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ANALYTICITY OF RELATIVE FUNDAMENTAL SOLUTIONS AND PROJECTIONS FOR LEFT INVARIANT OPERATORS ON THE HEISENBERG GROUP

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1. Introduction

We show that for certain classes of unsolvable, non-hypoelliptic differential operators on the Heisenberg group there exist left (respectively right) inverses modulo the orthogonal projection onto the L^2 nullspace of the operator (resp. the adjoint of the operator). We also show that these relative inverses and the projections preserve analyticity locally.

Let G be the Heisenberg group and let $X_1, X_2, \ldots, X_{2n}$, T be a basis for the Lie algebra $\mathscr{G}=\mathscr{G}_1+\mathscr{G}_2$ of G with $X_1, X_2, \ldots, X_{2n}$ a basis of $\mathscr{G}_1$, $\mathscr{G}_2$ spanned by (T) and $[\mathscr{G}_1, \mathscr{G}_1]=\mathscr{G}_2=$ the center of $\mathscr{G}$. A left invariant differential operator L on G is said to be *homogeneous of degree d* if there is a homogeneous non-commutative polynomial p such that $L=p(X_1, X_2, \ldots, X_{2n})$. L is *elliptic in the generating directions* if $p(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_{2n})$ is elliptic on $\mathbb{R}^{2n}$.

Our main result is the following.

(1.1) Theorem. — *Let* L *be a homogeneous, left invariant differential operator on the Heisenberg group* G *elliptic in the generating directions. Then there are distributions* k_1 *and* k_2 *such that:*

$$L f * k_1 = f - \Pi_1 f \tag{1.2}$$

$$L(f * k_2) = f - \Pi_2 f \tag{1.3}$$

for $f \in C_0^\infty(G)$, *where* Π_1 *and* Π_2 *are orthogonal projections onto the* L^2 *nullspaces of* L *and* L^* *respectively, and* $*$ *denotes group convolution. Furthermore, the operators* $f \to f * k_i$ *and* $f \to \Pi_i f$, $i=1, 2$, *all preserve analyticity locally.*

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COROLLARY. — *If* $u, f \in C_0^\infty(G)$ *and:*

$$(1.4) \qquad Lu = f \quad \text{in } U,$$

U *open, then* $u_1 = (I - \Pi_1) u$ *is analytic in every open subset of* U *where* f *is, and* u_1 *is again a solution of* (1.4).

Proof. — If $Kv = v * k_1$, then:

$$(I - \Pi_1) u = Kf + K(Lu - f).$$

By Theorem 1.1, the right hand side is analytic in U.

Theorem 1.1 was proved by Greiner, Kohn and Stein [4] for the case where $L = \Box_b$, the boundary Laplace operator. The analyticity of the projections Π_1 and Π_2 was proved by Geller [2], who also proved the existence of distributions k'_1, k'_2 satisfying (1.2) and (1.3) and preserving local smoothness. The general result was conjectured by Stein [2]. For the general case, Métivier, by the methods in [11], obtained a proof (unpublished). However, our method is a direct reduction to the hypoelliptic case.

A differential operator D is called C^∞ *hypoelliptic* (resp. *analytic hypoelliptic*) in U if $Du = f$ in U with f smooth (resp. analytic) in any open subset $V \subset U$ implies u is smooth in V (resp. u is analytic in V if u is smooth in V). Tartakoff [16], [17] and Trèves [18] have shown that for homogeneous left invariant differential operators on the Heisenberg group, analytic hypoellipticity is implied by C^∞ hypoellipticity. For C^∞ hypoellipticity, necessary and sufficient conditions for operators of the above type on groups with dilations have been given by Rockland [14] and Helffer and Nourrigat [6].

2. The self adjoint case

One can easily reduce the proof of Theorem 1.1 to the case where L is self adjoint and of large homogeneous degree d, with $d/2$ even. Indeed, suppose the result is known for $(L^* L)^n = L_1$ and $(LL^*)^n = L_2$. Then there exists k'_2 such that:

$$(2.1) \qquad L_2(f * k'_2) = f - \Pi_2 f,$$

since $\ker L_2 = \ker L^*$. Since for any left invariant vector field X we have $X(f * k) = f * Xk$, from (2.1):

$$L(f * L^*(LL^*)^{n-1} k'_2) = f - \Pi_2 f,$$

so that (1.3) follows with $k_2 = L^*(LL^*)^{n-1} k'_2$. Furthermore, if convolution with k'_2 preserves local analyticity, so does convolution with k_2. (1.2) is obtained similarly from L_1.

Theorem 1.1 is then a consequence of the following, which is partly based on an idea of Beals and Greiner [1].

(2.2) THEOREM. — *Let* L *be a self adjoint operator satisfying the hypotheses of Theorem* 1.1 *and of sufficiently high degree divisible by* 4. *Then there is a closed contour* Γ *around* 0 *in* $\mathbb{C}$ *such that* $L_\alpha = L - \alpha(-iT)^{d/2}$ *is hypoelliptic for all* $\alpha \in \Gamma$. *There exist distributions* k_α *satisfying* $L_\alpha k_\alpha = \delta$, *with* $\alpha \to \| D^\beta (f * k_\alpha) \|_{L_\infty}$ *bounded on* Γ *for all multi-indices and any* $f \in C_0^\infty(G)$. *Hence define:*

$$K, S: \quad C_0^\infty(G) \to C^\infty(G)$$

by:

$$Kf = \frac{1}{2\pi i} \int_\Gamma \alpha^{-1} f * k_\alpha \, d\alpha,$$

$$Sf = \frac{1}{2\pi i} \int_\Gamma (-iT)^{d/2} f * k_\alpha \, d\alpha.$$

Then:

$$LKf = K^* Lf = f - Sf, \qquad f \in C_0^\infty(G), \tag{2.3}$$

and $S = \Pi$, *the orthogonal projection onto the* L^2 *kernel of* L. *Furthermore,* K *and* Π *preserve local analyticity.*

The proof of Theorem 2.2 will proceed as follows. First, one must construct the k_α. For this we use the construction given by Métivier [11] for a single operator and check that the k_α vary well with α. The first equality in (2.3) is an immediate consequence of the self adjointness of L and Π, while the second is easily obtained by writing $L = L_\alpha + \alpha(-iT)^{d/2}$. The proof that $S = \Pi$ will be obtained by applying the irreducible unitary representations of G to both operators and then using the Plancherel theorem for G.

Finally, to prove that K and S preserve analyticity, it suffices to obtain local estimates for derivatives of $f * k_\alpha$ independent of α. For this we use the methods of the second author [16], checking that the constants obtained in the L^2 estimates can be chosen independent of α.

3. Unitary representations and the Plancherel formula for G

We summarize some facts about the irreducible unitary representations of G which will be used in the construction of k_α and in the proof that $S = \Pi$. Let X_i', X_i'', $i = 1, 2, \ldots, n$, T be a basis for $\mathscr{G}$ with $[X_i', X_j''] = \delta_{ij} T$, all other commutators zero. For every $\lambda \in \mathbb{R} - \{0\}$, let π_λ be the irreducible unitary representation of G on $L^2(\mathbb{R}^n)$ defined by:

$$\pi_\lambda(x', x'', t) f(u) = e^{i((\operatorname{sgn}\lambda)|\lambda|^{1/2} x''.u + \lambda t + \lambda x'.x''/2)} f(u - |\lambda|^{1/2} x'). \tag{3.1}$$

Here (x', x'', t) are the coordinates given by:

$$(x', x'', t) \leftrightarrow \exp(x'.X' + x''.X'' + tT),$$

where $x' \cdot X' = \sum_{i=1}^{n} x_i' X_i'$ and exp denotes the exponential map.

These induce the following on $\mathcal{G}$:

$$\pi_\lambda(T) = i\lambda,$$

$$\pi_\lambda(X_j') = |\lambda|^{1/2} \frac{\partial}{\partial u_j},$$

$$\pi_\lambda(X_j'') = i \operatorname{sgn} \lambda |\lambda|^{1/2} u_j.$$

If $\varphi \in C_0^\infty(G)$, let $\pi_\lambda(\varphi)$ be the bounded operator on $L^2(\mathbb{R}^n)$ given by:

$$\pi_\lambda(\varphi) = \int \varphi(g) \pi_\lambda(g^{-1}) \, du(g),$$

where $du(g) = dx' \, dx'' \, dt$ is a Haar measure on G. If $L \in U(\mathcal{G})$, the universal enveloping algebra of $\mathcal{G}$, then:

$$\pi_\lambda(L\varphi) = \pi_\lambda(L) \pi_\lambda(\varphi), \tag{3.2}$$

where:

$$\pi_\lambda: \quad \mathcal{G} \to \operatorname{End}(L^2(\mathbb{R}^n))$$

is the corresponding representation of $\mathcal{G}$.

It will be useful to know the distribution kernel $a_{\varphi, \lambda}(u, v)$ of the operator $\pi_\lambda(\varphi)$. By direct calculation, for $f \in L^2(\mathbb{R}^n)$:

$$\pi_\lambda(\varphi) f(u) = \int \varphi(x', x'', t) e^{-i(y_\lambda \cdot u + x_\lambda \cdot y_\lambda/2 + \lambda t)} f(u - x_\lambda) \, dx' \, dx'' \, dt$$

where $x_\lambda = |\lambda|^{1/2} x'$ and $y_\lambda = (\operatorname{sgn} \lambda) |\lambda|^{1/2} x''$. Since $dx' \, dx'' = (\operatorname{sgn} \lambda) |\lambda|^{-n} dx_\lambda \, dy_\lambda$:

$$\pi_\lambda(\varphi) f(u) = |\lambda|^{-n} \int \varphi_\lambda(x_\lambda, y_\lambda, t) e^{-i(y_\lambda \cdot u - x_\lambda \cdot y_\lambda/2 + \lambda t)} f(u - x_\lambda) \, dx_\lambda \, dy_\lambda \, dt$$

where:

$$\varphi_\lambda(x_\lambda, y_\lambda, t) = \varphi(x', x'', t).$$

Hence a simple change of variables shows that:

$$a_{\varphi,\lambda}(u,v)=|\lambda|^{-n}(\varphi_\lambda)^{\hat{}}\left(u-v, \frac{u+v}{2}, \lambda\right), \tag{3.3}$$

where $\hat{}$ denotes the Euclidean Fourier transform of φ_λ in the last two sets of variables. The reader is referred to Métivier [11] for a more detailed account of the above calculation for a more general class of groups.

We shall need two versions of the Plancherel theorem for G. The first is the following equality for $\varphi \in C_0^\infty(G)$:

$$\varphi(0)=\int_{\mathbb{R}-\{0\}} \operatorname{tr}(\pi_\lambda(\varphi))\, d\mu(\lambda), \tag{3.4}$$

where $d\mu(\lambda)=c|\lambda|^n d\lambda$, c constant, and tr denotes trace. The second version of the Plancherel theorem states that π_λ extends to a Hilbert space isomorphism:

$$\pi_\lambda:\ L^2(G)\to L^2(\mathbb{R}-\{0\}, H-S),$$

where $L^2(\mathbb{R}-\{0\}, H-S)$ is the space of all functions F from $\mathbb{R}-\{0\}$ to the space $H-S$ of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ satisfying:

$$\int_{\mathbb{R}-\{0\}} \operatorname{tr}(F(\lambda)\,F(\lambda)^*)\, d\lambda < \infty.$$

The norm is:

$$\| F(\lambda)\|^2_{L^2(\mathbb{R}-\{0\}, H-S)}=\int_{\mathbb{R}-\{0\}} \operatorname{tr}(F(\lambda)\,F(\lambda)^*)\, d\mu'(\lambda),$$

where $d\mu'(\lambda)=c'|\lambda|^n d\lambda$, where c' is a constant. In particular, for $f, g\in L^2(G)$;

$$(f, g)_{L^2}=\int_{\mathbb{R}-\{0\}} \operatorname{tr}(\pi_\lambda(f)\,\pi_\lambda(g)^*)\, d\mu'(\lambda). \tag{3.5}$$

The reader is referred to Kirillov [8] or Pukanszky [13] for a complete account of the Plancherel theorem for nilpotent groups.

4. Construction of the fundamental solutions k_α of L_α

(4.1) Lemma. — *Let* L *be as in Theorem* 2.2, *and let* $L_\alpha=L-\alpha(-iT)^{d/2}$. *Then if* $\varepsilon>0$ *is sufficiently small,* L_α *is hypoelliptic for all* α, $\varepsilon \leqq |\alpha| \leqq 2\varepsilon$.

Proof. — By Rockland [14], L_α is hypoelliptic if and only if $\pi_1(L_\alpha)$ and $\pi_{-1}(L_\alpha)$ are injective on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. Now:

$$\pi_{\pm 1}(L_\alpha)=\pi_{\pm 1}(L)\pm\alpha.$$

By Grušin [5], the eigenvalues of $\pi_{\pm 1}(L)$ are discrete, and so if:

$$\mu=\min_{\substack{\sigma \text{ eigenvalue of } \pi_{\pm 1}(L)\\ \sigma\neq 0}} |\sigma|,$$

then any $\varepsilon<\mu/2$ will satisfy the lemma.

A family σ_α of distributions on a manifold M will be called *uniformly bounded* if for every compact set $K\subset M$ there exist C and M independent of α such that:

$$|\sigma_\alpha(\varphi)|\leqq C\sup_{|\beta|\leqq M}|D^\beta\varphi(x)|$$

for all $\varphi\in C_0^\infty(K)$. Our proof of analyticity requires that the k_α be uniformly bounded.

(4.2) PROPOSITION. — *Let $\varepsilon>0$ be chosen as in Lemma 4.1. If d is sufficiently large and d/2 even, there is a uniformly bounded family of fundamental solutions k_α, $L_\alpha k_\alpha=\delta$, all $\alpha\in\mathbb{C}$, $\varepsilon\leqq|\alpha|\leqq 2\varepsilon$, such that $S_\alpha: C^\infty(G)\to C^\infty(G)$ defined by:*

$$S_\alpha\varphi=(-iT)^{d/2}(\varphi\star k_\alpha),$$

extends to a bounded mapping of $L^2(G)$ into itself satisfying the following:

$$\|S_\alpha\varphi\|_{L^2}\leqq C\|\varphi\|_{L^2} \tag{4.3}$$

C independent of α, and

$$\pi_\lambda(S_\alpha\varphi)=\begin{cases}\pi_1(L_\alpha)^{-1}\pi_\lambda(\varphi), & \lambda>0\\ \pi_{-1}(L_\alpha)^{-1}\pi_\lambda(\varphi), & \lambda<0'\end{cases} \tag{4.4}$$

for almost all $\lambda\in\mathbb{R}-\{0\}$.

To prove Proposition 4.2 we shall follow a similar construction in Métivier [11] (where one of the ideas is attributed to Lion [9]), keeping track of the dependence on α. We let $B_\varepsilon=\{\alpha\in\mathbb{C}:\varepsilon\leqq|\alpha|\leqq 2\varepsilon\}$.

(4.5) LEMMA. — *For $\alpha\in B_\varepsilon$, let $I_{\lambda,\alpha}(u,v)$ be the distribution kernel of the operator $\pi_\lambda(L_\alpha)^{-1}: L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$. Then if d is sufficiently large $I_{\lambda,\alpha}$ is continuous and satisfies, for $n/2<k<d-n/2$;*

$$|I_{\varepsilon_\lambda,\alpha}(v,u)|\leqq C_\lambda(1+|v|)^{n/2-d+k}(1+|u|)^{n/2-k} \tag{4.6}$$

for some constant C independent of $\alpha\in B_\varepsilon$, where $\varepsilon_\lambda=(-1)^{\operatorname{sgn}\lambda}$.

Proof. — The estimate (4.6) is given in [11] for α fixed in B_ε. Let

$$H^k=\{f\in L^2(\mathbb{R}^n): u^\beta D_u^\gamma f\in L^2, \quad \text{all } |\beta|+|\gamma|\leqq k\}$$

with norm $\|f\|_{H^k}^2=\sum_{|\beta|+|\gamma|\leqq k}\|u^\beta D_u^\gamma f\|_{L^2}^2$, and let H^{-k} be the dual space. By Grušin [5], each $\pi_{\pm 1}(L_\alpha)^{-1}$ is bounded from L^2 to H^d i.e., there exists C′ such that:

$$\|f\|_{H^d}\leqq C'\|\pi_{\pm 1}(L_\alpha)f\|_{L^2} \tag{4.7}$$

with C′ dependent on α. An easy perturbation argument (*see* [15], for details) shows that one can choose C′ independent of α for α varying in B_ε. From (4.7) one obtains for each j

$$\|f\|_{H_j}\leqq C_0\|\pi_{\pm 1}(L_\alpha)^{-1}f\|_{H^{d+j}}$$

for all $j\in Z$, as in [11], Lemma 12.

Now the proof is exactly as given in [11].

Proof of Proposition 4.2. — We shall follow the construction given in [11] for a more general class of groups. Put $\check{\varphi}(g)=\varphi(g^{-1})$, $\varphi\in C_0^\infty(G)$. First

$$\operatorname{tr}(\pi_\lambda(L_\alpha)^{-1}\pi_\lambda(\check{\varphi}))=\int I_{\lambda,\alpha}(v,u)\,a_{\check{\varphi},\lambda}(u,v)\,dv\,du,$$

for $\alpha\in B_\varepsilon$, where $a_{\check{\varphi},\lambda}(u,v)$ is the kernel of $\pi_\lambda(\check{\varphi})$, which by (3.3) is given by

$$a_{\check{\varphi},\lambda}(u,v)=|\lambda|^{-n}(\varphi_\lambda)^\wedge(v-u,\,-(u+v)/2,\,-\lambda). \tag{4.8}$$

Putting

$$J_{\lambda,\alpha}(u,v)=\int e^{-iv\cdot\xi}I_{\lambda,\alpha}\left(\frac{u}{2}-\xi,\,-\frac{u}{2}-\xi\right)d\xi$$

we obtain

$$\operatorname{tr}(\pi_\lambda(L_\alpha)^{-1}\pi_\lambda(\check{\varphi}))=|\lambda|^{-n}\int J_{\lambda,\alpha}(u,v)\,\varphi_{\hat{\lambda}}(u,v,-\lambda)\,du\,dv,$$

where $\varphi_{\hat{t}}$ denotes the Fourier transform in the t variable. Finally, let $u_\lambda=|\lambda|^{-1/2}u$, $v_\lambda=(\operatorname{sgn}\lambda)\,|\lambda|^{-1/2}v$, $u^\lambda=|\lambda|^{1/2}u$, $v^\lambda=(\operatorname{sgn}\lambda)\,|\lambda|^{1/2}v$ and put

$$K_{\lambda,\alpha}(u_\lambda,v_\lambda)=J_{\lambda,\alpha}(u,v).$$

In view of (3.2) and (3.4) we want to estimate tr $(\pi_\lambda(L_\alpha)^{-1}\pi_k(\check{\varphi}))$. By the above we have

$$\operatorname{tr}(\pi_\lambda(L_\alpha)^{-1}\pi_\lambda(\check{\varphi}))=\int K_{\lambda,\alpha}(u_\lambda,v_\lambda)\,\hat{\varphi}(u_\lambda,v_\lambda,-\lambda)\,du_\lambda\,dv_\lambda$$

by the definition of φ_λ. It is easy to check that

$$K_{\lambda,\alpha}(u, v)=|\lambda|^{-d/2} K_{\varepsilon_\lambda,\alpha}(u^\lambda, v^\lambda).$$

We shall need to show

(4.9) $$|K_{\varepsilon_\lambda,\alpha}(u^\lambda, v^\lambda)| \leqq C,$$

all $\alpha \in B_\varepsilon$, u, v. By definition

$$K_{\varepsilon_\lambda,\alpha}(u^\lambda, v^\lambda)=J_{\varepsilon_\lambda,\alpha}(u^\lambda, \pm v^\lambda)=\int e^{\mp i v^\lambda \xi} I_{\varepsilon_\lambda,\alpha}\left(\frac{u^\lambda}{2}-\xi, \frac{-u^\lambda}{2}-\xi\right)d\xi.$$

Hence by (4 . 6)

(4. 10) $$|K_{\varepsilon_\lambda,\alpha}(u^\lambda, v^\lambda)| \leqq C_0 \sup(1+|\xi|)^{n+1}\left(1+\left|\frac{u^\lambda}{2}-\xi\right|\right)^{n/2-d+k}\left(1+\left|-\frac{u^\lambda}{2}-\xi\right|\right)^{n/2-k}$$

for $n/2 < k < d-n/2$. Choose k to be the smallest integer larger than $(3n/2)+1$. Then for $d > 3n+4$, k is in the range $n/2 < k < d-n/2$. Now for any $a \in \mathbb{R}$

(4.11) $$(1+|\xi|) \leqq \sup((1+|a-\xi|), (1+|-a-\xi|)).$$

Then (4. 11), together with (4. 10), proves (4. 9).

Hence, if $\chi(u, v, \lambda) \in \mathscr{S}(\mathbb{R}^{2n+1})$, with $\chi(u, v, \lambda)$ vanishing in λ to order at least $d/2-n$ at $\lambda=0$, the integral

$$\int |K_{\lambda,\alpha}(u, v)|\ |\chi(u, v, \lambda)|\, du\, dv\, d\lambda$$

exists and is bounded, independent of $\alpha \in B_\varepsilon$.

To handle the singularity near $\lambda=0$ we proceed as in [11]. Let $\psi \in C_0^\infty(\mathbb{R})$ be chosen with

$$\psi(\lambda)=\begin{cases} 1 & \text{for } |\lambda| \leqq 1/2, \\ 0 & \text{for } |\lambda| \geqq 1. \end{cases}$$

Let $\mathscr{X}: \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ be defined by

$$\mathscr{X} f(\lambda)=f(\lambda)-\psi(\lambda) \sum_{k \leqq (d/2)-n} \frac{\lambda^k}{k!}\left(\frac{\partial}{\partial \lambda}\right)^k f(0).$$

Then

$$\mathscr{X} f(\lambda)=f(\lambda) \quad \text{for } |\lambda| \geqq 1,$$

and

$$\mathscr{X} f(\lambda) \text{ vanishes to order } d/2-n \text{ at } \lambda=0.$$

Now define $k_{\alpha, 1}$ by:

$$k_{\alpha, 1}(\varphi)=c \int \mathrm{K}_{\lambda, \alpha}(u, v, \lambda) \mathscr{X} \varphi^{\wedge}(u, v, -\lambda)|\lambda|^{n} \, du \, dv \, d\lambda,$$

with c as in (3.4). Then $k_{\alpha, 1}$ is a uniformly bounded family of distributions for $\alpha \in \mathrm{B}_{\varepsilon}$.

We will now construct a uniformly bounded family $k_{\alpha, 2}$ of distributions such that $k_{\alpha, 1}+k_{\alpha, 2}$ is a fundamental solution for L_{α}, i. e.,

$$\mathrm{L}_{\alpha}(k_{\alpha, 1}+k_{\alpha, 2})=\delta.$$

For this, let r_{α} be the distribution defined by

$$-r_{\alpha}=\mathrm{L}_{\alpha} k_{\alpha, 1}-\delta.$$

Clearly r_{α} is a uniformly bounded family, and we must find $k_{\alpha, 2}$ satisfying

$$\mathrm{L}_{\alpha} k_{\alpha, 2}=r_{\alpha}.$$

As in [11] we note that since $\mathrm{T}^{s} r_{\alpha}=0$ for $s \geqq d/2-n$ we may write r_{α} in the form:

$$r_{\alpha}=\sum_{|j| \leqq d/2-n} r_{\alpha, j}(x', x'') t^{j}$$

where $r_{\alpha, j}(x', x'')$ is a uniformly bounded family of distributions on $\mathbb{R}^{2n}$. Let L^{0} be the constant coefficient differential operator elliptic in $\mathbb{R}^{2n}$, corresponding to the principal symbol at 0 (in the classical sense) of L_{α} [i. e., $\mathrm{L}^{0}=p(\partial/\partial x'_{1}, \ldots, \partial/\partial x'_{n}, \partial/\partial x''_{1}, \ldots, \partial/\partial x''_{n})$. See the definition of "elliptic in the generating directions" on the first page. Note that L^{0} is independent of α, since the parameter occurs as a coefficient of a term of lower degree.] Now we may seek to find $k_{\alpha, 2}$ in the form

$$k_{\alpha, 2}=\sum_{j \leqq d/2-n} \mathrm{W}_{\alpha, j}(x', x'') t^{j}.$$

The $\mathrm{W}_{\alpha, j}$ may be found by downward recursion by writing

$$\mathrm{L}=\mathrm{L}^{0}+\sum_{0<j \leqq d} \mathrm{L}_{\alpha, j} \frac{\partial^{j}}{\partial t^{j}}$$

and solving recursively for $\mathrm{W}_{\alpha, j}$ satisfying

$$\mathrm{L}^{0} \mathrm{W}_{\alpha, j}(x', x'')+\sum_{k>j} \mathrm{L}_{\alpha, k} \frac{(j+k)!}{j!} \mathrm{W}_{\alpha, j+k}=r_{\alpha, j}. \tag{4.12}$$

with the convention that $W_{\alpha, j+k}=0$ for $j+k > d/2-n$. We must still show that (4.12) can be solved with $W_{\alpha, j}$ a uniformly bounded family. For this we use the following modification of [7], Theorem 3.6.4.

(4.13) LEMMA. — *Suppose that $\{f_\alpha\}$ is a uniformly bounded family of distributions on an open set $\Omega \subset \mathbb{R}^N$ which is strongly convex for the constant coefficient differential operator* P(D). *Then there exists a uniformly bounded family u_α on Ω such that*

$$P(D)\, u_\alpha = f_\alpha.$$

The proof of Lemma 4.13 is an easy modification of [7], Theorem 3.6.4, where the result is proved for fixed α.

Now we may complete the proof of Proposition 4.2 by verifying (4.3) and (4.4). For this, note that since T is bi-invariant, $(-i\,T)^{d/2}(\varphi * k_\alpha)=((-i\,T)^{d/2}\varphi) * k_\alpha$. One easily sees by the definition of $k_{\alpha,2}$ that $((-i\,T)^{d/2}\varphi) \star k_{\alpha,2}=0$. Hence

$$S_\alpha \varphi = (-i\,T)^{d/2}(\varphi * k_{\alpha,1}).$$

Now

$$\pi_\lambda(S_\alpha \varphi)=\pi_\lambda((-i\,T)^{d/2}\varphi \star k_{\alpha,1})=\pi_\lambda(L_\alpha)^{-1}\pi_\lambda(-i\,T)^{d/2}\pi_\lambda(\varphi)$$
$$=\lambda^{d/2}\,|\lambda|^{-d/2}\pi_{\varepsilon_\lambda}(L_\alpha)^{-1}\pi_\lambda(\varphi)=\pi_{\varepsilon_\lambda}(L_\alpha)^{-1}\pi_\lambda(\varphi), \qquad \text{since } d/2 \text{ is even,}$$

which proves (4.4).

To prove (4.3) it suffices, by (4.4) and the Plancherel Theorem (3.5), to show that

$$\|\pi_{\varepsilon_\lambda}(L_\alpha)^{-1}\pi_\lambda(\varphi)\|_{H-S} \leqq C'\,\|\pi_\lambda(\varphi)\|_{H-S},$$

where H−S denotes the Hilbert Schmidt norm. This follows immediately since $\pi_{\pm 1}(L_\alpha)^{-1}$ is bounded on L^2. This completes the proof of Proposition 4.2.

5. Proof that S = Π.

(5.1) PROPOSITION. — *Let k_α be defined as in Proposition* 4.2. *Then the operator:*

$$S f=(2\pi i)^{-1}\int_\Gamma (-i\,T)^{d/2}(f * k_\alpha)\,d\alpha, \qquad f \in C_0^\infty(G)$$

extends to a bounded operator on L^2 *and* $S=\Pi$, *the orthogonal projection onto the nullspace of* L.

The proof of Proposition 5.1 requires some preliminaries.

(5.2) LEMMA. — *For almost all $\lambda \in \mathbb{R}-\{0\}$, for all $f \in C_0^\infty(G)$,*

$$\pi_\lambda(\Pi f)=P_{\varepsilon_\lambda}\pi_\lambda(f)$$

where P_{ε_λ} is the orthogonal projection onto the nullspace of $\pi_{\varepsilon_\lambda}(L)$.

Proof. — This is very similar to Goodman [3]. First, it is clear that

$$\operatorname{Im} \pi_\lambda (\Pi f) \subset \ker \pi_\lambda (\mathrm{L}) = \ker \pi_{\varepsilon_\lambda} (\mathrm{L}).$$

Furthermore, if $h \in \ker \mathrm{L}$, then $\operatorname{Im} \pi_\lambda (h) \subset \ker \pi_\lambda (\mathrm{L})$ and hence $\pi_\lambda (h) = \mathrm{P}_{\varepsilon_\lambda} \pi_\lambda (h)$. Hence it suffices to show that if $g \perp \ker \mathrm{L}$, then $\operatorname{Im} \pi_\lambda (g) \subset (\ker \pi_\lambda (\mathrm{L}))^\perp$.

Now if $g \perp \ker \mathrm{L}$, then by the Plancherel formula (3.5),

$$\int \operatorname{tr} (\pi_\lambda (f) \pi_\lambda (g)^*)\, d\mu (\lambda) = 0 \tag{5.3}$$

for all $f \in \ker \mathrm{L}$. Let $\{ \varphi_j \}$ be an orthonormal basis of $\mathrm{L}^2 (\mathbb{R}^n)$ such that $\varphi_1, \varphi_2, \ldots, \varphi_\mathrm{N}$ is a basis of $\ker \pi_{+1} (\mathrm{L})$ (which is of finite dimension by Grušin [5]), and $\varphi_{\mathrm{N}+1}, \varphi_{\mathrm{N}+2}, \ldots$ is a basis of $(\ker \pi_{+1} (\mathrm{L}))$. Then

$$\operatorname{tr} (\pi_\lambda (f) \pi_\lambda (g)^*) = \sum_{i=1}^{\infty} (\pi_\lambda (f) \varphi_i, \pi_\lambda (g) \varphi_i) \tag{5.4}$$

for any $f \in \mathrm{L}^2 (\mathrm{G})$. Let the indices i and j be fixed with $1 \leqq j \leqq \mathrm{N}$ and let $c (\lambda) \in \mathrm{L}^1 (\mathbb{R}^1 - \{0\}, d\mu (\lambda))$ be arbitrary with support in $\mathbb{R}^+$. Then by the second version of the Plancherel formula define $h_{ij} \in \mathrm{L}^2 (\mathrm{G})$ by

$$\pi_\lambda (h_{ij}) \varphi_k = \begin{cases} c (\lambda) \delta_{ik} \varphi_j, & 1 \leqq i \leqq \mathrm{N}, \\ 0, & i > \mathrm{N}. \end{cases}$$

Then

$$\operatorname{tr} (\pi_\lambda (h_{ij}) \pi_\lambda (g^*)) = \sum_k (\pi_\lambda (h_{ij}) \varphi_k, \pi_\lambda (g) \varphi_k) = c (\lambda) (\varphi_j, \pi_\lambda (g) \varphi_i).$$

By (5.3), since $h_{ij} \in \ker \mathrm{L}$

$$\int c (\lambda) (\varphi_j, \pi_\lambda (g) \varphi_i)\, du (\lambda) = 0.$$

Since $c (\lambda)$ is arbitrary with support in $\mathbb{R}^+$, $(\varphi_j, \pi_\lambda (g) \varphi_i) = 0$ for almost all $\lambda > 0$, all i. The proof for $\lambda < 0$ is the same, obtained by using a basis adapted to the decomposition $\ker \pi_{-1} (\mathrm{L}) + (\ker \pi_{-1} (\mathrm{L}))^\perp$. Hence $\pi_\lambda (g) \varphi_i \perp \ker \pi_\lambda (\mathrm{L})$ as claimed.

From now on, Γ will denote a fixed simple contour in $\mathbb{C}$ lying in B_ε.

(5.5) LEMMA. — $\pi_\lambda \left(\int_\Gamma \mathrm{S}_\alpha f\, d\alpha \right) = \int_\Gamma \pi_\lambda (\mathrm{S}_\alpha f)\, d\alpha$ *for almost all* $\lambda \in \mathbb{R} - \{0\}$.

Proof. — Suppose $g \in \mathrm{L}^2 (\mathrm{G})$. Then by the Plancherel Theorem (3.5)

$$\left(\int_\Gamma \mathrm{S}_\alpha f\, d\alpha, g \right) = \int_{\mathbb{R} - \{0\}} \operatorname{tr} \left(\pi_\lambda \left(\int_\Gamma \mathrm{S}_\alpha f\, d\alpha \right) \pi_\lambda (g)^* \right) d\mu (\lambda).$$

Now let $\{\varphi_j\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then

$$(5.6)\quad \int \operatorname{tr}\left(\int_\Gamma \pi_\lambda(S_\alpha f)\,d\alpha\ \pi_\lambda(g)^*\right)d\mu(\lambda)=\int\sum_i\left(\int_\Gamma \pi_\lambda(S_\alpha f)\,d\alpha\ (\pi_\lambda(g))^*\varphi_i,\ \varphi_i\right)d\mu(\lambda).$$

Now since the infinite sum in the right hand side of (5.6) converges absolutely, by the dominated convergence theorem:

$$(5.7)\quad \int \operatorname{tr}\left(\int_\Gamma \pi_\lambda(S_\alpha f)\,d\alpha\,\pi_\lambda(g)^*\right)d\mu(\lambda)$$

$$=\iint_\Gamma \operatorname{tr}(\pi_\lambda(S_\alpha f)\,\pi_\lambda(g)^*)\,d\alpha\,d\mu(\lambda)$$

$$=\int_\Gamma\int_{\mathbb{R}-\{0\}} \operatorname{tr}(\pi_\lambda(S_\alpha f)\,\pi_\lambda(g)^*)\,d\mu(\lambda)\,d\alpha=\int_\Gamma (S_\alpha f,\ g)\,d\alpha$$

$$=\left(\int_\Gamma S_\alpha f\,d\alpha,\ g\right)=\int \operatorname{tr}\left(\pi_\lambda\left(\int S_\alpha f\,d\alpha\right)\pi_\lambda(g)^*\right)d\mu(\lambda).$$

Since g is arbitrary, (3.5) implies

$$\int_\Gamma \pi_\lambda(S_\alpha f)\,d\alpha=\pi_\lambda\left(\int_\Gamma S_\alpha f\,d\alpha\right)$$

for almost all λ by the Plancherel Theorem. This proves Lemma 5.5.

We may now prove Proposition 5.1. By Lemma 5.2, it suffices to show that

$$(5.8)\qquad \pi_\lambda(Sf)=P_{\varepsilon_\lambda}\pi_\lambda(f)\qquad \text{for}\quad \varphi\in C_0^\infty(G).$$

By Lemma 5.5 and (4.4)

$$(5.9)\qquad \pi_\lambda(S f)=\int_\Gamma \pi_\lambda(S_\alpha f)\,d\alpha=\int_\Gamma \pi_{\varepsilon_\lambda}(L_\alpha)^{-1}\,\pi_\lambda(f)\,d\alpha.$$

Suppose $\lambda>0$. Then from (5.9)

$$\pi_\lambda(S f)=(2\pi i)^{-1}\int_\Gamma \pi_1(L_\alpha)^{-1}\,\pi_\lambda(f)\,d\alpha=(2\pi i)^{-1}\int_\Gamma (\pi_1(L)-\alpha)^{-1}\,\pi_\lambda(f)\,d\alpha.$$

Since zero is an isolated point of the spectrum of $\pi_1(L)$ by [5],

$$(2\pi i)^{-1}\int_\Gamma (\pi_1(L)-\alpha)^{-1}\,d\alpha=P_1.$$

A similar argument holds for $\lambda<0$. Hence (5.8) is proved.

6. Analyticity

In this part we complete the proof of Theorem 1.1 by proving.

(6.1) THEOREM. — *The operators* K *and* S *constructed above preserve local analyticity.*

(6.2) LEMMA. — *Let* $u_\alpha = K_\alpha f$, $f \in C_0^\infty(G)$. *Suppose that for any bounded open set* U_0 *in which* f *is real analytic and any* V_0 *with compact closure in* U_0 *there exists a constant* C *such that*

$$\sup_{x\in V_0} |D^\gamma u_\alpha(x)| \leqq C^{|\gamma|+1} |\gamma|! \tag{6.3}$$

for all multi-indices γ *and all* $|\alpha|=\varepsilon$. *Then* K *and* Π *preserve local analyticity.*

Proof. — Suppose f is analytic in U_0. Then since $L_\alpha K_\alpha f = L_\alpha(f * k_\alpha) = f$,

$$\sup_{x\in V_0} |D^\gamma (K_\alpha f)(x)| \leqq C^{|\gamma|+1} |\gamma|!$$

and hence

$$\sup_{x\in V_0} |D^\gamma \int_\Gamma \alpha^{-1} K_\alpha f \, d\alpha| \leqq \int_\Gamma |\alpha|^{-1} \sup_{x\in V_0} |D^\gamma (K_\alpha f)(x)| \, d\alpha \leqq C' C^{|\gamma|+1} |\gamma|!$$

Hence Kf is analytic in V_0. The proof for Π is the same.

We shall now prove (6.3). For this we shall need the maximal estimate

$$\| X_{i_1} X_{i_2} \ldots X_{i_d} v \|_{L^2} \leqq C(\| L_\alpha v \|_{L^2} + \| v \|_{L^2}) \tag{6.4}$$

for all $v \in C_0^\infty(U_0)$, some constant C, which may be chosen independent of α, $|\alpha|=\varepsilon$. For each fixed α, $|\alpha|$ small but nonzero, the estimate (6.4) follows from the hypoellipticity [6] of L_α and is clearly preserved under sufficiently small changes in α on the circle $|\alpha|=\varepsilon$. Hence (6.4) follows by compactness for some C independent of α.

7. Proof of the uniform estimates on high derivatives

To demonstrate local bounds of the form:

$$|D^\beta u_\alpha(x)| \leqq C^{|\beta|+1} \beta! \qquad \forall \beta,\ x \in V_0$$

it is sufficient to obtain analogous L^2 bounds:

$$\| D^\beta u_\alpha \|_{L^2(V_1)} \leqq C_1^{|\beta|+1} |\beta|! \qquad \text{with} \quad V_0 \subset\subset V_1$$

and, as Nelson has shown, we may use the vector fields X_i and T instead of ordinary partial derivatives. Thus we write $X_I = X_{i_1} X_{i_2} \dots X_{i_{|I|}}$ (or $X^{|I|}$, abusively, for short) and shall show the bounds

$$\| X_I T^b u_\alpha \|_{L^2(V_1)} \leqq C_2^{|I|+b+1} (|I| + b)!$$

for all I and b, uniformly in α for $|\alpha| = \varepsilon$. Equivalently, we show:

$$(7.1) \qquad \| X_I T^b u_\alpha \|_{L^2(V_1)} \leqq C_3^{|I|+b+1} N^{|I|+b}$$

uniformly in α, $|\alpha| = \varepsilon$, N, I and b subject to $|I| + b \leqq N$, since Stirling's formula yields

$$N^N \leqq C_4^N N!$$

What follows is an extension of [16], but we feel much easier to read, to $d \geqq 2$ with attention given to the dependence of all estimates on α.

Clearly [*see* the *a priori* estimate (6.4)], estimating T derivatives is harder than estimating X derivatives, though one cannot, it appears, do one without the other. To use (6.4) effectively, we should at each stage try to retain at least d X's in our expressions, and yet this is no limitation, since high, pure T derivatives can yield the required X's by use of the commutation relations between the X's ($d/2$ times) and it is easy to see that if one has the desired bounds for $|I| \geqq d$, one also has them (with a different constant) for all smaller I.

To localize high T derivatives is not simple, for $[X_j, \varphi T^p]$ exhibits insufficient gain in p (at most a gain of 1/2 power, while a whole derivative lands on the localizing function). One could repeatedly replace X derivatives consumed in this fashion, but to do so would eventually transfer the p T-derivatives into derivatives of order $2p$ on φ, and this will not yield analyticity.

To overcome this obstacle, we introduce a rather complicated (looking) localization of T^p, i. e., a differential operator of order p, equal to T^p in any open set where $\varphi = 1$ and zero outside the support of φ. First, however, we must pick a new basis for $\mathcal{G}_1$. An analytic change of coordinates allows us to pick the basis:

$$X_j = X'_j = \partial/\partial x_j, \qquad j \leqq n,$$
$$X_{j+n} = X''_j = \partial/\partial y_j + x_j \partial/\partial t, \qquad j \leqq n,$$
$$T = \partial/\partial t,$$

where the X_j still generate $\mathcal{G}_1$, and T generates $\mathcal{G}_2$.

(7.2) DEFINITION. — *Let the* X_j, T *be defined as above. Then let:*

$$(T^p)_\varphi = T^p_\varphi = \sum_{r=|\beta+\gamma|\leqq p} \frac{(-1)^{|\beta|}}{\beta!\gamma!} (X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}.$$

(7.3) LEMMA. — *With* T_φ^p *defined as above, then modulo* C^p *terms of the form* $\varphi^{(p+1)} X^p/\beta!\,\gamma!$ *where* $|\beta+\gamma|=p$,

$$[X'_j, T_\varphi^p] \equiv 0,$$
$$[X''_j, T_\varphi^p] \equiv (T^{p-1})_{T\varphi} X''_j.$$

Proof. — From (7.2) and the obvious commutation relations $[X'_j, X''^\beta] = \beta_j$ terms, each $X''^{\beta-e_j}T$, where e_j is the multi-index of length one whose only non-zero entry is a 1 in the jth position,

(7.4)
$$[X'_j, T_\varphi^p] = \sum_{r=|\beta+\gamma|\leq p} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} (X'_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}$$
$$+ \sum_{r=|\beta+\gamma|\leq p} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} \beta_j (X'^\beta X''^\gamma \varphi) X'^\gamma X''^{\beta-e_j} T^{p-|\beta-e_j+\gamma|}.$$

Note that in the second sum, $r \geqq 1$, since for $r=0$, all $\beta_j=0$. But each term in the first sum, except those with $r=p$, is cancelled by a term in the second; a term in the first with $\beta=\beta_0$, $\gamma=\gamma_0$ is cancelled by a term in the second when $\beta=\beta_0+e_j$, $\gamma=\gamma_0$ unless $|\beta_0+\gamma_0|=p$. Only terms from the first sum with $r=p$ remain, and there are fewer than $(2n)^p$ of them.

For the second part of the Lemma, a similar cancellation takes place (with a shift of the γ index this time), the change of sign coming not from the power of -1, as it did with a shift of β, but from the observation that $[X''_j, X'^\gamma X''^\beta]$ consists of γ_j terms each $X'^{\gamma-e_j} X''^\beta [X''_j, X'_j]$ and $[X''_j, X'_j] = -T$. The more significant difference, however, is that in the first term in (7.4) the extra X'_j sits beside the other X' derivatives on φ, with X''_j it will be on the extreme left, while the others will sit beside φ. Thus what was literal cancellation for the first part of the Lemma will be a commutator here. To be precise:

$$[X''_j, T_\varphi^p] = \sum_{|\beta+\gamma|=p} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} (X''_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^0$$
$$+ \sum_{r=|\beta+\gamma|\leq p-1} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} (X''_j X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}$$
$$- \sum_{r=|\beta+\gamma|\leq p} \gamma_j \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} (X'^\beta X''^\gamma \varphi) X'^{\gamma-e_j} X''^\beta T^{p-|\beta+\gamma-e_j|}.$$

The last term may be rewritten, replacing $\gamma-e_j$ by γ, noting that this term is missing when $r=0$ (since then all γ_j are zero):

$$- \sum_{r=|\beta+\gamma|\leq p-1} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} (X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}$$

so that we have:

$$[X_j'', T_\varphi^p] = \sum_{|\beta+\gamma|=p} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!}(X_j'' X'^\beta X''^\gamma \varphi) X'^\gamma X''^\beta$$

$$+ \sum_{r=|\beta+\gamma|\leqq p-1} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!}([X_j'', X'^\beta] X''^\gamma \varphi) X'^\gamma X''^\beta T^{p-|\beta+\gamma|}$$

The first term above is the same type of error term as was discussed in proving the first part of the Lemma. $_{|\beta|}$The second term above may be written as:

$$-\sum_{r=|\beta+\gamma|=p-1} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!}\beta_j (X'^{\beta-e_j} X''^\gamma T\varphi) X'^\gamma X''^{\beta-e_j} T^{p-|\gamma+\beta-e_j|-1} \circ X_j''$$

$$= \sum_{r=|\beta+\gamma|\leqq p-1} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!}(X'^\beta X''^\gamma T\varphi) X'^\gamma X''^\beta T^{p-1-|\beta+\gamma|} \circ X_j''$$

(replacing $\beta - e_j$ by β). But this last is nothing but $(T^{p-1})_{T\varphi} \circ X_j''$.

Let N be fixed for now. We nest $[\![\log_2 N]\!]$ open sets:

$$V_1 = W_0 \subset\subset W_1 \subset\subset \ldots \subset\subset W_{[[\log_2 N]]} = U_0$$

(where $[\![\log_2 N]\!]$ denotes the integral part of $\log_2 N$), and choose functions ψ_j, φ_j, and χ_j in $C_0^\infty(W_{j+1})$ with $\psi_j = 1$ near $\overline{W_j}$, $\varphi_j = 1$ near supp ψ_j, and $\chi_j = 1$ near supp φ_j with specified bounds on their derivatives up to order $2N_j$ where

$$N_j = N/2^j.$$

Namely, we choose the W_j in such a way that if $d = \operatorname{dist}(V_1, U_0^{comp})$, then $d_j = \operatorname{dist}(W_j, W_{j+1}^{comp}) = d/2^{j+1}$. Then the ψ_j, φ_j and χ_j may be chosen (*cf.* [16]) so that

$$|D^\gamma(\psi_j, \varphi_j, \text{or } \chi_j)| \leqq (K d_j^{-1})^{|\gamma|} N_j^{|\gamma|} \quad \text{if} \quad |\gamma| \leqq 2N_j$$

with K independent of N (but depending on V_1 and U_0). These families of cut-off functions are just dilations of those introduced by Ehrenpreis and used by Hõrmander, Andersson, and others.

Since $\psi_0 = 1$ in $V_1 (= W_0)$, for $a+b$ less than N_0 but $a > d$, we estimate

$$\|X^a T^b u_\alpha\|_{L^2(V_1)} \quad \text{by} \quad \|X^d \psi_0 X^{a-d} T^b u_\alpha\|_{L^2}$$

and use the *a priori* estimate (6.4) on this. On the right there will be fewer X's:

(7.5) PROPOSITION. — *There exists a constant* $\tilde{C}$ *depending on f but independent of* α, $|\alpha| = \varepsilon$ *and* N *so that if* $a+b \leqq N_0$,

$$\|X^a T^b u_\alpha\|_{L^2(W_0)}/N_0^{a+b} \leqq \tilde{C}^{N_0}(d_0^{-1})^{N_0}\Big(1 + \sup_{\substack{d'+b' \leqq N_0 \\ d' \leqq d \\ b'-b \leqq (a-d')/2}} \|X^{d'} T^{b'} u_\alpha\|_{L^2(\text{supp}\,\psi_0)}/N_0^{d'+b'}\Big).$$

Proof. — Using (6.4) we have

$$(7.6)\quad \|X^d\psi_0 X^{a-d}T^b u_\alpha\|_{L^2} \leqq C(\|L_\alpha\psi_0 X^{a-d}T^b u_\alpha\|_{L^2} + \|\psi_0 X^{a-d}T^b u_\alpha\|_{L^2})$$

$$\leqq C(\|\psi_0 X^{a-d}T^b L_\alpha u_\alpha\|_{L^2} + \|\psi_0 X^{a-d}T^b u_\alpha\|_{L^2}$$

$$+ \sum_{|I|=d}(c'_I + c''_I|\alpha|)\,\|[X_I, \psi_0 X^{a-d}]T^b u_\alpha\|_{L_2}).$$

Here we have written (non-uniquely)

$$(7.7)\qquad L_\alpha = \sum_{|I|=d}(c'_I X_I + c''_I \alpha X_I)$$

with constants c'_I and c''_I. Next

$$[X^d, \psi_0 X^{a-d}] = [X^d, \psi_0]X^{a-d} + \psi_0[X^d, X^{a-d}]$$

consists of terms of the form $X^i\psi'_0 X^{a-i-1}$ ($i < d$, one for each i) arising from the first term on the right above and at most d times $a-d$ terms $\psi_0 X^{a-2}T$ from the second. To avoid constantly commuting X's to the left, we note that for $i < d$:

$$\|X^i\psi_0 X^{a-i}T^b u_\alpha\|_{L^2} \leqq \|X^d\psi_0 X^{a-d}T^b u_\alpha\|_{L^2} + \sum_{j=0}^{d-1}\|X^j\psi'_0 X^{a-j-1}T^b u_\alpha\|_{L^2}.$$

Thus we may generalize (7.6) to:

$$(7.8)\quad \sum_{i\leqq d}\|X^i\psi_0 X^{a-i}T^b u_\alpha\|_{L^2} \leqq C(\|\psi_0 X^{a-d}T^b f\|_{L^2}$$

$$+ \sum_{i<d}\|X^i\psi_0^{(1)}X^{a-i-1}T^b u_\alpha\|_{L^2} + C(a-d)\sum_{i\leqq d}\|X^i\psi_0 X^{a-i-2}T^{b+1}u_\alpha\|_{L^2})$$

with a new constant C (depending on d, but uniform in $|\alpha|=\varepsilon$), and now X^e may denote any X_I with $|I| \leqq e$.

We iterate this process (with a replaced by $a-1$, ψ_0 by ψ'_0, or with a by $a-2$ and b by $b+1$) on each term which still has at least $d+1$ X's, [except the first term, of course, since once a term contains $f(x)$, there is no need to iterate further]. One type of term, after a iterations, will be (bounded by)

$$C^a(a-d)^k\sum_{i\leqq d}\|X^i\psi_0^{(r)}X^{a-r-2k-i}T^{b+k}u_\alpha\|_{L^2}$$

for some k, r with $a-r-2k \leqq d$, and there will be at most $(2d+1)^a$ such terms.

The other terms will all contain f. These, again at most $(2d)^a$ of them, will be of the form

$$C^a(a-d)^k\|\psi_0^{(r)}X^{a-r-2k-d}T^{b+k}f\|_{L^2}.$$

In view of the bounds on derivatives of ψ_0, and the real analyticity of f in U_0, then, (7.8) yields:

$$\sum_{i\leq d} \| X^i \psi_0 X^{a-i} T^b u_\alpha \|_{L^2}$$

$$(7.9) \qquad \leq C^a \underset{0\leq a-2k-r=d'\leq d}{\text{supremum}}\ a^k (K d_0^{-1})^r N_0^r \| X^{d'} T^{b+k} u_\alpha \|_{L^2(\text{supp}\,\psi_0)}$$

$$+ C^a \underset{2k+r\leq a-d}{\text{supremum}}\ a^k (K d_0^{-1})^r N_0^r K_f^{a-r-k-d+b+1} (a-r-k-d+b)!$$

(The value of C, it should be clear by now, will change from estimate to estimate, but remain uniform in α, $|\alpha|=\varepsilon$ and independent of a, b and N as well as f.) This leads quickly to (7.5) since in the first term on the right in (7.9) we may observe that $a^k N_0^r N_0^{-(a+b)} \leq N_0^{r+k-a-b} \leq N_0^{-d'-b-k}$ if $d'=a-r-2k$ and for the second term on the right in (7.9) we use $a^k N_0^r (a-r-k-d+b)!\, N_0^{-a-b} \leq N_0^{k+r+a-r-d+b-a-b} \leq 1$. The strings of constants that build up, $C^a K^r$ for the first term in (7.9) and $C^a K^r K_f^{a-r-k-d+b+1}$ for the second, are both bounded by C'^{N_0} for a suitable new constant uniform in α, $|\alpha|=\varepsilon$, a, b, and N.

For $d' < d$, further iterations of this type will be useless in proving analyticity, since effective use of (6.4) requires essentially the presence of at least d X's. Using (7.2), however, we may continue profitably. For we have:

$$(7.10) \qquad \| X^{d'} T^{b'} u_\alpha \|_{L^2(\text{supp}\,\psi_0)} \leq \| X^{d'} (T^{b'})_{\varphi_0} u \|_{L^2}$$

since $\varphi_0 = 1$ near supp ψ_0, so $(T^{b'})_{\varphi_0} = T^{b'}$ in supp ψ_0.

(7.11) PROPOSITION. — *There exists a constant $\tilde{C}$ depending on f but not on α, $|\alpha|=\varepsilon$ or N so that if $a+b \leq N_0$:*

$$\underset{\substack{d'\leq d\\ b'+d\leq N_0\\ 0\leq b'-b\leq (a-d')/2}}{\text{supremum}} \| X^{d'} T^{b'} u_\alpha \|_{L^2(\text{supp}\,\psi_0)} / N_0^{b'+d'}$$

$$\leq \tilde{C}^{N_0} (d_0^{-1})^{N_0} (1 + \underset{a''\leq b+d+(a-d')/2}{\text{supremum}} \| X^{a''} u_\alpha \|_{L^2(\text{supp}\,\varphi_0)} / N_1^{a''})$$

Proof. — Since $X_k^* = -X_k$, integration by parts allows us to improve (6.4) by including terms with fewer X derivations on the left:

$$(6.4)' \qquad \sum_{d'\leq d} \| X^{d'} v \|_{L^2} \leq C(\| L_\alpha v \|_{L^2} + \| v \|_{L^2}), \qquad v \in C_0^\infty(U_0).$$

If we apply this to $v = (T^{b'})_{\varphi_0} u_\alpha$, we obtain, uniformly in α,

$$(7.12) \quad \sum_{d'\leq d} \| X^{d'} (T^{b'})_{\varphi_0} u_\alpha \|_{L^2} \leq C(\| L_\alpha (T^{b'})_{\varphi_0} u_\alpha \|_{L^2} + \| (T^{b'})_{\varphi_0} u_\alpha \|_{L^2})$$

$$\leq C(\| (T^{b'})_{\varphi_0} f \|_{L^2} + \| [L_\alpha, (T^{b'})_{\varphi_0}] u_\alpha \|_{L^2} + \| (T^{b'})_{\varphi_0} u_\alpha \|_{L^2}).$$

The commutator may be expanded using Lemma (7.3) and (7.7):

$$[L_\alpha, (T^{b'})_{\varphi_0}] = \sum_{|I|=d} (c_I' + c_I'' \alpha)[X_I, (T^{b'})_{\varphi_0}]$$

and an application of Lemma (7.3) gives:

$$[X_I, (T^{b'})_{\varphi_0}] = [X_{i_1} X_{i_2} \dots X_{i_d}, (T^{b'})_{\varphi_0}]$$

(7.13)
$$= \sum_{j=0}^{d-1} X_{i_1} \dots X_{i_j} [X_{i_{j+1}}, (T^{b'})_{\varphi_0}] X_{i_{j+2}} \dots X_{i_d}$$

$$= -A \sum_{j=0}^{d-1} X_{i_1} \dots X_{i_j} (T^{b'-1})_{T\varphi_0} X_{i_{j+1}} \dots X_{i_d} + C^{b'} \text{ terms } \sum_{j=0}^{d-1} X^j \varphi_0^{(b'+1)} X^{d-j-1+b'}/b'!$$

where A may be 0 or 1, depending on whether $X_{i_{j+1}}$ is an X' or an X''. We shall assume that $A=1$ below; when it is zero, that term just doesn't appear. Now to continue to use (6.4) or (6.4)′ on the right hand side above we would have to commute all X's to the left of $(T^{b'-1})_{T\varphi_0}$. This would introduce more terms of the same type, with X's on both sides. So we choose to estimate generally all divisions of the X's; i.e., we choose to estimate $\sum_{i \leq d' \leq d} \| X^i (T^{b'})_{\varphi_0} X^{d'-i} u_\alpha \|_{L^2}$. In doing so, we first attempt to bring all X's to the left [and then use (7.12)]—the above expansion of the bracket will yield an error which can be estimated by such a *sum* (over $i \leq d'$) but with smaller b', together with terms free of T altogether and then of course the right hand side of (7.12) followed by another use of (7.13). This gives:

(7.14)
$$\sum_{i \leq d' \leq d} \| X^i (T^{b'})_{\varphi_0} X^{d'-i} u_\alpha \|_{L^2} \leq C\Big(\sum_{i \leq d' \leq d} \| X^i (T^{b'-1})_{T\varphi_0} X^{d'-i} u_\alpha \|_{L^2}$$
$$+ \| (T^{b'})_{\varphi_0} f \|_{L^2} + \| (T^{b'})_{\varphi_0} u_\alpha \|_{L^2} + C^{b'} \sum_{j \leq d} \| X^j \varphi_0^{(b'+1)} X^{d-j-1+b'} u_\alpha \|_{L^2} / b'!$$

We want to iterate this to reduce b' still further. But first we must handle the third term on the right. By the definition;

$$(T^{b'})_{\varphi_0} = (T^{b'-1})_{\varphi_0} T + C^{b'} \text{ terms } (\varphi_0^{(b'+1)} X^{b'}/b'!)$$
$$= 2 \text{ terms } (T^{b'-1})_{\varphi_0} X^2 + C^{b'} \text{ terms } (\varphi_0^{(b'+1)} X^{b'}/b'!).$$

If we (abusively) now write φ_0' for $T\varphi_0$, $X\varphi_0$, or φ_0 itself, this expansion of $(T^{b'})_{\varphi_0}$ allows the third term in (7.14) to be absorbed by the first and fourth terms (with a new constant):

(7.15)
$$\sum_{i \leq d' \leq d} \| X^i (T^{b'})_{\varphi_0} X^{d'-i} u_\alpha \|_{L^2} \leq C\Big(\sum_{i \leq d' \leq d} \| X^i (T^{b'-1})_{T\varphi_0} X^{d'-i} u_\alpha \|_{L^2}$$
$$+ \| (T^{b'})_{\varphi_0} f \|_{L^2} + C^{b'} \sum_{j \leq d} \| X^j \varphi_0^{(b'+1)} X^{d-j-1+b'} u_\alpha \|_{L^2} / b'! \Big)$$

Now we may iterate (7.15) by subjecting the first term on the right to (7.15) again, with b' reduced by one and φ_0 replaced by φ_0'. After at most b' iterations, we obtain $C^{b'}$ terms, each either

$$(7.16)\qquad C^k \| (T^{b'-k})_{\varphi_0^{(k)}} f \|_{L^2} \text{ or } C^k C^{b'} \sum_{j \leqq d} \| X^j \varphi_0^{(b'+1)} X^{d-j-1+b'-k} u_\alpha \|_{L^2} / (b'-k)\,!$$

for some $k \leqq b'$.

For the first type [in (7.16)] we have [see Definition (7.2) where we have $|\beta+\gamma|=r \leqq p=b'-k$]:

$$(T^{b'-k})_{\varphi_0^{(k)}} f = C^{b'-k} \text{ terms, each } \varphi_0^{(k+r)} X^r T^{b'-k-r} f/\rho\,!$$

for some multi-index ρ with $|\rho|=r$, $r \leqq b'-k$. Since X derivatives of f have the same type of bounds as ordinary partial derivations,

$$|X_I T^{b_0} f| \leqq K_f^{b_0+|I|+1} (b_0+|I|)\,!$$

in a compact set (despite the coefficients in the X's) we have:

$$C^k \| (T^{b-k})_{\varphi_0^{(k)}} f \|_{L^2} \leqq C^{b'} \sup_{k+r \leqq b'} (K\, d_0^{-1})^{k+r+1} N_0^{k+r} K_f^{b'-k} (b'-k)\,!/r\,! \leqq (CKK_f d_0^{-1})^{b'+1} N_0^{b'}$$

[recall that $r+k \leqq b'$ so that $(b'-k)\,!/r\,! \leqq C^{b'} N_0^{b'-k-r}$ for $b' \leqq N_0$].

Thus we obtain, from (7.16) and the above,

$$(7.17)\qquad \sum_{d' \leqq d} \| X^{d'} (T^{b'})_{\varphi_0} u_\alpha \|_{L^2} \leqq (CKK_f d_0^{-1})^{b'+1} N_0^{b'}$$
$$+ C^{b'} \sup_{k \leqq b'} \sum_{j \leqq d} \| X^j \varphi_0^{(b'+1)} X^{d-j-1+b'-k} u_\alpha \|_{L^2} / (b'-k)\,!$$

To bring this last term into a clearer form we commute $\varphi_0^{(b'+1)}$ to the left and bring it out of the norm. Since $[X^j, \varphi_0^{(b'+1)}] = C^j$ terms, each $\varphi_0^{(b'+1+j')} X^{j-j'}$, $j' \leqq j$ we have:

$$C^{b'} \| X^j \varphi_0^{(b'+1)} X^{d-j-1+b'-k} u_\alpha \|_{L^2} / (b'-k)\,!$$
$$\leqq C^{d+b'} \sup_{j' \leqq j} (K\, d_0^{-1})^{b'+1+j'} N_0^{b'+1+j'} \| X^{d-j'-1+b'-k} u_\alpha \|_{L^2(\operatorname{supp} \varphi_0)} / (b'-k)\,!$$
$$\leqq C^{N_0} (K\, d_0^{-1})^{N_0} N_0^{b'+d} \| X^{d-j'-1+b'-k} u_\alpha \|_{L^2(\operatorname{supp} \varphi_0)} / N_0^{d-j'-1+b'-k}$$

since $N_0^{b'+1+j'} N_0^{d-j'-1+b'-k} / N_0^{b'+d} (b'-k)\,! \leqq e^{N_0}$ if $b'+d \leqq N_0$.

Together with (7.10) and (7.17), this proves Proposition (7.11), since $b'-b \leqq (a-d')/2$ implies $d+b'(-j'-1-k) \leqq d+b+(a-d')/2$.

Next, we once more reduce X derivatives. Actually, this could all have been done at once, as in [16], but breaking it down into three stages should render the proof more readable; this third stage is needed to reduce the total order by half. An application of (7.5) to the right hand side of (7.11) gives:

(7.18) PROPOSITION. — *There exists a constant $\tilde{C}$ depending on f but not on α, $|\alpha|=\varepsilon$ or N so that if $a+b \leqq N_0$:*

$$\underset{\substack{d' \leqq d \\ b'+d \leqq N_0 \\ b'-b \leqq (a-d')/2}}{\text{supremum}} \|X^{d'} T^{b'} u_\alpha\|_{L^2(\operatorname{supp} \psi_{0,1})} / N_0^{b'+d} \leqq \tilde{C}^{N_0} (d_0^{-1})^{2N_0} \left(1 + \underset{\substack{d' \leqq d \\ b''=(b+d+(a-3d')/2)/2}}{\text{supremum}} \|X^{d'} T^{b''} u_\alpha\|_{L^2(\operatorname{supp} \chi_0)} / N_1^{b''+d'}\right)$$

Combining (7.18) with (7.5) gives, for $a_0+b_0 \leqq 2N_0$

$$(7.19)\quad \|X^{a_0} T^{b_0} u_\alpha\|_{L^2(W_0)} / N_0^{a_0+b_0} \leqq \tilde{C}^{N_0} (d_0^{-1})^{3N_0} \Big(1 + \underset{a_1+b_1 \leqq d+(a_0+b_0)/2}{\text{supremum}} \|X^{a_1} T^{b_1} u_\alpha\| \quad / N_1^{a_1+b_1}\Big)$$

Actually, one calculates $a_1+b_1 \leqq (2b_0+a_0+3d)/4$, but $d+(a_0+b_0)/2$ will suffice.

(7.20) PROPOSITION. — *There exists a constant $\tilde{C}$, depending on f but not on α, $|\alpha|=\varepsilon$ or N such that for $j \leqq [[\log_2 N]]$ and for $a_j+b_j \leqq 2N_j$:*

$$\|X^{a_j} T^{b_j} u_\alpha\|_{L^2(W_j)} / N_j^{a_j+b_j}$$
$$\leqq \tilde{C}^{N_j} (d_j^{-1})^{3N_j} \Big(1 + \underset{a_{j+1}+b_{j+1} \leqq d+(a_j+b_j)/2}{\text{supremum}} \|X^{a_{j+1}} T^{b_{j+1}} u_\alpha\|_{L^2(W_{j+1})} / N_{j+1}^{a_{j+1}+b_{j+1}}\Big).$$

Proof. — Exactly the same proof as the proof of (7.19) applies, everything starting with a_j, b_j, W_j, N_j, d_j, etc. instead of a_0, b_0, W_0, N_0, d_0, etc.

If we now start with $a_0+b_0 \leqq N$ and apply (7.20) repeatedly, we obtain:

$$\underset{a_0+b_0 \leqq N_0}{\text{supremum}} \|X^{a_0} T^{b_0} u_\alpha\|_{L^2(W_0)} / N_0^{a_0+b_0} \leqq \tilde{C}^{N_0} (d_0^{-1})^{3N_0} (1 + \tilde{C}^{N_1} (d_1^{-1})^{3N_1} (1 + \tilde{C}^{N_2} (d_2^{-1})^{3N_2} (1 + \ldots$$
$$+ (1 + \tilde{C}^{2d} (d_{[[\log_2 N]]}^{-1})^{3[[\log_2 N]]} \underset{a+b<2d+1}{\text{supremum}} \|X^a T^b u_\alpha\|_{L^2(U_0)}) \ldots)$$
$$\leqq (2C)^{\Sigma N_j} \Pi (d_j^{-1})^{3N_j} \left(1 + \underset{\substack{a+b \leqq 2d+1 \\ |\alpha|=\varepsilon}}{\text{supremum}} \|X^a T^b u_\alpha\|_{L^2(U_0)}\right),$$

since $(\ldots((((a_0+b_0)/2+d)/2+d)/2+d)\ldots)/2+d \leqq 2d+1$ after $[[\log_2 N_0]]$ iterations.

Only in this last line does a supremum over α, $|\alpha|=\varepsilon$ enter. Now $\Sigma N_j \leqq 2N$ and we also have the bound:

$$\Pi (d_j^{-1})^{3N_j} \leqq C^N$$

since $\Pi (2^j)^{1/2^j} \leqq C$. The supremum over $|\alpha|=\varepsilon$ and $a+b \leqq 2d$ of $\|X^a T^b u_\alpha\|_{L^2(U_0)}$ is easily seen to be finite in view of the uniform boundedness of the k_α (*see* the definition following Lemma 4.1), and this finishes the proof of (7.1).

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