y-x_0|^2)^{n/2}} = \sum_{j=0}^{N}(-1)^j\frac{(n/2)(n/2+1)\cdots(n/2+j-1)}{j!}\frac{s^{2j+1}}{|y-x_0|^{2j+n}}$$
$$+(-1)^{N+1}\frac{(n/2)(n/2+1)\cdots(n/2+N)}{N!}$$
$$\times\left(\int_0^1\frac{(1-\tau)^N}{(\tau s^2+|y-x_0|^2)^{n/2+N+1}}\,d\tau\right)s^{2N+3}.$$

By substituting (1.7) into (1.5), we obtain the desired expansion for (B) with (1.2), provided we show that the integral

$$(1.8)\qquad \int_S\int_0^1\frac{(1-\tau)^N}{(\tau s^2+|y-x_0|^2)^{n/2+N+1}}f(y)\,d\tau\,d\sigma(y)$$

is bounded independently of $s \geqslant 0$. The latter follows from the fact that the integrand in (1.8) is dominated by the function $|f(y)||y-x_0|^{-(n+2N+2)}$, which is integrable on $S$ by condition (A). The proof of Proposition 1.1 is complete. ■

*Proof of Proposition 1.3.* We shall assume that $f(y)\geqslant 0$ and that the expansion of (B) holds. We write $v(s)=(t^{(n-1)/2}/(1+t))w(tx_0)$, with $s=(1-t)/\sqrt{t}$, $s\geqslant 0$. Condition (B) can then be written

$$(1.9)\qquad v(s)=\sum_{j=0}^{N}a_js^{2j+1}+O(s^{2N+3}),\qquad s>0,\, s\to 0.$$

We shall show that the coefficients $a_j$ are necessarily given by (1.2), which will prove condition (A). For this we shall use an induction on $j$. Note that, by (1.9), $v(0)=0$ and $\lim_{s\to 0^+} v(s)/s=a_0$. On the other hand, from (1.5), we have, for $s>0$,

$$(1.10)\qquad \frac{v(s)}{s}=\frac{1}{n\omega_n}\int_S\frac{f(y)}{(s^2+|y-x_0|^2)^{n/2}}\,d\sigma(y).$$

Hence, by the assumption $f\geqslant 0$, we can use the monotone convergence theorem to conclude (1.2) for $j=0$. Now assume by induction that (1.2) holds for $j\leqslant N$; then we must prove it for $j=N+1$. By the induction formula and (1.7) we have

$$(1.11)\quad v(s)=\sum_{j=0}^{N}a_js^{2j+1}+(-1)^{N+1}\frac{M_{N+1}(N+1)}{n\omega_n}$$
$$\times\left(\int_S\int_0^1\frac{(1-\tau)^N}{(\tau s^2+|y-x_0|^2)^{n/2+N+1}}f(y)\,d\tau\,d\sigma(y)\right)s^{2N+3}.$$

Hence we have

$$(1.12)\qquad a_{N+1} = \lim_{s\to 0^+}\left(v(s) - \sum_{j=0}^{N} a_j s^{2j+1}\right)s^{-2N-3}$$

$$= \lim_{s\to 0^+} (-1)^{N+1}\frac{M_{N+1}(N+1)}{n\omega_n}$$

$$\times \int_S\int_0^1 \frac{(1-\tau)^N}{(\tau s^2 + |y-x_0|^2)^{n/2+N+1}} f(y)\,d\tau\,d\sigma(y).$$

Again making use of the monotone convergence theorem for the integral on the right-hand side of (1.12), we conclude

$$(1.13)\quad a_{N+1} = (-1)^{N+1}\frac{M_{N+1}(N+1)}{n\omega_n}\int_0^1 (1-\tau)^N\,d\tau\int_S \frac{f(y)}{|y-x_0|^{n+2N+2}}\,d\sigma(y).$$

Now (1.2) for $j = N+1$ follows immediately from (1.13), which completes the induction and hence the proof of Proposition 1.3. ■

**2. Local results.** Recall that if $(r, \theta)$ are the usual polar coordinates in $\mathbf{R}^n$, and $\Delta = \sum_i(\partial^2/\partial x_i^2)$ is the Laplacian, then

$$(2.1)\qquad r^2\Delta = r^{-n+3}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \Delta_\theta,$$

where $\Delta_\theta$ is the Laplacian on the sphere $S$. For $x \in \mathbf{R}^n\backslash\{0\}$, i.e., $0 < r < \infty$, let $s = (1-r)/\sqrt{r}$. Then a direct calculation left to the reader shows that, in the coordinates $(s, \theta)$, $s \in \mathbf{R}$, $\theta \in S$, the Laplacian acting on a function $h(x)$ is given by

$$(2.2)\quad \frac{4r^{(n+5)/2}}{(1+r)^3}\Delta h = \left[\frac{\partial^2}{\partial s^2} + \frac{3s}{s^2+4}\frac{\partial}{\partial s} + \frac{1}{s^2+4}(4\Delta_\theta - 3 + 4n - n^2)\right]\left(\frac{r^{(n-1)/2}}{1+r}h\right).$$

If $\Omega$ and $V$ are as in §0, then the following is an immediate consequence of (2.2).

LEMMA 2.1. *Let $h \in C^\infty(\overline{\Omega})$, harmonic in $\Omega$, and $h|_V = 0$; then, if $s = (1-r)/\sqrt{r}$, the following holds:*

$$\frac{\partial^{2j}}{\partial s^{2j}}\left(\frac{r^{(n-1)/2}}{1+r}h\right)\bigg|_V = 0, \qquad j = 0, 1, 2, \ldots.$$

Let $x_0$, $\Omega$, and $V$ be as in §0. We shall always assume in what follows that $u$ is in $C^0(\overline{\Omega})$, is real valued, and is harmonic in $\Omega$. In this section we shall prove local analogues of Propositions 1.1 and 1.3. First we introduce conditions similar to (A) and (B) of §1.

(A′) For every positive $N$, the function $y \mapsto |u(y)||y - x_0|^{-N}$ is integrable on $V$.

(B′) There is a sequence of real numbers $a_0, \ldots, a_j, \ldots$ such that, for every positive integer $N$, the following holds:

$$\frac{t^{(n-1)/2}}{1+t} u(tx_0) = \sum_{j=0}^{N} a_j \left(\frac{1-t}{\sqrt{t}}\right)^{2j+1} + O((1-t)^{2N+3}), \qquad t \to 1^-.$$

We have the following results.

THEOREM 1. *If $u$ satisfies (A′), then (B′) holds and there is $C \geqslant 0$ such that*

$$\left| a_j - \frac{(-1)^j M_j}{n\omega_n} \int_V \frac{u(y)}{|y - x_0|^{n+2j}} d\sigma(y) \right| \leqslant C^{j+1}, \qquad j = 0, 1, \ldots, \tag{2.3}$$

*where $M_j$ and $\omega_n$ are as in Proposition (1.1).*

THEOREM 2. *If $u(y) \geqslant 0$ for $y \in V$, then (B′) implies (A′).*

*Proof of Theorem 1.* Let $\chi \in C_0^\infty(\mathcal{O})$ with $0 \leqslant \chi(x) \leqslant 1$ and $\chi(x) = 1$ for $x$ near $x_0$. Let $f(x) = \chi(x)u(x)$ for $x \in S$, and let $w \in C^0(\bar{B})$ be the harmonic function in $B$ with $w|_S = f$. Note that the function $u - w$ is harmonic in $\Omega$ and vanishes in a neighborhood of $x_0$ in $S$ contained in $V$. Hence by a classical result (see, e.g., [5] and the references contained therein), $u - w$ is real analytic in a neighborhood of $x_0$ in $\bar{B}$. By Lemma 2.1, we conclude that we have, for $1 - t$ sufficiently small and $t < 1$,

$$\frac{t^{(n-1)/2}}{1+t} (u(tx_0) - w(tx_0)) = \sum_{j=0}^{\infty} b_j \left(\frac{1-t}{\sqrt{t}}\right)^{2j+1}, \tag{2.4}$$

where $|b_j| \leqslant C^{j+1}$ with some constant $C > 0$.

Note that $M_j \leqslant C'^{j+1}$ for some $C'$, depending only on $n$, and that

$$\left| \int_S \frac{f(y)}{|y - x_0|^{n+2j}} d\sigma(y) - \int_V \frac{u(y)}{|y - x_0|^{n+2j}} d\sigma(y) \right| \leqslant C''^{j+1}, \qquad j = 0, 1, \ldots, \tag{2.5}$$

for some $C''$, since $u$ and $f$ agree in a neighborhood of $x_0$ in $S$. Writing $u = (u - w) + w$, applying Proposition 1.1 to $w$, and using the remarks above, we obtain (B′) and (2.3) for some constant $C$. This completes the proof of Theorem 1. ∎

*Proof of Theorem 2.* Let $\chi$, $f$, and $w$ be as in the proof of Theorem 1. We write $w = (w - u) + u$. Again by Lemma 2.1 the expansion (2.4) holds. Hence, since $u$ satisfies (B′), $w$ satisfies condition (B) of §1. Since $f \geqslant 0$, we can apply Proposition (1.3) to conclude that (A) holds for $w$, which is equivalent to (A′). This proves Theorem 2. ∎

THEOREM 3. *If $u$ vanishes of infinite order in the normal direction at $x_0$ and $u(x) \geqslant 0$ for $x \in V$, then the following hold.*

(i) *$u$ vanishes identically in a neighborhood of $x_0$ in $V$.*

(ii) *$u$ vanishes identically in the normal direction at $x_0$, i.e., $u(tx_0) = 0$ for all $t \leqslant 1$ such that the segment $[tx_0, x_0]$ is contained in $\bar{\Omega}$.*

Before proving Theorem 3, we state some corollaries.

COROLLARY 2.6. *If $u$ vanishes of infinite order at $x_0$ and $u(x) \geqslant 0$ for $x \in V$, then $u$ vanishes identically in $\Omega$.*

*Proof of Corollary 2.6.* By Theorem 3, $u$ vanishes in a neighborhood of $x_0$ in $V$. It is then real analytic in a neighborhood of $x_0$ in $\bar{\Omega}$. Since $u$ vanishes of infinite order at $x_0$, it must vanish identically in $\Omega$. ■

The following corollaries are immediate consequences of the above results.

COROLLARY 2.7. *If the restriction of $u$ to $V$ reaches an extremum at $x_0$, then either $u$ is constant in a neighborhood of $x_0$ in $V$ or there exists a positive integer $N_0$ such that*

$$\limsup_{t \to 1^-} \frac{|u(tx_0) - u(x_0)|}{(1-t)^{N_0}} > 0.$$

COROLLARY 2.8. *If $u$ is nonconstant in $\Omega$, and the restriction of $u$ to $V$ reaches an extremum at $x_0$, then there exists a positive integer $N_0$ such that*

$$\limsup_{\substack{x \in \Omega \\ x \to x_0}} \frac{|u(x) - u(x_0)|}{|x - x_0|^{N_0}} > 0.$$

*Proof of Theorem 3.* Under the assumptions of Theorem 3, (B′) holds with $a_j = 0$ for $j = 0, 1, \ldots$. Since $u(y) \geqslant 0$ for $y \in V$, we can apply Theorem 2 to conclude that (A′) holds. Using Theorem 1, we conclude that (2.3) holds with $a_j = 0$. Hence, since $u|_V$ is nonnegative, we have

$$\frac{M_j}{n\omega_n} \int_V \frac{u(y)}{|y - x_0|^{n+2j}} \, d\sigma(y) \leqslant C^{j+1}, \qquad j = 0, 1, \ldots. \tag{2.9}$$

We reason now by contradiction. If $u|_V$ does not vanish in any neighborhood of $x_0$, then for every positive $\varepsilon$ sufficiently small we would have

$$\int_{y \in S, |y - x_0| < \varepsilon} u(y) \, d\sigma(y) > 0. \tag{2.10}$$

On the other hand, it follows from (2.9) and the nonnegativity of $u|_V$ that we have, for every $\varepsilon > 0$ sufficiently small,

$$\frac{1}{n\omega_n \varepsilon^{n+2j}} \int_{y\in S, |y-x_0|<\varepsilon} u(y)\, d\sigma(y) \leqslant C^{j+1}. \tag{2.11}$$

Taking the $j$th root of both sides of (2.11), making use of (2.10), and letting $j$ go to infinity, we obtain $1 \leqslant C\varepsilon^2$. Since $\varepsilon$ can be taken arbitrarily small, we reach a contradiction, which proves (i) of Theorem 3. To prove (ii), observe that, since (i) holds, $u$ is real analytic in a neighborhood of $x_0$ in $\bar{B}$; since it vanishes of infinite order in the normal direction at $x_0$, it must vanish identically on any segment $[tx_0, x_0]$ contained in $\bar{\Omega}$. This completes the proof of Theorem 3. ■

*Remark 1.* If, in Theorem 3, the condition $u|_V \geqslant 0$ is dropped, then (B′) no longer implies (A′) (e.g., take $n = 2$, $x_0 = (1, 0)$ and $u(x) = x_2$).

*Remark 2.* In Theorem 3, if the condition $u|_V \geqslant 0$ is replaced by the stronger condition $u(x) \geqslant 0$ for $x \in \bar{\Omega}$, then the conclusion of the theorem follows from the classical Hopf lemma (see, e.g., [5]). Similarly, in Corollary 2.6, if the condition $u|_V \geqslant 0$ is replaced by $u|_V \equiv 0$, then the conclusion follows immediately from the local real analyticity of $u$.

*Remark 3.* Note that the assumptions of Theorem 3 (and hence (i) and (ii) of Theorem 3) do not imply that $u$ must vanish identically as shown by the following example with $n = 2$ and $x_0 = (1, 0)$. Let $z = x + iy$, $x, y \in \mathbf{R}$, and

$$u(x, y) = \operatorname{Im}\left(\frac{1-z}{1+z}\right)^2 = \frac{4y(x^2+y^2-1)}{(1+x^2+y^2+2x)^2};$$

then $u$ is harmonic in $\mathbf{R}^2 \setminus \{(-1, 0)\}$ and vanishes on $\{y = 0\}$ as well as on the unit circle $\{x^2 + y^2 = 1\}$.

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