ANALYTIC APPROXIMATION FOR HOMOGENEOUS SOLUTIONS OF INVARIANT DIFFERENTIAL OPERATORS ON LIE GROUPS

M. S. Baouendi* Linda Preiss Rothschild**

0. Introduction and Statements of Results.

A classical result by Malgrange [3] states that if $P(D)$ is a differential operator with constant coefficients in $\mathbb{R}^n$, then any solution u of the homogeneous equation $P(D)u = 0$ is a limit of exponential-polynomials solutions of the same equation.

Suppose now that $P(x,D)$ is a differential operator with analytic coefficients in an open set of $\mathbb{R}^n$. Assume that the principal symbol is nowhere identically zero. It is natural to ask the following question:

Is it true that any solution of $P(x,D)u = 0$ is locally a limit of real analytic solutions of the same equation?

The answer to this question is not known. However an affirmative answer is given in Baouendi-Treves [2] when P has *simple* (complex) characteristics. (See also [1] for first order overdetermined systems). We prove in this paper that the answer is also affirmative for left invariant operators defined on a general Lie group.

Theorem 1. *Let* L *be a left invariant differential operator defined on a Lie group* G. *For every open set* $U \subset G$, *neighborhood of the identity* $e \in G$, *there exists another open neighborhood of* e, $W \subset G$, *such that if* u *is a distribution on* G $(u \in \mathcal{D}'(G))$ *satisfying* $Lu = 0$ *in* U, *then there exists a sequence* u_ν *of real analytic functions defined in* W *and satisfying*:

(i) $Lu_\nu = 0$ *in* W
(ii) $\lim u_\nu = u$ *in* $\mathcal{D}'(W)$.

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Furthermore if u is of class C^k, $k \geq 0$, then the convergence in (ii) is in $C^k(W)$.

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$, the Lie algebra of G. If α is a multi-index, $\alpha \in \mathbb{Z}_+^n$, as usual set

$$|\alpha| = \sum_{j=1}^{n} \alpha_j \; , \; X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n} \; .$$

Note that a left invariant differential operator on G is of the form

$$L = \sum_{|\alpha| \leq m} a_\alpha X^\alpha \; , \qquad a_\alpha \in \mathbb{C} \; . \tag{0.1}$$

We can state a somewhat more general result than Theorem 1. Consider a differential operator on $(-T,T) \times G$, $(T > 0)$, of the form

$$P = \partial_t^m + \sum_{\substack{|\alpha|+j \leq m \\ j<m}} a_{j,\alpha}(t) X^\alpha \partial_t^j \; , \tag{0.2}$$

where $a_{j,\alpha}$ are real analytic functions defined on $(-T,T)$.

Theorem 2. Let P be a differential operator on $(-T,T) \times G$ of the form (0.2). For every open set $U \subset G$, neighborhood of e, there exists W, another open neighborhood of e, and $\varepsilon \in (0,T)$, such that if $u \in \mathcal{D}'((-T,T) \times G)$ and satisfies $Pu = 0$ in $(-T,T) \times U$, then there exists a sequence u_ν of real analytic functions in $(-\varepsilon,\varepsilon) \times W$ satisfying

(i) $Pu_\nu = 0$ in $(-\varepsilon,\varepsilon) \times W$,

(ii) $\lim u_\nu = u$ in $\mathcal{D}'((-\varepsilon,\varepsilon) \times W)$.

Furthermore if u is of class C^k, then the convergence in (ii) is in $C^k((-\varepsilon,\varepsilon) \times W)$.

I. Proof of Theorem 1.

Before starting the proof we need to introduce some notation. Denote by dg a right Haar measure on G. If $f, h \in L^1(G,dg)$ define the convolution $f*h$ by the integral

(I.1)
$$(f*h)(x) = \int_G f(xg^{-1})h(g)dg.$$

If we set

(I.2)
$$\check{f}(x) = f(x^{-1}), \qquad \forall x \in G,$$

then making the change of variable $g' = gx^{-1}$, we also get

(I.3)
$$(f*h)(x) = \int_G \check{f}(g)h(gx)dg.$$

Note that if f is a smooth function defined in an open neighborhood V of the identity e, and h is a distribution with compact support in V^{-1} then (I.1) (or (I.3)) is defined for x in an open neighborhood W of e (depending only on V and the support of h, we may take W satisfying $W(\text{supp } h)^{-1} \subset\subset V$).

If L is a left invariant operator on G, using (I.3) we see that

(I.4)
$$L(f*h) = f*(Lh).$$

Recall that $X_1,\ldots,X_n$ is a basis of $\mathfrak{g}$. Let V be a sufficiently small open neighborhood of the identity in G such that the exponential map Exp is an analytic diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto V. For simplicity we assume $V = V^{-1}$. For $x \in V$ we may write

$$x = \text{Exp}(s_1X_1 + \ldots + s_nX_n) = \text{Exp}(s.X)$$

with $s = (s_1,\ldots,s_n) \in \mathbb{R}^n$. The map

(I.5)
$$S: V \to \mathbb{R}^n, \quad S(x) = s,$$

is then an analytic diffeomorphism of V onto a neighborhood $\tilde{V}$ of the origin in $\mathbb{R}^n$.

There exists an analytic function σ, $\sigma \neq 0$, defined in $\tilde{V}$ such that if u is, say a continuous function with compact support in V, then

(I.6)
$$\int_G u(g)dg = \int_{\mathbb{R}^n} u(S^{-1}(t))\sigma(t)dt.$$

For $\nu \in \mathbb{Z}_+$ and $x \in V$, set

(I.7) $$f_\nu(x) = \left(\frac{\nu}{\sqrt{\pi}}\right)^n \sigma(0)^{-1} e^{-\nu^2(S(x))^2}.$$

(If $s \in \mathbb{R}^n$, $s^2 = \sum_{j=1}^{n} s_j^2$). Note that f_ν is an analytic function defined in V and satisfies $\check{f}_\nu = f_\nu$.

<u>Lemma 1</u>. <u>Let</u> h <u>be a distribution with compact support in</u> V. <u>There is an open neighborhood of</u> e, $W \subset G$, <u>depending only on the support of</u> h, <u>such that</u>

$$\lim_{\nu \to \infty} (f_\nu * h)|_W = h|_W \quad \underline{\text{in}} \quad \mathcal{D}'(W).$$

<u>Moreover if</u> h <u>is in</u> C^k <u>then the convergence is in</u> $C^k(W)$.

<u>Proof</u>: Let W_1 be an open neighborhood of the support of h satisfying

$$\overline{W}_1 \subset V.$$

We may choose an open neighborhood W of e in G satisfying

(I.8) $$W.W_1^{-1} \subset\subset V.$$

(Recall that $V = V^{-1}$).

Assume first that h is a continuous function (with compact support in W_1). Using (I.3), (I.7) and the fact that $\check{f}_\nu = f_\nu$ we get for $x \in \overline{W}$.

$$(f_\nu * h)(x) = \left(\frac{\nu}{\sqrt{\pi}}\right)^n \sigma(0)^{-1} \int_G e^{-\nu^2(S(g))^2} h(gx)dg ,$$

and making use of (I.5) and (I.6), we obtain for $x \in \overline{W}$

$$(f_\nu * h)(x) = \left(\frac{\nu}{\sqrt{\pi}}\right)^n \sigma(0)^{-1} \int_{R^n} e^{-\nu^2 s^2} h((\mathrm{Exp}\ s.X)x)\sigma(s)ds.$$

Changing variables in the latter $(\nu s = t)$ yields

(I.9) $$(f_\nu * h)(x) = \frac{\sigma(0)^{-1}}{\pi^{n/2}} \int_{R^n} e^{-t^2} h((\mathrm{Exp}\ \tfrac{t}{\nu}\ .X)x)\sigma\left(\tfrac{t}{\nu}\right)dt .$$

A limiting argument in (I.9) easily shows that $(f_\nu * h)|_{\overline{W}}$ converges uniformly to $h|_{\overline{W}}$.

If in addition h is of class C^k, $k > 0$, since we have

$$X^\alpha(f_\nu * h) = f_\nu * (X^\alpha h), \qquad \forall \alpha \in \mathbb{Z}_+^n ,$$

we also get the convergence in $C^k(\overline{W})$.

Assume now that h is a distribution with compact support in W_1. Let $\phi \in C_0^\infty(W)$. Since $V = V^{-1}$ we get from (I.8)

$$W_1 . W^{-1} \subset\subset V .$$

Therefore it follows from the first part of the proof of this lemma that $f_\nu * \phi$ converges to ϕ in $C^\infty(W_1)$. On the other hand, using (I.1) and (I.3) we have

$$\int_G (f_\nu * h)(x)\phi(x)dx = \int_G h(g)(f_\nu * \phi)(g)dg.$$

This shows that $f_\nu * h$ converges to h in $\mathcal{D}'(W)$. Q.E.D.

Lemma 2. If the open set V in (I.5) is small enough then for every pair of open neighborhoods of e, V_0 and V_1, $V_1 \subset\subset V_0 \subset\subset V$, there exists an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^n$ such that if h is a distribution with compact support in V_0, and $h \equiv 0$ in V_1, then for every $\nu \in \mathbb{Z}_+$,

$$(f_\nu * h) \circ S^{-1}$$

extends holomorphically to $\mathcal{O}$, and converges uniformly to zero in $\mathcal{O}$ as $\nu \to \infty$.

Proof: Let us first state the Baker-Campbell-Hausdorff formula in a form which will be needed further (see Varadarajan [4] for example). For $s, t \in \mathbb{R}^n$ sufficiently small we have

(I.10) $$\mathrm{Exp}(s.X).\mathrm{Exp}(-t.X) = \mathrm{Exp}(u.X)$$

with $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, and for $j = 1, \ldots, n$, (I.11) $$u_j = u_j(s,t) = s_j - t_j + \sum_{\substack{|\alpha|\geq 1\\ |\beta|\geq 1}} c_{\alpha,\beta,j} t^\alpha s^\beta ,$$

where $c_{\alpha,\beta,j} \in \mathbb{R}$ and satisfy

$$|c_{\alpha,\beta,j}| \leq M^{|\alpha|+|\beta|+1} .$$

Let V be the open set in (I.5) $(V = V^{-1})$. We may assume that V is small enough so that for all $x, g \in V$, if

$$S(x) = s, \quad S(g) = t ,$$

then the power series (I.11) is absolutely convergent.

Now let $h \in \mathcal{E}'(V_0)$, $h \equiv 0$ in V_1, with $V_1 \subset\subset V_0 \subset\subset V$. Using (I.1) and (I.7) we get, for x near e

$$h_\nu(x) = (f_\nu * h)(x) = C_\nu \int_G e^{-\nu^2(S(xg^{-1}))^2} h(g)dg$$

with $C_\nu = \left(\frac{\nu}{\sqrt{\pi}}\right)^n \sigma(0)^{-1}$. Writing $x = \mathrm{Exp}(s.X)$, $g = \mathrm{Exp}(t.X)$ $\tilde{h} = h\circ S^{-1}$, $\tilde{h}_\nu = h_\nu \circ S^{-1}$ and using (I.6) we obtain

$$\tilde{h}_\nu(s) = C_\nu \int_{R^n} e^{-\nu^2[S(\mathrm{Exp}(s.X)\mathrm{Exp}(-t.X))]^2} \tilde{h}(t)\sigma(t)dt.$$

Making use of (I.10) yields

(I.12) $$\tilde{h}_\nu(s) = C_\nu \int_{R^n} e^{-\nu^2 u^2} \tilde{h}(t)\sigma(t)dt ,$$

where $u = (u_1,\ldots,u_n)$ is given by (I.11). Since h vanishes in V_1, we may assume that

$$\mathrm{supp}\, \tilde{h} \subset \{t \in \mathbb{R}^n,\ A < |t| < B\}, \qquad A > 0.$$

We must show that $\tilde{h}_\nu$ defined by (I.12) extends holomorphically to a neighborhood of 0 in $\mathbb{C}^n$ (independent of ν), and there converges to 0 as $\nu \to \infty$.

Indeed for $s, \tilde{s} \in \mathbb{R}^n$, sufficiently small, we get from (I.12)

$$\tilde{h}(s+i\tilde{s}) = C_\nu \int_{A \le |t| \le B} e^{-\nu^2 v^2} \tilde{h}(t)\sigma(t)dt, \tag{I.13}$$

with $v = (v_1,\dots,v_n)$, and v_j is the expression obtained by putting $s_j + i\tilde{s}_j$ instead of s_j in (I.11), i.e.

$$v_j = s_j + i\tilde{s}_j - t_j + \sum_{\substack{|\alpha| \ge 1 \\ |\beta| \ge 1}} c_{\alpha,\beta,j} t^\alpha (s+i\tilde{s})^\beta. \tag{I.14}$$

Note that the latter is absolutely convergent for $|t| \le B$ and s and $\tilde{s}$ sufficiently small. Set

$$Q = \operatorname{Re} v^2 = \operatorname{Re}\left(\sum_{j=1}^n v_j^2\right).$$

It is easy to check that there is $\delta_0 > 0$ and $C > 0$ such that if $\delta \in (0,\delta_0)$ then for $|s| \le \delta$, $|\tilde{s}| \le \delta$ and $A \le |t| \le B$ we have

$$Q \ge (A-\delta)^2 - C\delta.$$

Choosing $\delta \in (0,\delta_0)$ small enough we get

$$Q \ge \frac{A^2}{2}. \tag{I.15}$$

Since $\tilde{h}$ is a distribution with compact support in $\{A < |t| < B\}$ it follows from (I.13) that there exists $C > 0$ and $\ell \in \mathbb{Z}_+$ such that for $|s| \le \delta$, $|\tilde{s}| \le \delta$

$$|\tilde{h}_\nu(s+i\tilde{s})| \le CC_\nu \sup_{\substack{|\alpha| \le \ell \\ A \le |t| \le B \\ |s|,|\tilde{s}| \le \delta}} |\partial_t^\alpha e^{-\nu^2 v^2}|. \tag{I.16}$$

It is clear that the right hand side of (I.16) may be bounded by

$$C'\nu^N \sup_{\substack{A \le |t| \le B \\ |s|,|\tilde{s}| \le \delta}} (e^{-\nu^2 Q})$$

where $C' > 0$ and $N \in \mathbb{Z}_+$ are independent of ν. Therefore (I.15) and (I.16) imply that for $|s| \le \delta$, $|s'| \le \delta$

$$(I.17) \qquad |\hat{h}_\nu(s + i\tilde{s})| \le C'\nu^N e^{-\nu^2 A^2/2} .$$

(I.17) yields the desired result by taking

$$\mathcal{O} = \{s + i\tilde{s} \in \mathbb{C}^n,\ |s| < \delta\ |\tilde{s}| < \delta\}. \qquad \text{Q.E.D.}$$

We are now ready to prove Theorem 1. Let u be as in Theorem 1 i.e.

$$u \in \mathcal{D}'(G),\ Lu = 0 \ \text{ in } \ U, \quad e \in U \subset G.$$

Let V be a sufficiently small open neighborhood of e, $V \subset G$, in which Lemmas 1 and 2 are valid. Take $\zeta \in C_0^\infty(V)$, $\zeta \equiv 1$ near e. Set

$$(I.18) \qquad h = \zeta u, \qquad r = Lh.$$

Both h and r are distributions with compact supports in V. Furthermore $r \equiv 0$ in some neighborhood V_1 of e, $V_1 \subset\subset V$. Since L commutes with the convolution with f_ν we get from (I.18).

$$(I.19) \qquad L(f_\nu * h) = f_\nu * r .$$

By Lemma 1, we know that $f_\nu * h$ converges to h in a neighborhood W of e. Lemma 2 implies that $f_\nu * r$ extends holomorphically to a complex neighborhood of e (independent of h and ν) and there converges to zero. By the Cauchy-Kovalevski theorem and by shrinking W if needed, we may find a sequence k_ν of analytic functions in W converging to 0 (in the space of analytic functions in W) and satisfying

$$(I.20) \qquad Lk_\nu = f_\nu * r.$$

[In fact we can require that the Cauchy data of k_ν be zero on a non-characteristic analytic hypersurface passing through e].

Put

$$u_\nu = f_\nu * h - k_\nu .$$

It follows from (I.19) and (I.20) that

$$Lu_\nu = 0 \quad \text{in} \quad W.$$

On the other hand

$$\lim u_\nu = h \quad \text{in} \quad \mathcal{D}'(W) \; ; \tag{I.21}$$

since $h = u$ near e (where $\zeta \equiv 1$, see (I.8)), the proof of Theorem 1, when u is a distribution, is complete.

If u is of class C^k, it follows from Lemma 1 that the convergence in (I.21) is in $C^k(W)$. Q.E.D.

II. Proof of Theorem 2.

The proof of Theorem 2 is similar to the proof of Theorem 1. Let $u \in \mathcal{D}'((-T,T) \times G)$ satisfying

$$Pu = 0 \quad \text{in} \quad (-T,T) \times U, \qquad e \in U \subset G.$$

Without loss of generality, by shrinking U and the interval $(-T,T)$ if needed, we may assume

$$u \in C^m((-T,T); H^{-N}(U)) \tag{II.1}$$

($N \in \mathbb{Z}_+$, $H^{-N}(U)$ is the usual negative Sobolev space in U).

Let V be an open neighborhood of e in which Lemmas 1 and 2 are valid. Take $\zeta \in C_0^\infty(V)$, $\zeta \equiv 1$ near e, and set

$$\zeta u = h, \quad Ph = r. \tag{II.2}$$

It follows from (II.1) and (II.2) that we have

$$h \in C^m((-T,T), H^{-N}_{comp}(V)), \; r \in C^0((-T,T), H^{-N-m}_{comp}(V)),$$

furthermore

$$r(t,\cdot) \equiv 0 \quad \text{near} \quad e.$$

Let f_ν be defined by (I.7), since P (defined by (0.2)) commutes with the convolution with f_ν (convolution on G, t being a parameter) we get from (II.2)

$$P(f_\nu * h) = f_\nu * r \; . \tag{II.3}$$

Inspection of the proofs of Lemmas 1 and 2 shows that

$$\lim f_\nu * h = h \quad \text{in} \quad \mathcal{D}'((-T,T) \times W), \tag{II.4}$$

and that $f_\nu * r$ extends as an element of

$$C^0((-T,T),\mathcal{K}(\mathcal{O}))$$

and converges to 0 in this space ($\mathcal{K}(\mathcal{O})$ is the space of bounded holomorphic functions in $\mathcal{O} \subset \mathbb{C}^n$).

Using a refinement of the Cauchy-Kovalevsky theorem, and contracting W if needed, we may find $\varepsilon > 0$ (independent of h and ν) and a sequence

$$k_\nu \in C^m((-\varepsilon,\varepsilon),\mathcal{A}(W))$$

($\mathcal{A}(W)$ is the space of real analytic functions in W) converging to zero in that space and satisfying

$$\begin{cases} Pk_\nu = f_\nu * r & \text{in} \quad (-\varepsilon,\varepsilon) \times W \\ \partial_t^j k_\nu \big|_{t=0} = 0, & j = 0,\ldots,m-1. \end{cases} \tag{II.5}$$

If we set

$$u_\nu = f_\nu * h - k_\nu \; ,$$

it follows from (II.3) and (II.5) that we have

$$Pu_\nu = 0 \qquad \text{in} \quad (-\varepsilon,\varepsilon) \times W. \tag{II.6}$$

On the other hand we have

$$u_\nu \in C^m((-\varepsilon,\varepsilon),\mathcal{A}(W)).$$

Since $\partial_t^j u_\nu|_{t=0} = f_\nu * (\partial_t^j h)|_{t=0} \in \mathcal{A}(W)$, uniqueness for the Cauchy problem, in conjunction with (II.6), implies that u_ν is analytic in $(-c,c) \times W$.

Q.E.D.

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M. S. Baouendi
Department of Mathematics
Purdue University
West Lafayette, IN 47907
USA

Linda Preiss Rothschild
Department of Mathematics
University of California
San Diego, CA 92023
USA