Analytic approximability of solutions
of partial differential equations

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1. Introduction

Let $P(x, D_x)$ be a linear partial differential operator with analytic coefficients defined in a neighborhood of a point $x_0 \in \mathbb{R}^n$. We shall call $P$ locally approximable at $x_0$ if for any distribution $u$ for which $P u = 0$ in a neighborhood of $x_0$, there is a neighborhood $U$ of $x_0$ and a sequence of distributions $u_j$ real analytic in $U$ such that

$$u_j \to u \quad \text{in} \quad U,$$

$$P u_j \equiv 0 \quad \text{in} \quad U.$$  

The property of local approximability was studied by Baouendi and Treves [2], who showed that $P$ is locally approximable at $x_0$ if its complex characteristics at $x_0$ are simple. Métivier [7] has proved approximability for a class of first order nonlinear equations. Baouendi and the second author [1] showed that any left invariant differential operator on a Lie group is locally approximable.

The class of locally approximable differential operators contains that of analytic hypoelliptic differential operators. (Recall that $P$ is analytic hypoelliptic at $x_0$ if $P u$ real analytic in a neighborhood of $x_0$ implies that $u$ is real analytic near $x_0$.) The notion of analytic hypoellipticity has been microlocalized in an obvious way, but the notion of microlocal approximability is less clear. In §2 we give a definition of microlocal approximability and also extend the definition of local approximability to pseudodifferential operators. These definitions are based on the constants for the Fourier–Bros–Iagolnitzer transform of a distribution (see e.g. [11]). We show that when $\text{char}_{x_0} P$ is contained in a line then local approximability is equivalent to microlocal approximability in all directions.

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In §3 we follow the method of Sjöstrand [10] and Helffer [4] (see also [3] as well as the references to Grušin’s work given in [10]) to study a class of differential operators with symplectic characteristic variety. For these we show that the question of microlocal approximability for \( P \) is equivalent to that of a system of analytic pseudodifferential operators in one variable. In §4 we use the machinery developed by Métivier [6] to prove the analyticity of the operators defined in §3. We refer the reader to [12] and [6], respectively for the definitions of classical analytic pseudodifferential operators of type \((1/2, 1/2)\).

In §5 we give the first example of a differential operator, not totally characteristic at \( x_0 \), which is not locally approximable at \( x_0 \). This operator is
\[
P = \frac{\partial^2}{\partial t^2} + t^2 \frac{\partial^2}{\partial y^2} + i \frac{\partial}{\partial y} + y
\]
on \( \mathbb{R}^3 \) with \( x_0 = (0, 0) \). The proof of non approximability uses the reduction to a pseudodifferential operator in one variable given in §3 and §4, and the connection between microlocal and local approximability proved in §2.

2. Microlocal analytic approximability

We shall microlocalize the definition of analytic approximability. Recall that for a distribution \( u \) defined near \( x_0 \in \mathbb{R}^n \), an \textit{FBI (Fourier—Bros—Jagannathan)} transform of \( u \) is an integral of the form
\[
\mathcal{F}u(x, \xi) = \int e^{i(x-y)\cdot \xi - |x-y|} \chi(y) u(y) \, dy,
\]
where \( \chi \in C_0^\infty (\mathbb{R}^n) \), \( \chi = 1 \) near \( x_0 \). Then \( u \) is \textit{analytic} at \( (x_0, \xi_0) \), \( \xi_0 \neq 0 \), or \( (x_0, \xi_0) \in WF_\mathcal{F}u \), if there is a conic neighborhood \( \Gamma \) of \( \xi_0 \), a neighborhood \( \mathcal{U} \) of \( x_0 \) and a constant \( C \) such that
\[
|\mathcal{F}u(x, \xi)| \leq Ce^{-|\xi|/C} \quad \text{for all} \quad (x, \xi) \in \mathcal{U} \times \Gamma.
\]
We shall write \( u \sim 0 \) at \( (x_0, \xi_0) \) if \( (x_0, \xi_0) \notin WF_\mathcal{F}u \). If \( \{u_j\} \) is a sequence of distributions, we write \( \{u_j\} \sim 0 \) at \( (x_0, \xi_0) \) if (2.2) holds for all \( u_j \) with the same \( C \), \( \mathcal{U} \) and \( \Gamma \).

(2.3) \textit{Definition.} A classical analytic pseudodifferential operator \( Q \) defined in a conic neighborhood of \( (x_0, \xi_0) \) is \textit{microlocally approximable} at \( (x_0, \xi_0) \) if for any distribution \( v \) for which \( Qv \sim 0 \) at \( (x_0, \xi_0) \) there is a sequence of distributions \( v_j \) such that
i) \( v_j \rightarrow v \)
ii) \( \{Qv_j\} \sim 0 \) at \( (x_0, \xi_0) \)
iii) \( v_j \sim 0 \) at \( (x_0, \xi_0) \) with the conic neighborhood independent of \( j \).
In order to show that these definitions make sense we need the following lemma. For an analytic pseudodifferential operator (either classical or of type \((\frac{1}{2}, \frac{1}{2})\)) we write \(R \sim 0\) at \((x_0, \xi_0)\) if \(R\) is of order \(-\infty\) in a conic neighborhood of \((x_0, \xi_0)\).

**Lemma.** If \(R\) is an analytic pseudodifferential operator and \(\{w_j\}\) is a sequence of distributions with \(\{w_j\} \sim 0\) at \((x_0, \xi_0)\) then \(\{Rw_j\} \sim 0\) at \((x_0, \xi_0)\). If \(R \sim 0\) and \(\{w_j\}\) is any bounded sequence of distributions, then \(\{Rw_j\} \sim 0\) at \((x_0, \xi_0)\).

**Proof.** The second statement may be proved by following the constants in the FBI transform. The first is also refinement of the statement that an analytic pseudodifferential operator does not extend the wave front set.

By abuse of notation we shall also write \(Q_1 \sim Q_2\) at \((x_0, \xi_0)\) if \(Q_1 - F_1 Q_2 F_2\) is of order \(-\infty\) at \((x_0, \xi_0)\), where \(F_1\) and \(F_2\) are elliptic pseudodifferential operators. By the above it follows that \(Q_1\) is microlocally approximable at \((x_0, \xi_0)\) if and only if \(Q_2\) is.

We may also generalize local approximability to locally defined analytic pseudodifferential operators as follows.

**Definition.** If \(Q\) is an analytic pseudodifferential operator defined in a neighborhood of \(x_0\) then \(Q\) is *locally approximable* at \(x_0\) if for every distribution \(v\) for which \(Qv\) is real analytic in a neighborhood \(\mathcal{U}\) of \(x_0\) there is a sequence \(v_j\) of functions, real analytic in a neighborhood \(\mathcal{U}'\) of \(x_0\), such that

i) \(v_j \rightarrow v\)

ii) \(\{Qv_j\}\) extends to a convergent sequence of holomorphic functions in a neighborhood of \(x_0\) in \(C^n\).

It is easy to check, using the Cauchy—Kovalevsky Theorem, that if \(Q\) is an analytic differential operator not totally characteristic at \(x_0\), then this definition agrees with that of Baouendi and Treves [2].

We write \(w \sim 0\) or \(\{w_j\} \sim 0\) or \(Q \sim 0\) at \(x_0\) if the equivalence defined above holds for every \(\lambda \in \mathbb{R}^n \setminus 0\). We note that if \(\{w_j\} \sim 0\) at \(x_0\), then (see, e.g., [11]) there is a neighborhood of \(x_0\) in \(C^n\) to which all the \(w_j\) extend as uniformly bounded holomorphic functions \(W_j\). Hence, by the well known theorem for holomorphic functions, the \(W_j\) have a convergent subsequence. This is the connection between the two conditions (ii) in the local and microlocal definitions of analytic approximability.

In a special case microlocal approximability in all directions is equivalent to local approximability.

**Theorem.** Let \(Q\) be a classical analytic pseudodifferential operator for which \(\text{char}_{x_0}Q\) is contained in a line. Then \(Q\) is locally approximable at \(x_0\) if and only if \(Q\) is microlocally approximable at \((x_0, \xi)\) for all \(\xi \in \mathbb{R}^n \setminus \{0\}\).
Proof. We may assume that at $x_0$, $Q$ is elliptic away from $\{(0, \ldots, 0, \xi_n) : \xi_n \neq 0\}$. Suppose first that $Q$ is microlocally approximable at $(x_0, \xi)$ for all $\xi$ and that $Qv \sim 0$ at $x_0$. Then $v$ is analytic at $(x_0, \xi)$, $\xi \neq (0, \ldots, 0, \pm 1)$ and by using appropriate cut-off functions we may write $v = v_1 + v_2$, $v_1 \sim 0$ except at $(x_0, 0, \ldots, 0, \xi_n)$, $\xi_n \neq 0$ and $v_2 \sim 0$ except at $(x_0; 0, \ldots, 0, \xi_n)$, $\xi_n < 0$. Since $Qv_1 + Qv_2 \sim 0$ at $x_0$ and $Qv_2 \sim 0$ at $(x_0; 0, \ldots, 0, 1)$ we have $Qv_1 \sim 0$. Hence we may assume $v \sim 0$ at $(x_0, \xi)$, $\xi \neq (0, \ldots, 0, 1)$. By assumption of microlocal approximability there is a sequence $\{v_j\}$ such that $v_j \rightarrow v$, $\{Qv_j\} \sim 0$ and $v_j \sim 0$ in a conic neighborhood $\Sigma \times \Gamma$ of $(x_0; 0, \ldots, 0, 1)$. We claim first that $\{v_j\} \sim 0$ at $(x_0, \xi)$, $\xi \neq (0, \ldots, 0, 1)$. Indeed, since $Q$ is elliptic at such points, there is an analytic pseudodifferential operator $P$ such that $PQ \sim I$ at $(x_0, \xi)$. The claim then follows by Lemma (2.4). Now we may find an analytic pseudodifferential operator $\Psi(D)$ such that $\Psi(D) \sim 0$ at $(x_0, \xi)$ if $\xi_n \neq 0$ and $\Psi(D) \sim I$ at $(x_0, \xi)$, $\xi$ near $(0, \ldots, 0, 1)$. Let $v_j = \Psi(D)\psi_j + (I - \Psi(D))v$. Then $v_j \rightarrow v$, since $\Psi(D)\psi_j - \Psi(D)v \sim 0$. Also, $\Psi(D)\psi_j \sim 0$ for all $j$ and $(I - \Psi(D))v \sim 0$, since $\Psi(D)v \sim v$ near $(x_0; 0, \ldots, 0, 1)$ and $v \sim 0$ at $(x_0, \xi)$, $\xi \neq (0, \ldots, 1)$. Finally,

$$\{Qv_j\} = \{Q\Psi(D)v_j\} + \{Q(I - \Psi(D))v\},$$

so it suffices to show $\{Q\Psi(D)v_j\} \sim 0$. We have

$$Q\Psi v_j = Q\Psi v_j + [Q, \Psi]v_j$$

and since $\{Qv_j\} \sim 0$, $\{\Psi Qv_j\} \sim 0$ by Lemma (2.4), while $\{[Q, \Psi]v_j\} \sim 0$ also by the lemma, because $[Q, \Psi]$ is of order $-\infty$ near $\text{char}_{x_0}(Q)$ and $\{v_j\} \sim 0$ away from that set. Hence, $Q$ is locally approximable at $x_0$.

For the converse, assume $Q$ is locally approximable at $x_0$ and let $v$ be such that $Qv \sim 0$ at $(x_0; 0, \ldots, 0, 1)$. We write $v = v_1 + v_2$ as above. Then $Qv_1 \sim 0$ at $x_0$, so there exists a sequence $v_j \rightarrow v_1$ so that $v_j \sim 0$ at $x_0$ and $\{Qv_j\} \sim 0$ at $x_0$. Since $v_2 \sim 0$ at $(x_0; 0, \ldots, 0, 1)$ we may take $v_j = v_2 + v_2$ which proves $Q$ is microlocally approximable at $(x_0; 0, \ldots, 0, 1)$. The proof of microlocal approximability at $(x_0; 0, \ldots, 0, -1)$ is the same, and it is clear that $Q$ is microlocally approximable at any noncharacteristic point.

3. A criterion of microlocal approximability for some differential operators

We consider here a differential operator of degree $m > n$ in the variables $(t, y) \in \mathbb{R}^n \times \mathbb{R}$ of the form

$$A(t, y, D_t, D_y) = \sum_{|\beta| + |\gamma| \leq m} a_{\beta\gamma}(t, y) D_t^\beta D_y^\gamma$$

where

$$a_{\beta\gamma}(t, y) = \sum_{|\beta| + |\gamma| \leq (m - |\beta| - |\gamma|)} a_{\beta\gamma}(t, y) D_t^\beta D_y^\gamma$$
is an analytic differential operator of degree \( \equiv (m-|\alpha|-|\beta|)/2 \) with \( a_{\alpha \beta \gamma \delta} \) constant for \( |\alpha|+|\beta| \leq m \) and \( |\gamma|= (m-|\alpha|-|\beta|)/2 \). We let

\[
P_m = \sum_{|\alpha|+|\beta| \leq m, |\gamma|= (m-|\alpha|-|\beta|)/2} a_{\alpha \beta \gamma \delta} D_{\alpha}^\gamma D_{\beta}^\delta.
\]

We shall assume the condition of transversal ellipticity, i.e., that

\[
\sum_{|\alpha|+|\beta| = m} a_{\alpha \beta \gamma \delta} \tau^\alpha \tau^\beta \neq 0 \quad \text{for all} \quad (\tau', \tau') \in \mathbb{R}^{2n} \setminus \{0\}.
\]

The operators considered here are in a more restricted class than those studied in [4], [6] and [10]. Following the approach of Grušin and that of [4] and [10] (for the \( C^\omega \) case) we shall reduce the question of microlocal analytic hypoellipticity and microlocal approximability to that of a system of pseudodifferential operators.

We first use a result from [3] applied to \( P_m \). We fix a point \((0; 0, \ldots, \eta_0)\) in \( \text{char } P_m \) (determined by \( \eta_0 > 0 \) or \( \eta_0 < 0 \)) and let

\[
P_m(\eta) = \sum_{|\alpha|+|\beta| = m, |\gamma|= (m-|\alpha|-|\beta|)/2} a_{\alpha \beta \gamma \delta} \eta^{\alpha+\beta} D_{\gamma}^\delta.
\]

Then \( P_m(\eta) P_m(\eta) \) has a kernel in \( L^2(\mathbb{R}^n) \) of finite dimension \( q_1 \) and \( P_m(\eta) P_m^*(\eta) \) a kernel of dimension \( q_2 \). In [3] it is shown that there exist microlocal systems of analytic pseudodifferential operators \( J_1, J_2, Q, L \) defined

\[
J_1: \mathcal{S}'(R^+_{t, y}) \to \mathcal{D}'(R^+_{t, y}), \quad i = 1, 2
\]

\[
Q: \mathcal{S}'(R^+_{t, y}) \to \mathcal{D}'(R^+_{t, y})
\]

\[
L: \mathcal{S}'(R^+_{t, y}) \to \mathcal{D}'(R^+_{t, y})
\]

such that

\[
\begin{pmatrix} P_m & J_3 \\ J_1 & 0 \end{pmatrix} \begin{pmatrix} Q & J_1 \\ J_3 & -L \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & I_{q_1} \end{pmatrix}
\]

and

\[
\begin{pmatrix} Q & J_1 \\ J_3 & -L \end{pmatrix} \begin{pmatrix} P_m & J_3 \\ J_1 & 0 \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & I_{q_1} \end{pmatrix}
\]

near \((0; 0, \ldots, \eta)\). Here the analyticity of the above systems means that if \( u \in \mathcal{S}'(R^+_{t, y}) \) and \( \eta \in \mathcal{S}'(R^+_{t, y}) \) then

\[
WF_a(Qu) \subset WF_a(u),
\]

\[
WF_a(Lu) \subset WF_a(u),
\]

\[
WF_a(J_1 v) \subset \{(t, y; \tau, \eta)| t = \tau = 0, (y, \eta) \in WF_a(v)\},
\]

\[
WF_a(J_1^* u) \subset \{(y, \eta)|(0, y; 0, \eta) \in WF_a(u)\}.
\]

By the construction of Helffer [4], following Sjöstrand [10], there exist \( C^\omega \)-microlocal operators \( E, E^+, E^- \) and \( E^\pm \) so that if

\[
\mathcal{E} = \begin{pmatrix} E & E^+ \\ E^- & E^\pm \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} P & J_3 \\ J_1 & 0 \end{pmatrix}
\]
then $\mathcal{D} \sim I$, $\mathcal{D} \sim I$ in the $C^n$ sense. In fact, these operators may be obtained from

$$\mathcal{D}_0 = \begin{pmatrix} Q & J_1 \\ J^*_1 & -L \end{pmatrix}$$

by a Neumann series: by (3.3) $\mathcal{D}_0 = \mathcal{I} + \mathcal{A}$, where

$$\mathcal{A} = \begin{pmatrix} (P - P_m)Q & (P - P_m)J_1 \\ 0 & 0 \end{pmatrix}$$

and formally

$$\mathcal{D}^{-1} = \left( \mathcal{D}_0^{-1} \right) = \left( \sum_{j=0}^{\infty} (-1)^j ((P - P_m)Q)^j - \sum_{j=0}^{\infty} (-1)^j ((P - P_m)Q)^j (P - P_m)J_1 \right)$$

so, $\mathcal{D} = \mathcal{D}_0 (\mathcal{D}_0^{-1})^{-1}$. It is shown in [4] that the operators $E$, $E^+$, $E^-$, $E^\pm$ can be obtained from (3.5) in this fashion and that they have the appropriate $C^n$ behavior. In particular, $E^\pm$ is semiclassical and $E$ is of type $\left( \frac{1}{2}, \frac{1}{2} \right)$. We shall show, using the machinery of Métivier [6] that they are also analytic.

(3.6) Theorem. Let $P$ be of the form (3.1), transversally elliptic and such that $a_{\alpha\beta}v_0$ is constant for $|\alpha| + |\beta| \leq m$, $|\gamma| = (m - |\alpha| - |\beta|)/2$. Let $\omega = (0; 0, \ldots, 0, \eta)$. Then the operators $E$, $E^+$, $E^-$, $E^\pm$ are all microlocally analytic at $\omega$. $P$ is microlocally analytic hypoelliptic at $\omega$ if and only if $E^\pm$ is at $\omega' = (0; \eta) \in \mathbb{R} \times (\mathbb{R} \setminus 0)$. Furthermore, $P$ is microlocally approximable at $\omega$ if and only if $E^\pm$ is microlocally approximable at $\omega'$.

Proof. The first statement will be proved in § 4. We shall assume it here. The statement about microlocal analytic hypoellipticity is proved by standard arguments (see [1], [4] and [10]). For the statement about microlocal approximability, suppose first that $P$ is microlocally approximable at $\omega$ and that $E^\pm v \sim 0$ at $\omega'$. If $v = E^+ v$ then $Pv \sim 0$ at $\omega$ since $PE^+ + J_2 E^\pm \sim 0$. Let $\{w_j\}$ be such that $w_j \rightarrow w$ and $\{Pw_j\} \sim 0$ at $\omega$, and set $v_j = J^*_1 w_j$. Since $E^- P + E^\pm J^*_1 \sim 0$ and $\{Pw_j\} \sim 0$, $\{E^\pm J^*_1 w_j\} \sim 0$, that is, $\{E^\pm v_j\} \sim 0$ at $\omega'$. Finally, if $v_0 = v - J^*_1 E^+ v$ then $v_0 \sim 0$ at $\omega'$, $v_j + v_0 \sim 0$ at $\omega'$ and $v_j + v_0 \sim 0$.

Conversely, if $E^\pm$ is microlocally approximable at $\omega'$ and $Pv \sim 0$ at $\omega$, let $v = J^*_1 u$. Then $Pv \sim 0$ at $\omega$ and $E^- P + E^\pm J^*_1 \sim 0$ imply $E\pm v \sim 0$ at $\omega'$. Let $v_j \rightarrow v$, $\{E^\pm v_j\} \sim 0$ at $\omega'$, $u_j \rightarrow E^+ v_j$. Then $\{Pu_j\} \sim 0$ at $\omega$ since $PE^+ + J_2 E^\pm \sim 0$. Since $E^+ P + E^\pm J^*_1 \sim I$, $E^+ J^*_1 u = u - u_0$ with $u_0 \sim 0$ at $\omega$ so $u_j + u_0 \rightarrow E^+ v_j + u_0 \sim 0$. This completes the proof of theorem.
4. Proof of the analyticity of $E$, $E^+$, $E^-$, $E^\pm$

We shall base the proof of the analyticity of the operator $\mathcal{E}$ on Métivier [6], where it is proved an operator of the form (3.1) (in fact, those in a more general class) has a left analytic parametrix if $\ker P_m(\eta)$ is trivial. In what follows we use the material contained in Chapter II and III of [6] and will recall only essential definitions and results as needed to make this section readable.

After some analysis of the Neumann series expansion of $(\mathcal{P}\mathcal{E}_0)^{-1}$ we see that the only thing to do is find the sequence of operators $Q_j$, $j=0, 1, \ldots$ defined by taking $Q_0=Q$, $Q_j=-Q\hat{P}Q_{j-1}$, $j\geq 1$, where $\hat{P}=P(t, \gamma, D_1, D_2)-P_m(D_2)(t, D_1)$. If this is done and we let $E=E_1J_1$, $E^+=-EPJ_1+J_1$, $E^-=J_3^*-J_2^*\hat{P}E$ and $E^\pm=E-E\hat{P}J_1$ then using (3.3) and the definition of $E$ one sees that at least formally the matrix $\mathcal{E}$ thus obtained is a right inverse for $\mathcal{P}$. We shall show that $E$ can be found in this way and that it is an analytic operator.

We begin by recalling from [6] that for a nonnegative integer $k$, the space $\mathcal{H}^k(\mathbb{R}^n)$ consists of those $u\in H^k(\mathbb{R}^n)$ such that $u\in H^k(\mathbb{R}^n)$, where $H^k$ is the usual Sobolev space based on $L^2(\mathbb{R}^n)$. In $\mathcal{H}^k(\mathbb{R}^n)$ we have the norm $\|\cdot\|_k$ given by $\|u\|_k=\sup_{T_i}|T_i u|$, where the sup is taken over all $T_i=T_1\ldots T_k$, each $T_i$ being either a multiplication by a $t_j$ or a partial derivative $\partial_{t_j}$. The dual of $\mathcal{H}^k$ is $\mathcal{H}^{-k}$. For a positive real number $R$ and a nonnegative integer $m\mathcal{L}_R^m(\mathbb{R}^n)$ is the space of operators $K: \mathcal{P}^p\rightarrow \mathcal{P}^p$ such that for $p=0, \ldots, m+|\gamma|$ (ad $T$) $K: \mathcal{H}^{-p}\rightarrow \mathcal{H}^{-p+m+|\gamma|}$ with norm $\|K\|_{\mathcal{L}_R^m}$ for some $C$ independent of $\gamma$ and $p$. The best of such constants $C$ is denoted by $\|K\|_{\mathcal{L}_R^m}$.

In this definition, $\gamma$ is a multi-index $(\gamma_1, \ldots, \gamma_m)$, ad $T$ $K=$ $\partial_{t_1}^{\gamma_1}\ldots$ $\partial_{t_n}^{\gamma_m}$ $K$ and as usual (ad $T$) $(K)=$ $TK-KT$.

One has (see [6])

$$\|\text{ad } T^r K\|_{\mathcal{L}_R^m} \leq \sqrt{|\gamma|! (2R)^{|\gamma|}} \|K\|_{\mathcal{L}_R^m}.$$  

With this one can show: If $K\in \mathcal{L}_R^m$ then its left symbol $\sigma(K)(r, \varphi)=e^{-br}K(e^{br})$ satisfies

$$|r^x \partial_{\varphi}^\delta \partial_{\varphi}^\beta \sigma(K)(r, \varphi)| \leq C_0 (CR)^{|x|+|\beta|+|\gamma|+|\delta|} \sqrt{x! \beta! \gamma! \delta!} \|K\|_{\mathcal{L}_R^m}$$  

if $|x|+|\beta| \leq m+|\gamma|+|\delta|$, for some $C_0$ and $C$.

Now let $\varphi_1: \mathbb{R}^n\times \mathbb{R}^n\rightarrow \mathbb{R}$ be given by $\varphi_1(t, s)=t_3+s_3^2$ if $t_3s_3 \geq 0$, $\varphi_1(t, s)=t_3^2-s_3^2$ if $t_3s_3 \leq 0$. For $\varepsilon>0$ $B_\varepsilon$ is the space of operators $K: L^2(\mathbb{R}^n)\rightarrow L^2(\mathbb{R}^n)$ whose Schwartz kernel, denoted as $K(t, s)$, satisfies

$$\|e^{\varphi_1(t, s)}K(t, s)\|_{L^2} < \infty, \quad \|e^{\varphi_1(t, s)}K(t, -s)\|_{L^2} < \infty, \quad j = 1, \ldots, n.$$  

$K(t, -s)$ is essentially the Schwartz kernel of $K$, the operator given by $K\hat{u}=(Ku)^\ast$. The norm $\|K\|_s$ of $K$ is the maximum of the numbers in (4.2).
Proposition 2.10 of [6] states that if \(0 < \varepsilon' \leq \varepsilon \leq 1\) then \(K \in B_\varepsilon\) implies (ad \(T\)) \(K \in B_{\varepsilon'}\), where \(T\) is either \(t_j\) or \(\partial_t j\), and

\[
\|(\text{ad } T) K\|_{\varepsilon'} = \left(\frac{M_\varepsilon}{\varepsilon - \varepsilon'}\right)^{1/2} \|K\|_{\varepsilon},
\]

with \(M_\varepsilon\) independent of \(\varepsilon, \varepsilon'\) and \(K\). It follows from this, Plancherel's theorem and the fact that \(B_\varepsilon \subset L^2(\mathbb{R}^{2n})\) with norm independent of \(\varepsilon\) that if \(K \in B_\varepsilon, \varepsilon \leq 1\) and \(\varepsilon' < \varepsilon\), the left symbol

\[
\sigma(K)(r, \varrho) = e^{-ir\varrho} \int e^{is\varrho} K(r, s) \, ds
\]

of \(K\) satisfies

\[
\|\partial_\varrho^\alpha \partial_\eta^\beta \sigma(K)\|_{L^2} \leq C \left(\frac{M_\varepsilon}{\varepsilon}\right)^{|\alpha| + |\beta|/2} (|\alpha| + |\beta|)^{(|\alpha| + |\beta|)/2} \|K\|_\varepsilon
\]

with \(C\) independent of \(\alpha, \beta, \varepsilon, \varepsilon'\) and \(K\). Therefore

\[
|\partial_\varrho^\alpha \partial_\eta^\beta \sigma(K)(r, \varrho)| \leq C \left(\frac{M_\varepsilon}{\varepsilon}\right)^{|\alpha| + 1 + (|\alpha| + |\beta|)/2} (|\alpha| + |\beta|)^{(|\alpha| + |\beta|)/2} \|K\|_\varepsilon
\]

with another constant \(C\).

Let \(\Omega\) be a complex neighborhood of \((0, y_0) \in \mathbb{R}^n \times \mathbb{R}\) and \(\Gamma_0 \subset \mathbb{R}^n \times \mathbb{R} \setminus 0\) a conic neighborhood of \((0, \eta_0)\) such that \((\tau, \eta) \in h_0 \Rightarrow \eta \sim |\eta|\). We shall assume \(\eta_0 > 0\). For \(c > 0\) small let \(\Gamma = \{(\tau, \eta) \in C \times C: \text{Re}(\tau, \eta) \in \Gamma_0, |\text{Im}(\tau, \eta)| < c|\text{Re}(\tau, \eta)|\}\.\]

Modifying slightly the definition given in [6] we let \(G^\mu_\varepsilon(\Omega \times \Gamma)\) be the space of holomorphic functions \(a: \Omega \times \Gamma \rightarrow B_\varepsilon\) such that

\[
\|a(t, y, \tau, \eta)\|_\varepsilon \leq C |\text{Re}(\tau, \eta)|^\mu \quad \text{if} \quad (t, y, \tau, \eta) \in \Omega \times \Gamma.
\]

Similarly \(F^\mu_\varepsilon(\Omega \times \Gamma)\) consists of the holomorphic functions \(a: \Omega \times \Gamma \rightarrow \mathcal{L}^0_\varepsilon\) such that (4.5) holds with the \(\mathcal{L}^0_\varepsilon\)-norm.

(4.6) Lemma. For \(j = 0, 1, \ldots\) let \(A_j \in F^{m - j/2}_\varepsilon(\Omega \times \Gamma)\) (respectively \(G^{m - j/2}_\varepsilon(\Omega \times \Gamma)\)) be such that

\[
\|A_j(t, y, \tau, \eta)\| \leq C_\varepsilon R^j \|\text{Re}(\tau, \eta)|^{m - j/2}
\]

where the norm is that of \(\mathcal{L}^0_\varepsilon\) (respectively \(B_\varepsilon\)) and let \(a_j(t, y, \tau, \eta, r, \varrho) = \sigma(A_j(t, y, \tau, \eta))(r, \varrho)\). Then there exist \(C_1, R_1\) such that for \((t, y) \in \Omega, (\tau, \eta) \in \Gamma \subset \subset \Gamma_0, (r, \varrho) \in \mathbb{R}^{2n}\):

\[
|\tau^\alpha \varrho^\beta \partial_\varrho^\delta \partial_\eta^\gamma a_j(t, y, \tau, \eta, r, \varrho)| \leq C_1 |\alpha|! |\beta|! |\gamma|! |\delta|! |\mu|!
\]

\[
\times R^{|\alpha| + |\beta| + |\gamma| + |\mu| + |\eta|^{m - j/2} - |\eta|^{m - j/2}}
\]

for any \(\alpha, \beta, \ldots, \mu\) with \(|\alpha + \beta| = |\gamma + \delta|\) (respectively \(\alpha = \beta = 0\)). If \(\chi_j \in C^{\infty}(\mathbb{R}^{2n})\) is a sequence such that \(0 \leq \chi_j \leq 1, \chi_j(\tau, \eta) = 0\) if \(|(\tau, \eta)| < j, \chi_j(\tau, \eta) = 1\) if \(|(\tau, \eta)| > 2j\)
and \( |\partial_x^\alpha \partial_{\eta}^\beta \chi_j(\tau, \eta)| \leq C^{[|\alpha| + |\beta|]} \) when \(|\alpha| + |\beta| = j\) for some \(C\) independent of \(j\) and if \(a = \sum \chi_j((\tau, \eta) / |\lambda|) a_j\) then for some \(C_2, \lambda, R_2\) and indices \(\alpha, \ldots, \mu\) as above (respectively \(\alpha = \beta = 0\))

\[
|r^\alpha q^\beta \partial_x^\gamma \partial_{\eta}^\delta \partial_x^\alpha \partial_{\eta}^\beta a(t, y, x, \tau, \eta, r, q) | \leq C_2 \sqrt{\alpha! \beta! \gamma! \delta! 0! \mu!} \times R_2^{\alpha + \beta + \gamma + \delta + |\beta| |\eta| m - |\alpha| - |\beta|}
\]

if \((t, y, \eta) \in \Omega, (\tau, \eta) \in \Gamma_1\) \(\) and \(|(\tau, \eta)| > R_2(|\theta| + \mu)\).

For the proof of the lemma we only point out that (4.8) is a consequence of (4.7), (4.1) (respectively (4.4)) and Cauchy’s integral formula, and (4.9) follows from (4.8) by the argument given in the proof of Lemma 3.2 of [6].

**Lemma.** If \(a: \Omega \times \Gamma_1 \to C\) satisfies (4.9) for some \(C_3, R_2\) and all \(\alpha, \ldots, \mu\) with \(|\alpha| + |\beta| \leq |\gamma| + |\delta|\) (respectively \(\alpha = \beta = 0\)) then there are constants \(C_3, R_3\) such that for \(|t| + |\eta| > R_3(|\theta| + \mu)\)

\[
|\partial_x^\gamma \partial_{\eta}^\delta \partial_x^\alpha \partial_{\eta}^\beta a(t, y, x, \tau, \eta, \lambda^{1/2} r, \lambda^{-1/2} q) | \leq C_3 \sqrt{\alpha! \beta! \gamma! \delta! 0! \mu!} \times R_3^{\alpha + \beta + \gamma + \delta + |\beta| |\eta| m - |\alpha| - |\beta|}
\]

if \((t, y, \eta) \in \Omega, (\tau, \eta) \in \Gamma_1, \lambda > 0, r \in C^a \) and \(|q| < c \lambda\). For some small \(c\). (Respectively

\[
|\partial_x^\gamma \partial_{\eta}^\delta \partial_x^\alpha \partial_{\eta}^\beta a(t, y, x, \tau, \eta, \eta^{1/2} r, \eta^{-1/2} r) | \leq C_3 \sqrt{\alpha! \beta! \gamma! \delta! 0! \mu!} \times R_3^{\alpha + \beta + \gamma + \delta + |\beta| \eta m - |\alpha| - |\beta| - |\gamma| - |\delta|}
\]

if \((t, y, \eta) \in \Omega, (\tau, \eta) \in \Gamma, r \in C^a \) and \(|\text{Re} r| < 1\). In particular, for \((t, y, x, \tau, \eta) \in \text{Re} \Omega \times \Gamma_1 a(t, y, x, \tau, \eta, \eta^{1/2} r, \eta^{-1/2} r)\) is an analytic symbol of type \((1/2, 1/2)\).

This Lemma is a direct consequence of the estimates (4.9).

Let us now consider the operator

\[
P_m(\eta)(r, D_r) = \sum_{|\alpha| + |\beta| \leq m} a_{\beta \gamma \delta \eta} r^{\alpha} D_r^\beta
\]

of the previous section with the ellipticity condition (3.2). It satisfies \(P_m(\eta)(r, D_r) = \eta^{m/2} P_m(1)(\eta^{1/2} r, \eta^{-1/2} D_r)\). Let \(\Pi_1: L^2 \to L^2\) be the orthogonal projection onto the kernel of \(P_m(1)\) as an operator on \(L^2\) and let \(I - \Pi_2: L^2 \to L^2\) be the orthogonal projection onto the range, which is known to be closed; \(\Pi_1\) and \(\Pi_2\) have finite range (see [3] and references given there) because of the ellipticity of \(P_m(1)\), which implies also the existence of an operator \(K: L^2 \to \mathcal{H}_{m}\) such that \(P_m K = I - \Pi_2, KP_m = I - \Pi_1, K \in \mathcal{L}_m\) for some \(m > 0\) so \(K \in B_{\epsilon_0}\) for some \(\epsilon_0 > 0\) (see [6]) since \(m > n\). Let \(k(r, q)\) be the symbol of \(K\). Then \(\eta^{-m/2} k(\eta^{1/2} r, \eta^{-1/2} r)\) is an analytic symbol of type \((1/2, 1/2)\) and order \(-m/2\).

We will now begin the construction of the operators \(Q_j\) mentioned at the beginning of this section. Let \(g(\tau, \eta) \in C^\infty(\mathbb{R}_+^{n+1})\) be supported in a conic neighborhood of \((0, \eta_0)\) contained in \(\Gamma_0\) such that \(g(\tau, \eta) = 1\) if \(|(\tau, \eta)| \leq 1\) and \((\tau, \eta) \in \Gamma_1 \subset \subset \Gamma_0\).
another conic neighborhood of \((0, \eta_0)\), vanishing near 0 and such that
\[
|\partial_{\eta}^\mu g(\tau, \eta)| \lesssim C^{|\alpha| + \mu} (|z|/(\tau, \eta))^{\alpha(|\alpha| + \mu)}
\]
as in Lemma 3.1 of [6].

The operator \(Q\) appearing in \(\mathcal{O}_0\) in § 3 is given by
\[
Qf(t, y) = (2\pi)^{-n-1} \int e^{it(s-y) + i(y-y')s} g(s, \eta)\eta^{-m/2} \times k(\eta^{1/2}, \eta^{-1/2}) f(s, y') ds dy' d\tau d\eta
\]
where \(g\) is as in (4.14) with \(1/2 < q < 1\). If \(\bar{P}(t, y, D_t, D_y) = P(t, y, D_t, D_y) - P_m(D_y)(t, D_y)\) its left symbol can be written as \(\bar{P}(t, y, \tau, \eta, \eta^{1/2}, \eta^{-1/2})\) where \(\bar{P}(t, y, \tau, \eta)(r, \theta)\) is a polynomial in \((r, \theta)\) of degree \(\leq m\) with coefficients which are holomorphic functions in a complex conic neighborhood \(\Omega \times \Gamma\) of \((0, \eta_0, 0, \eta_0)\) and bounded there by \(C|\text{Re}(\tau, \eta)|^{m/2-1}\).

Since the symbol of \(Q\) is of type \((1/2, 1/2)\) and that of \(\bar{P}\) of type \((1, 0)\) the usual formula for the symbol of the composition holds (see [6]). Using it we obtain, after some rearrangement of terms, that \(Q \bar{P}\) has a symbol of the form \(h(t, y, \tau, \eta, \eta^{1/2}, \eta^{-1/2})\) and for \((t, y, \tau, \eta)\) close to \((0, 0, 0, \eta_0)\)
\[
h(t, y, \tau, \eta, \eta, r, \theta) \sim \sum_{k=0}^{m} \sigma(H_k(t, y, \tau, \eta))(r, \theta)
\]
where
\[
H_k(t, y, \tau, \eta)(r, D_t, D_y) = \eta^{-m/2} \sum_{2\mu + |\alpha| = k} \sum_{|\mu| \leq k} C_{\alpha, \mu, \nu} J \times [(\text{ad}(-ir))^{|\nu|} L^1 K](r, D_t) \circ D_t^\mu D_y^\nu \bar{P}(t, y, \tau, \eta, r, D_y).
\]
Here \(L = 1/2(\sum r_j \text{ad} \partial_j + i\partial_j \text{ad} r_j)\) and \(|C_{\alpha, \mu, \nu, \eta}| \leq 2^{m/2 + k + \mu} \times (\eta - j)/|\theta|!\mu!\). Since \(K \in \mathcal{L}_m^m R_\eta^m\) there exists \(R_1, C_1\) such that \((\text{ad} r)^\eta L^1 K \in \mathcal{L}_m^m\) with norm bounded by \(C_1|\theta|!|J|(C_1 R_1)^{|\alpha| + k} \times K\|_{L_\eta^m}\) and with this and the fact that \(\bar{P}\) is a polynomial in \((r, \theta)\) of degree \(\leq m\) whose coefficients satisfy specific bounds one shows \(H \in \mathcal{F}^{m/2 - 1} R_\eta^m\) with
\[
\|H(t, y, \tau, \eta)\|_{L_\eta^m} \leq C_0 R_0 \sqrt{\int |\text{Re}(\tau, \eta)|^{-m/2}},
\]
for some \(C_0, R_0\). Let \(W \in \mathcal{L}^m R_\eta^m\) be such that
\[
\|W(t, y, \tau, \eta)\|_{L_\eta^m} \leq C_0 R_0 \sqrt{\int |\text{Re}(\tau, \eta)|^{-m/2}}.
\]
Let \(h_j = \sigma(H_j), w_j = \sigma(W_j)\) and set
\[
h(t, y, \tau, \eta, r, \theta) = g(\tau, \theta) \sum \chi_{j+1}(\tau, \eta) h_j(t, y, \tau, \eta, r, \theta)
\]
(likewise \(w(t, y, \tau, \eta, r, \theta)\)) with \(\chi_j\) chosen as in lemma (4.6) and \(g\) as in the definition of \(Q\). For a symbol such as \(h\) or \(w\) let
\[
\text{op}(h)f(t, y) = (2\pi)^{-n-1} \int e^{it(s-y) + i(y-y')s} h(t, y, \tau, \eta, \eta^{1/2}, \eta^{-1/2}) f(s, y') ds dy' d\tau d\eta.
\]
We also write \( \text{op}(\Sigma H_j) \) for this operator. Of course different realizations of the formal symbol \( \Sigma H_j \) yield operators differing only by an analytic-regularizing operator near \((0, y_0, 0, \eta_0)\).

\[(4.18)\] Proposition. Let \( h \) and \( w \) be as above. Then \( H = \text{op}(h) \) and \( W = \text{op}(w) \) are analytic pseudodifferential operators in a neighborhood \( \Omega_0 \) of \((0, y_0) \in \mathbb{R}^{n+1} \).
If \( \varphi \in C_0(\Omega_0) \) and \( \varphi = 1 \) near \((0, y_0) \) then \( H\varphi W = C \) where \( C \) is an operator which in a neighborhood of \((0, y_0, 0, \eta_0) \) is of the form \( \text{op}(c) \) with

\[(4.19)\]

\[
c(t, y, \tau, \eta, \eta^{1/2} t, \eta^{-1/2} \tau) = \sum_{\alpha \beta} \frac{(2\pi)^{-n}}{\alpha! \beta! \gamma! \delta!} \int e^{i(t-s)(\tau-\sigma)} \eta^{\beta/2} \times \partial_\eta^{\alpha}[h(t, y, \tau, \xi^{1/2} \eta, \xi^{-1/2} \tau)] \times (D_\tau D_\rho + \sigma) W(t, y, \tau, \eta, \eta^{1/2} s, \eta^{-1/2} \sigma) \; ds \; d\tau.
\]

Outline of Proof. The analyticity of the operators \( H \) and \( W \) follows from Lemma 4.10 above and Lemma 3.3 of [6]. The expansion of the symbol of the composition is obtained by taking the Taylor expansion of \( w(s, y', \sigma, \eta, r, \varphi) \) in \( s \) at \( s = t \) and that of \( h(t, y, \tau, \xi, \xi^{1/2} \eta, \xi^{-1/2} \tau) \) in \( \tau = \sigma \) and \( \xi = \eta \). After carrying out the \( \eta \) derivatives in (4.19) we see that \( c \sim \sum_0 c_k \) with \( c_k(t, y, \tau, \eta, r, \varphi) \) the left symbol of \( C_k \in G_\varepsilon^{r-k/2-1} \) (\( \varepsilon < \varepsilon \), see (4.3)) given by

\[(4.20)\]

\[
C_k = \sum_{\alpha \beta} \frac{1}{\alpha!} H_{\alpha \beta \gamma \delta}(t, y, \tau, \eta) \partial_{\tau}^{\gamma} \partial_{\rho}^{\delta} \big(-i \text{ad } r\big)^{\beta/2} W(t, y, \tau, \eta)
\]

where the sum extends over the indices \( \alpha, \beta, \gamma, \delta, i, j \) such that \( |\alpha| + |\beta| + 2(|\gamma| + |\delta|) + i + j = k \) and \( H_{\alpha \beta \gamma \delta} \in F_0^{r-k/2-1} (\mathcal{S} \times \mathcal{R}) \) for some \( R > 0 \) with

\[(4.21)\]

\[
\|H_{\alpha \beta \gamma \delta}(t, y, \tau, \eta)\|_{L_2} \equiv \sqrt{\prod_{i,j} \| \partial_{\tau}^{\gamma} \partial_{\rho}^{\delta} C_0^{(i+1+|\beta|+2|\gamma|+|\delta|)+j+1} \times \text{Re}(\tau, \eta)\|^{r/2-k/2-1}}
\]

in \( \mathcal{S} \times \mathcal{R} \), for some \( C_0 \), because of (4.16). Here \( \mathcal{S} \times \mathcal{R} \) is any sufficiently small neighborhood of \((0, y_0, 0, \eta_0) \) in \( \Omega \times \Gamma \).

Now Proposition 2.9 of [6] states that given \( R \) there are \( \varepsilon_0 \) and \( C > 0 \) such that \( H \in \mathcal{S}_0^0 \) and \( W \in \mathcal{B}_\varepsilon \) implies \( HW \in \mathcal{B}_\varepsilon \) and

\[(4.22)\]

\[
\|HW\|_{\varepsilon} \leq C \|H\|_{L_2} \|K\|_{L_2}.
\]

Using this, the bounds (4.21) for the \( H_j \) and (4.17) for the \( W_j \) one shows easily that the \( C_k \) satisfy the bounds (4.17) also, with \( m'-1 \) in place of \( m' \).

The expansion (4.19) uses only \( \varphi' \Delta' \) behavior with \( \varphi'<\Delta \), see Lemma (4.10), and because of this, the estimates for the \( C_k \) and Lemma (4.10) again, the proofs of Propositions 3.6 and 3.7 of [6] give, with minor modifications, the proof that \( C \) has the expansion stated.
We will now find the operator \( \Sigma Q_f \) mentioned at the beginning of this section by finding an operator \( W \) such that \((I+Q P)W=Q\) in a neighborhood of \((0,\gamma_0,0,\eta_0)\). \( Q \) is given by (4.15). If \( Q P=\text{op}(\Sigma H_j) \) where \( H_j \in F^H_{p-1}(\Omega \times \Gamma) \) satisfies (4.16) in a neighborhood of \((0,\gamma_0,0,\eta_0)\) and \( W=\text{op}(\Sigma W_f) \) with \( W \in G^{m/2-\eta} \) satisfying (4.17), then \((I+Q P)W=Q\) if \( W_0 \) is given by
\[
W_0(t, \gamma, \tau, \eta) = \eta^{-m/2} K(r, D_e)
\]
and \( W_{k+2} \) is the operator \( C_k \) in (4.21) above, by proposition (4.18). These conditions in turn determine the \( W_k \) and we will show using standard techniques and Propositions 2.9 and 2.10 of M'étivier [6], that the \( W_i \) obtained in this manner satisfy (4.17) in a neighborhood \( \Omega \times \Gamma \) of \((0,\gamma_0,0,\eta_0)\) in which the estimates (4.21) already hold.

Let \( M_0 \equiv 1 \) be the constant in (4.3), \( \varepsilon_0=(0,\gamma_0) \), \( 0<\varepsilon_0 \equiv 1 \) such that \( K \in B_{\varepsilon_0} \), let \( 0<r<1/M_0 \) be such that the polydisc with radius \( \varepsilon_0 r \) and center \( z_0, D(z_0, \varepsilon_0 r) \), is contained in \( \Omega \) and let \( \varepsilon_0 = D(z_0, \varepsilon_0 r) \). Suppose that

\[
||W_i(t, \gamma, \tau, \eta)||_e \leq M_1 \sum_{\gamma+i=1} \frac{(v/2+1)^{v/2+1/2}}{\mu^{\gamma/2}(\varepsilon_0 - \varepsilon)} \left| \text{Re}(\tau, \eta) \right|^{-\eta/2 - m/2}
\]

(4.23) holds for some \( M_1 \), all \( \varepsilon<\varepsilon_0 \), all \( (t, \gamma, \tau, \eta) \in \Omega_0 \times \Gamma \) when \( i \equiv k \). (4.23) is already true if \( i=0 \). Using Proposition 2.10 of [6] (outlines in (4.3) above) we get

\[
||\partial_\tau^{v+i} \partial_\eta^{j} (\text{ad } r)^k W_{i}||_e \leq M_1 \sum_{\gamma+i=1} \frac{\left[(v+|\beta|+2(|\gamma|+|\delta|)/2+|\alpha|+|\mu|+1)^{(v+|\beta|+2(|\gamma|+|\delta|)/2+|\alpha|+1)}}{\mu^{\gamma/2}(\varepsilon_0 - \varepsilon)} \left| \text{Re}(\tau, \eta) \right|^{-\eta/2 - 1}
\]

in \( \Omega_0 \times \Gamma \) if \( \varepsilon<\varepsilon_0 \). Using now the definition of \( W_{k+2} \), Proposition 2.10 of [6] (equation (4.22) here) and the estimates (4.21) we obtain

\[
||W_{k+2}||_e \leq M_1 \sum_{\mu+i=1} \frac{(\mu/2+s)^{v/2+s}}{\mu^{\gamma/2}(\varepsilon_0 - \varepsilon)} \left| \text{Re}(\tau, \eta) \right|^{-\eta/2 - 1}
\]

where the inner sum extends over the indices \( \alpha, \beta, \gamma, \delta, i, j, \nu, l \) such that \( v+|\beta|+2(|\gamma|+|\delta|)+j=\mu, \ |\alpha|+l=s, \ v+i \equiv k \equiv \eta - |\beta| - j - 2(|\gamma|+|\delta|) \). The inner sum, divided by

\[
\sum_{\nu+i=1} \frac{(M_0 e)^{0/2+p}}{\varepsilon_0 - \varepsilon} \frac{0/2+p)^{v/2+p}}{p^{v/2+p}}
\]

and
will be less than \( \frac{1}{C} \) if \( M_1 \) is large compared with \( C_0 \), since
\[
\frac{s^{1/2} p^{1/3}}{|\alpha|^2 |s|^{1/2} |\theta|^2 (\theta/2 + p)^{2/3}} \leq 1
\]
if \( s = |\alpha| + t (\leq k) \), \( p = s \) and \( \theta + p \leq k + 2 \), and \( M_0 c/|\alpha_0 - s| \leq e \). This shows that 
\( W_k \) satisfies (4.23) for all \( k \).

Now let \( W = \text{op} \left( \sum_0^\infty W_j \right) \). Because \( (I + Q^0)W = Q \), we have \( W = Q - Q^0 W \) and
\[ P_mW = (I - \Pi)(I - \bar{P}W) \]
near \( (0, \eta, 0, \eta_0) \), where \( \Pi = J_2 J_2^* \), because of (3.3). With these properties of the operator \( W \) and (3.3) one verifies that if \( E = W, E^+ = J_1 - W \bar{P}J_1, E^- = J_2^* - J_2 \bar{P}E \) and \( E^\pm = -E^\mp \bar{P}J_1 \) then \( \mathscr{P} \) is a right inverse for \( \mathcal{P} \) near \( (0, \eta, 0, \eta_0) \). These operators are compositions of analytic operators (in the sense of the previous section) so they are analytic. Finally, \( \mathcal{P}^* \) also has an analytic right inverse near \( (0, \eta, 0, \eta_0) \) which is then given by \( \sigma^* \) near that point. This completes the proof of Theorem (3.6).

5. An example of a locally nonapproximable differential operator

(5.1) Theorem. The operator
\[
P = \partial^2/\partial t + t^2 \partial/\partial y^2 + i \partial/\partial y + y
\]
is not locally approximable in any neighborhood of 0. In fact, a solution \( v \) of \( P v \sim 0 \) is approximable at 0 if and only if \( v \sim 0 \) at 0.

To prove the theorem we first use Theorem (2.6) to reduce the problem of approximability at 0 to that of microlocal approximability in all directions at 0. We claim that \( P \) is not microlocally approximable at \( \omega = (0, 0, 0, -1) \) in \( T^* \mathbb{R}^2 \). By Theorem (3.6) this will follow if \( E^\pm \) is not microlocally approximable at \( \omega' = (0, 0, -1) \in T^* \mathbb{R}^2 \). We shall show that \( E^\pm \) is equivalent, in a neighborhood of \( \omega' \), to the operator with symbol \( y \) via conjugation by elliptic analytic operators. Assume this has been done. The following lemma will then complete the proof of the first statement of the Theorem.

(5.3) Lemma. Let \( Q \) be an analytic pseudodifferential operator on \( \mathbb{R} \) with total symbol \( y \) on \( \eta < 0 \). If \( Qv \sim 0 \) at \( \omega' = (0, 0, -1) \) and there is a sequence of distributions \( \{v_j\} \) such that \( v_j \sim v, v_j \sim 0, (Qv_j) \sim 0 \) at \( \omega' \) then \( v \sim 0 \) at \( \omega' \). Hence \( Q \) is not microlocally approximable at \( \omega \).

The proof is given below. To prove the second statement of the theorem we observe that if \( P v \sim 0 \) and \( v \) is approximable at 0 then \( J_2^* v \sim 0 \) at \( \omega' \) by Lemma (5.3) so \( v \sim 0 \) at \( \omega \). But \( v \sim 0 \) also at all points \( (0, 0; \tau, \eta) \) with \( \tau \neq 0 \) or \( \eta > 0 \) since \( P \)
is analytic hypoelliptic at these points (see Métivier [6] for the points with \( \tau = 0 \)). Thus, \( v \sim 0 \) at 0. This completes the proof of the theorem.

**Proof of Lemma (5.3).** Let \( v \) and \( v_j \) be as stated, \( w = yv \), \( w_j = yv_j \). Then \( \{Qv_j\} \sim 0 \) at \( \omega' \) and Lemma (2.4) imply \( \{w_j\} \sim 0 \) at \( \omega' \) so there is a neighborhood \( \mathcal{U} \) of 0 in \( \mathbb{R} \) and \( a > 0 \) such that \( w \) and the \( w_j \) extend as holomorphic functions \( W \) and \( W_j \) to \( \Omega = \mathcal{U} + i(0, a) \subset \mathbb{C} \), and \( |W_j(z)| \leq C |\text{Im} \, z|^{-N} \) for all \( j \) and \( z \in \Omega \) and some \( C, N \) independent of \( j \). Passing to a subsequence we may assume that the \( W_j \) converge uniformly locally in \( \Omega \), to \( W \) since \( v_j \to v \). Now each \( v_j \), being analytic at \( \omega' \), is the boundary value of a holomorphic function defined on a set \( \Omega + i(0, a_j) \), \( a_j > 0 \), and \( yv_j = w_j \) implies \( zV_j = W_j \) on the common domain. Thus the \( V_j \) can be taken to be defined on \( \Omega \), the sequence \( \{V_j\} \) converges uniformly locally there and we have estimates of the form \( |V_j(z)| \leq C |\text{Im} \, z|^{-N - 1} \), \( C, N \) independent of \( j \). Thus the limit \( V \) also satisfies such an estimate, and has as boundary value at \( \text{Im} \, z = 0 \) the distribution \( v \), again because \( v_j \to v \). Thus \( v \sim 0 \) at \( \omega' \). This proves the lemma.

(5.4) **Lemma.** \( E^\pm \) is a classical pseudodifferential operator with symbol \( y + r(y, \eta) \) on \( \eta < 0 \), where \( r \) is of order \(-1\).

**Proof.** The analysis of the operator \( P \) in the \( C^\infty \) category carried out using the techniques of Helffer [4] and Proposition 3.2.2 there give that \( E^\pm \) is a classical pseudodifferential operator, since \( \partial_t^2 - t^2 \eta^2 - \eta \) is a selfadjoint second order operator for \( \eta \) real and the eigenvectors, for \( \eta < 0 \), are even. It only remains to find the principal symbol. For our operator \( P \) we have \( J_1 = J_2 = J \), \( J^* = J^* \), where

\[
Jv(t, y) = (2\pi)^{-1} \pi^{-1/4} \int_{\eta < 0} e^{i(y-y')\eta} |\eta|^{1/2} e^{-t|\eta|^{1/2}} v(y') \, dy' \, d\eta
\]

\[
J^* f(y) = (2\pi)^{-1} \pi^{-1/4} \int_{\eta > 0} \int_{\mathbb{R}} e^{i(y-y')\eta} |\eta|^{1/2} e^{-t|\eta|^{1/2}} f(t, y') \, dt \, dy' \, d\eta
\]

if \( v \in C_0^\infty (\mathbb{R}) \) and \( f \in C_0^\infty (\mathbb{R}^2) \). Since \( \bar{P} = y, E^\pm = J^* y J - J^* y E y J \). But \( J^* y E y J \) has order \(-1\) while \( J^* y J = y + [J^*, y] J \) and \([J^*, y] J \) has order \(-1\). Thus \( E^\pm = y + \text{op}(r) \) with \( r \) classical of order \(-1\) as stated.

(5.5) **Lemma.** Let \( R \) be a classical analytic pseudodifferential operator of order \(-1\) in a neighborhood of 0 in \( \mathbb{R} \). Then there exist analytic elliptic pseudodifferential operators \( F_1, F_2 \) such that

\[
F_1 (y + R) F_2 \sim y.
\]

**Proof.** It follows from a known result [8] in \( \mathbb{R}^n \) that one can find \( F_1', F_2' \) with \( F_1'(y + R) F_2' \sim \text{op}(y + c n \eta^{-1}) \), \( c \) constant (see Lebeau [5], Théorème 1.4). Hence it suffices to take \( R = \text{op}(c n \eta^{-1}) \). Now take \( F_1 \) with symbol \( \eta^{-1} c \) and \( F_2 = F_1^{-1} \) to get (5.6). This proves the lemma.
References


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