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006

ANALYTICITY OF SMOOTH CR MAPPINGS OF GENERIC SUBMANIFOLDS*

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Dedicated to Salah Baouendi for his seventieth birthday

Abstract. We consider a smooth CR mapping f from a real-analytic generic submanifold M in $\mathbb{C}^N$ into $\mathbb{C}^N$. For M of finite type and essentially finite at a point $p \in M$, and f formally finite at p, we give a necessary and sufficient condition for f to extend as a holomorphic mapping in some neighborhood of p. In a similar vein, we consider a formal holomorphic mapping H and give a necessary and sufficient condition for H to be convergent.

Key words.

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1. Introduction. In this paper, we consider a smooth CR mapping f from a real-analytic generic submanifold M in $\mathbb{C}^N$ into $\mathbb{C}^N$. Assuming that M is of finite type and essentially finite at a point $p \in M$, and that f is formally finite at p (see below), we give a necessary and sufficient condition for f to extend as a holomorphic mapping (Theorem 1.1) in some neighborhood of p (or equivalently to be real-analytic near p in M). In a similar vein, we consider a formal holomorphic mapping H and give a necessary and sufficient condition for H to be convergent (Theorem 1.2).

Before stating the main results, we shall recall some definitions. Let M be a real-analytic submanifold of codimension d in $\mathbb{C}^N$. Recall that M is said to be *generic* if M is defined locally near any point $p \in M$ by defining equations $\rho_1(Z,\bar Z) = \ldots = \rho_d(Z,\bar Z) = 0$ such that $\partial\rho_1 \wedge \ldots \wedge \partial\rho_d \neq 0$ along M. A generic submanifold M is said to be of *finite type* at $p \in M$ (in the sense of Kohn [K72] and Bloom-Graham [BG77]) if the (complex) Lie algebra $\mathfrak{g}_M$ generated by all smooth $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(p_0) = \mathbb{C}T_pM$, where $\mathbb{C}T_pM$ is the complexified tangent space to M. For the definition of *essentially finite*, the reader is referred to [BER99a]; see also Section 2 for an equivalent formulation in a slightly more general setting.

A (C^∞) smooth mapping $f\colon M \to \mathbb{C}^N$ is called CR if the tangent mapping df sends the CR bundle $T^{0,1}M$ into $T^{0,1}\mathbb{C}^N$. In particular, the restriction to M of a holomorphic mapping $H\colon U \to \mathbb{C}^N$, where U is some open neighborhood of M, is CR. To define the notion of *formally finite* at a point $p \in M$, we may assume, without loss of generality, that $p = f(p) = 0$. If $f\colon (M,0) \to (\mathbb{C}^N,0)$ is a germ of a smooth CR mapping, then one may associate to f a formal mapping $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ as follows. Let x be a local coordinate on M near 0 and $x \mapsto Z(x)$ the local embedding of M into $\mathbb{C}^N$ near 0. Then, there is a unique formal mapping H such that the Taylor series of $f(x)$ at 0 equals $H(Z(x))$ (see e.g. [BER99a], Proposition 1.7.14). Recall that a formal holomorphic (or simply formal) mapping $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ is an N-tuple $H = (H_1,\ldots,H_N)$ with $H_j \in \mathbb{C}[\![Z_1,\ldots,Z_N]\!] = \mathbb{C}[\![Z]\!]$ such that each component H_j has no constant term. We shall use the notation $I(H_1,\ldots,H_N)$ (or

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simply $I(H)$) for the ideal in $\mathbb{C}[\![Z]\!]$ generated by the components $H_1(Z), \ldots, H_N(Z)$. The formal mapping H is said to be *finite* if the ideal $I(H)$ is of finite codimension in $\mathbb{C}[\![Z]\!]$, i.e. if $\mathbb{C}[\![Z]\!]/I(H)$ is a finite dimensional vector space over $\mathbb{C}$. We shall say that the CR mapping f is *formally finite* if the associated formal mapping H is finite. The reader is referred to [BER99a] for further basic notions and properties of generic submanifolds in $\mathbb{C}^N$ and their mappings.

We may now state our first main result.

THEOREM 1.1. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of dimension m that is essentially finite and of finite type at 0. Let $f\colon (M,0) \to (\mathbb{C}^N, 0)$ be a smooth formally finite CR mapping. The following are equivalent.*

(i) *There exists an irreducible real-analytic subvariety $\tilde{X} \subset \mathbb{C}^N$ of dimension m at 0 such that $f(M) \subset \tilde{X}$ as germs at 0.*

(ii) *f is real-analytic in a neighborhood of 0.*

Our second result concerns the convergence of a formal holomorphic mapping sending a real-analytic generic submanifold into a real-analytic subvariety of the same dimension. Recall that a formal mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is said to send a real-analytic generic submanifold M through 0 in $\mathbb{C}^N$ into a real-analytic subvariety $\tilde{X}$ through 0 in $\mathbb{C}^N$, denoted $H(M) \subset \tilde{X}$, if $\sigma(H(Z(x)), \overline{H(Z(x))}) \equiv 0$ as a power series in x, where x is a local coordinate on M near 0, $x \mapsto Z(x)$ the local embedding of M into $\mathbb{C}^N$ near 0, $\sigma(Z, \bar{Z}) = (\sigma_1(Z, \bar{Z}), \ldots, \sigma_k(Z, \bar{Z}))$, and $\sigma_1(Z, \bar{Z}), \ldots, \sigma_k(Z, \bar{Z})$ generate the ideal of germs at 0 of real-analytic functions vanishing on $\tilde{X}$.

THEOREM 1.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of dimension m that is essentially finite and of finite type at 0. Let $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a finite formal holomorphic mapping. The following are equivalent.*

(i) *There exists an irreducible real-analytic subvariety $\tilde{X} \subset \mathbb{C}^N$ of dimension m at 0 such that $H(M) \subset \tilde{X}$.*

(ii) *H is convergent in a neighborhood of 0.*

The proofs of (i) $\Longrightarrow$ (ii) in Theorems 1.1 and 1.2 rest on general criteria for analyticity of smooth CR mappings and convergence of formal mappings given in [MMZ02] and [Mi02a], respectively, and a geometric result given in Lemma 3.1 below. A special case of (i) $\Longrightarrow$ (ii) in Theorem 1.1, in which the additional hypothesis that the target $\tilde{X}$ is a real-analytic generic submanifold $\widetilde{M}$ of dimension m is imposed, can be proved by using known results as follows. In [Me95], it was shown that f is analytic provided that either f is CR transversal to $\widetilde{M}$ at 0, or $\widetilde{M}$ is essentially finite at 0. The desired conclusion then follows from a result in [ER05], where the CR transversality of f was proved under the hypotheses given. One of the difficulties in the general case addressed here is the fact that the target $\tilde{X}$ need not be smooth at 0 and, consequently, there is no notion of tranversality of the mapping. This is overcome by showing directly that the target must satisfy a generalized essential finiteness condition (see Lemma 3.1), and applying the result from [MMZ02] mentioned above.

The case where M and $\tilde{X}$ are real-analytic (non-singular) hypersurfaces has a long history, beginning with the work of Lewy [Le77] and Pinchuk [P77]. There are also many subsequent results implying analyticity of a smooth CR mapping when the target $\tilde{X}$ is a real-analytic generic submanifold $\widetilde{M}$ of $\mathbb{C}^{N'}$ under various hypotheses on $\widetilde{M}$ and f. We mention here only, in addition to [MMZ02], the works [DW80], [Han83], [BJT85], [BR88], [DF88], [F89], [Pu90], [BHR96], [H96], [CPS00], [D01], [MMZ03b] and refer the reader to the bibliographies of these for further references. Previous results on convergence of formal mappings were given e.g. in the papers [BER00], [Mi00], [BMR02], [Mi02a], [Mi02b], [MMZ03a].

2. The essential variety of a real-analytic subvariety in $\mathbb{C}^N$. We begin by defining the notion of essential finiteness for a real-analytic subvariety X through 0 in $\mathbb{C}^N$. Let C_0^ω denote the ring of germs of real-valued real-analytic functions at 0 and $I_{\mathbb{R}}(X)$ be the ideal in C_0^ω of functions vanishing on X. Let $\sigma(Z,\bar Z) := (\sigma_1(Z,\bar Z),\ldots,\sigma_d(Z,\bar Z))$ be (representatives of) generators of $I_{\mathbb{R}}(X)$. We may assume that $\sigma(Z,\zeta)$ is defined in $B\times B$, where B is a sufficiently small open ball in $\mathbb{C}^N$ centered at the origin. Define the Segre variety $\Sigma_p\subset B$ of X at p, for $p\in B$, by the holomorphic equations $\sigma(Z,\bar p)=0$. Note that Σ_0 is a complex analytic variety through 0 and, by the reality of the functions $\sigma(Z,\bar Z)$, the variety Σ_p passes through 0 for every $p\in\Sigma_0$. Let $U\subset B$ be an open neighborhood of 0 and define

$$E_0^U := \bigcap_{p\in\Sigma_0\cap U}\Sigma_p. \tag{2.1}$$

Observe that E_0^U is a complex analytic variety through 0, and that $E_0^{U_1}\subset E_0^{U_2}$ (as a germ at 0) if $U_2\subset U_1$. Moreover, the germ of E_0^U at 0 depends only on the germs at 0 of the subvarieties Σ_p for $p\in\Sigma_0$ and, hence, does not depend on the ball B. We say that X is *essentially finite* at 0 if E_0^U has dimension 0 (as a germ at 0) for every open neighborhood U of 0.

We shall show (see Proposition 2.1 below) that, even if X is not essentially finite at 0, there exists a neighborhood U_0 of 0 such that $E_0^U=E_0^{U_0}$ (as germs at 0) for every $0\in U\subset U_0$. (This is well known in the case where X is a CR manifold.) Moreover, we shall give an alternative characterization of the stabilized germ $E_0^{U_0}$ that will be used in the proof of Theorem 1.1. Let A be the subvariety in B defined by

$$A := \{z\in B\colon \Sigma_0\subset\Sigma_z \text{ as germs at } 0\}. \tag{2.2}$$

PROPOSITION 2.1. *Let $X\subset\mathbb{C}^N$ be a real-analytic subvariety with $0\in X$ and let A be defined by (2.2), where B is a sufficiently small ball centered at 0. Then, there exists an open neighborhood $U_0\subset\mathbb{C}^N$ of 0 such that $E_0^U=A$, as germs at 0, for every open U with $0\in U\subset U_0$. In particular, $E_0^U=E_0^{U_0}$ as germs at 0.*

Proof. [Proof of Proposition 2.1] Let U be an open neighborhood of 0 contained in B. Let us temporarily, in order to distinguish between germs and subvarieties, introduce the notation D^U for the following representative in U of the germ at 0 of E_0^U

$$D^U := \bigcap_{p\in\Sigma_0\cap U}(\Sigma_p\cap U). \tag{2.3}$$

Observe that there exists an open neighborhood U_0 of 0, contained in B, with the property that if V is a subvariety in B and U is an open neighborhood of 0 with $U\subset U_0$, then

$$\Sigma_0\subset V \text{ as germs at } 0 \iff \Sigma_0\cap U\subset V\cap U. \tag{2.4}$$

Indeed, it suffices to take U_0 so small that only the irreducible components of Σ_0 in B that contain the origin meet U_0. As a consequence, if U is an open neighborhood of 0 with $U \subset U_0$, then the subvariety $A \cap U$ can be expressed as being those points $Z \in U$ for which $\Sigma_0 \cap U \subset \Sigma_Z \cap U$, i.e.

$$A \cap U = \{Z \in U \colon \sigma(W, \bar{Z}) = 0 \ \forall W \in U \text{ such that } \sigma(W, 0) = 0\}. \tag{2.5}$$

On the other hand, the subvariety D^U, defined by (2.3), can be expressed in equations as follows

$$D^U = \{Z \in U \colon \sigma(Z, \bar{W}) = 0 \ \forall W \in U \text{ such that } \sigma(W, 0) = 0\}. \tag{2.6}$$

Since $\sigma(Z, \bar{W}) = \overline{\sigma(W, \bar{Z})}$, we conclude that $A \cap U = D^U$ and, hence, that $A = E_0^U$ as germs at 0. This completes the proof of Proposition 2.1 □

We shall refer to the germ at 0 of $E_0^{U_0}$ as the *essential variety* of X at 0 and denote this germ by E_0. Thus, X is essentially finite at 0 if and only if $E_0 = \{0\}$. When X is a generic submanifold, these two notions coincide with the standard notions of essential variety and essential finiteness (see e.g. [BER99a]).

We shall need the following reformulation of essential finiteness. In what follows, we shall let $\mathcal{X} \subset \mathbb{C}^N \times \mathbb{C}^N$ denote the (local) complexification of a real-analytic subvariety X through 0 in $\mathbb{C}^N$, i.e. the complex variety through 0 in $\mathbb{C}^N \times \mathbb{C}^N$ defined as the set of points $(Z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N$, near the origin, such that

$$\sigma_1(Z, \zeta) = \ldots = \sigma_d(Z, \zeta) = 0$$

where $\sigma_1(Z, \bar{Z}), \ldots \sigma_d(Z, \bar{Z})$ are generators of the ideal $I_{\mathbb{R}}(X)$. For a complex analytic subvariety V through 0 in $\mathbb{C}^k$, we shall also denote by $I_{\mathcal{O}}(V)$ the ideal of germs at 0 of holomorphic functions vanishing on V. We shall let $I(V)$ denote the ideal generated by $I_{\mathcal{O}}(V)$ in the corresponding ring of formal power series. If I is an ideal in a ring R, then we shall write $d(I)$ for the dimension of I in R, i.e. the dimension of the ring R/I (see [Ei95]; some texts, e.g. [AM69], refer to the number $d(I)$ as the *depth* of I).

LEMMA 2.2. *Let X be a real-analytic variety through* 0 *in $\mathbb{C}^N$ and let Σ_0 be its Segre variety at* 0. *The following are equivalent:*

(a) *X is not essentially finite at* 0.

(b) *There is a positive dimensional complex variety Γ through* 0 *in $\mathbb{C}^N$ such that $\Gamma \times \Sigma_0^* \subset \mathcal{X}$ as germs at* 0, *where $\mathcal{X} \subset \mathbb{C}^N \times \mathbb{C}^N$ denotes the complexification of X and * denotes the complex conjugate of a set (i.e. $S^* := \{Z \in \mathbb{C}^N \colon \bar{Z} \in S\}$).*

(c) *There is an ideal $J \subset \mathbb{C}[[Z]]$ with positive dimension such that $I(\mathcal{X}) \subset I(J) + I(\mathbb{C}^N \times \Sigma_0^*)$.*

(d) *There is a non-trivial formal holomorphic mapping $\mu\colon (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ and a neighborhood $U \subset \mathbb{C}^N$ such that $\mu(\mathbb{C}) \subset E_0^U$ (where E_0^U is defined by* (2.1)*), i.e. if $\sigma(Z, \bar{Z}) = (\sigma_1(Z, \bar{Z}), \ldots, \sigma_d(Z, \bar{Z}))$ are generators of the ideal $I_{\mathbb{R}}(X)$, then, for every fixed $p \in \Sigma_0 \cap U$,*

$$s \mapsto \sigma(\mu(s), \bar{p}) \text{ is identically zero as a power series.} \tag{2.7}$$

Proof. Recall that we use the coordinates (Z, ζ) in $\mathbb{C}^N \times \mathbb{C}^N$. Observe that for $p \in \mathbb{C}^N$, we have

$$\mathcal{X} \cap \{\zeta = \bar{p}\} = \Sigma_p \times \{\bar{p}\}, \quad \mathcal{X} \cap \{Z = p\} = \{p\} \times \Sigma_p^*. \tag{2.8}$$

Hence, $\Gamma \times \Sigma_0^* \subset \mathcal{X}$ means that for every $p \in \Sigma_0$,

$$\Gamma \times \{\bar{p}\} \subset \mathcal{X} \cap \{\zeta = \bar{p}\} = \Sigma_p \times \{\bar{p}\},$$

i.e. $\Gamma \subset \Sigma_p$. The equivalence of (a) and (b) is a simple consequence of this observation.

The implication (b) $\Longrightarrow$ (c) is easy. Simply observe that $\Gamma \times \Sigma_0^* = (\Gamma \times \mathbb{C}^N) \cap (\mathbb{C}^N \times \Sigma_0^*)$. The conclusion of (c) now follows from (b) by taking $J = I(\Gamma)$.

To prove the implication (c) $\Longrightarrow$ (d), we let $\mu\colon (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ be a germ at 0 of a non-trivial holomorphic mapping such that $f(\mu(t)) \equiv 0$ for all $f \in J$ (such exist, by [BER00], Lemma 3.32, since $d(J) \geq 1$). It follows from the hypothesis in (c) that there are formal power series $d_{ij}(Z, \zeta)$ such that

$$\sigma_i(\mu(t), \zeta) = \sum_{j=1}^{k} d_{ij}(\mu(t), \zeta) h_j(\zeta), \tag{2.9}$$

where $h_1(\zeta), \ldots, h_k(\zeta)$ generate the ideal $I(\Sigma_0^*) \subset \mathbb{C}[[\zeta]]$. Let us Taylor expand $\sigma_i(\mu(t), \zeta)$ in t,

$$\sigma_i(\mu(t), \zeta) = \sum_{l=0}^{\infty} a_{il}(\zeta) t^l, \tag{2.10}$$

and note that the coefficients $a_{il}(\zeta)$ are all holomorphic functions of ζ in some open neighborhood V of 0. If we Taylor expand both sides of (2.9) in t and compare coefficients, we conclude that the coefficents a_{il} all belong to the ideal $I(\Sigma_0^*) \subset \mathbb{C}[[\zeta]]$. Consequently, they vanish on $\Sigma_0^* \cap U$ for some open neighborhood $U \subset V$ of 0. This proves (d).

To show that (*d*) implies (*a*), let $\mu\colon (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ be a non-trivial formal mapping such that $\mu(\mathbb{C}) \subset E_0^U$ for some U. Then $I(E_0^U)$ does not have finite codimension (see [BER00], Lemma 3.32) and, hence, the dimension of E_0^U is positive, i.e. (a) holds. □

3. Main lemma. Recall that a formal mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is said to send (a germ at 0 of) a real-analytic subvariety $X \subset \mathbb{C}^N$ into another $\tilde{X} \subset \mathbb{C}^N$, denoted $H(X) \subset \tilde{X}$, if

$$\tilde{\sigma}\big(H(Z), \overline{H(Z)}\big) = a(Z, \bar{Z}) \sigma(Z, \bar{Z}) \tag{3.1}$$

holds as formal power series in Z and $\bar{Z}$, where $\sigma = (\sigma_1, \ldots, \sigma_d)^t$ and $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{\tilde{d}})^t$ generate $I_{\mathbb{R}}(X)$ and $I_{\mathbb{R}}(\tilde{X})$, respectively, and $a(Z, \bar{Z})$ is a $\tilde{d} \times d$ matrix of formal power series.

Before stating our main lemma, we review some notation and results concerning homomorphisms induced by mappings. If $h : (\mathbb{C}_z^k, 0) \to (\mathbb{C}_{\tilde{z}}^k, 0)$ is a formal mapping, then h induces a homomorphism $\varphi_h : \mathbb{C}[[\tilde{z}]] \to \mathbb{C}[[z]]$ defined by $\varphi_h(\tilde{f})(z) := \tilde{f}(h(z))$. If J is an ideal in $\mathbb{C}[[z]]$, then $\varphi_h^{-1}(J)$ is an ideal in $\mathbb{C}[[\tilde{z}]]$, and $\varphi_h^{-1}(J)$ is prime if J is prime. If $\tilde{J}$ is an ideal in $\mathbb{C}[[\tilde{z}]]$, we denote by $I(\varphi_h(\tilde{J}))$ the ideal in $\mathbb{C}[[z]]$ generated by $\varphi_h(\tilde{J})$. If $\operatorname{Jac} h \not\equiv 0$, where $\operatorname{Jac} h$ denotes the Jacobian determinant of h, then φ_h is injective. Indeed, in that case any $f \in \mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!]$ for which $f \circ h \equiv 0$ must be identically zero (see e.g. [BER99a], Proposition 5.3.5).

We may now state the geometric result needed to prove Theorem 1.1.

LEMMA 3.1. *Let $X, \tilde{X} \subset \mathbb{C}^N$ be irreducible real-analytic subvarieties of dimension m through 0 and Σ_0, $\tilde{\Sigma}_0$ their Segre varieties at 0. Let $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a finite formal mapping such that $H(X) \subset \tilde{X}$. Then the following hold.*

(i) *$\varphi_H^{-1}(I(\Sigma_0)) = I(\tilde{\Sigma}_0)$, where φ_H denotes the homomorphism $\varphi_H\colon \mathbb{C}[[\tilde{Z}]] \to \mathbb{C}[[Z]]$ induced by H. In particular, $\dim \tilde{\Sigma}_0 = \dim \Sigma_0$, and if Σ_0 is irreducible, then so is $\tilde{\Sigma}_0$.*

(ii) *If Σ_0 is irreducible at 0, then X is essentially finite at 0 if and only if $\tilde{X}$ is essentially finite at 0.*

Proof. [Proof of Lemma 3.1] We begin by proving statement (i). We denote by $\mathcal{H}\colon (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^N \times \mathbb{C}^N, 0)$ the complexified formal mapping $\mathcal{H}(Z, \zeta) = (H(Z), \bar{H}(\zeta))$ and by $\Phi = \varphi_{\mathcal{H}}\colon \mathbb{C}[[\tilde{Z}, \tilde{\zeta}]] \to \mathbb{C}[[Z, \zeta]]$ the induced homomorphism, i.e. $\Phi(\tilde{f})(Z, \zeta) := \tilde{f}(\mathcal{H}(Z, \zeta))$. The fact that H sends X into $\tilde{X}$ can be rephrased as saying that $I(\Phi(I(\tilde{\mathcal{X}}))) \subset I(\mathcal{X})$ or equivalently $I(\tilde{\mathcal{X}}) \subset \Phi^{-1}(I(\mathcal{X}))$, where $\tilde{\mathcal{X}}$ denotes the complexification of $\tilde{X}$. Since $\operatorname{Jac} \mathcal{H} \not\equiv 0$, Φ is injective. We also claim that $\mathbb{C}[\![Z, \zeta]\!]$ is integral over $\Phi(\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!])$. To see this, note, as is well known, that $\mathbb{C}[\![Z, \zeta]\!]$ is finitely generated over $\Phi(\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!])$ (any $f_1, \ldots, f_p \in \mathbb{C}[\![Z, \zeta]\!]$ such that their images form a basis for the finite dimensional vector space $\mathbb{C}[\![Z, \zeta]\!]/I(\mathcal{H}_1, \ldots, \mathcal{H}_{2N})$ generate $\mathbb{C}[\![Z, \zeta]\!]$ over $\Phi(\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!])$). The fact that $\mathbb{C}[\![Z, \zeta]\!]$ is integral over $\Phi(\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!])$ is now a general fact about finitely generated modules (see e.g. [AM69], Proposition 5.1). It follows that $d(\Phi^{-1}(I(\mathcal{X}))) = d(I(\mathcal{X}))$ (see e.g. [Ei95], Proposition 9.2). Moreover, $\Phi^{-1}(I(\mathcal{X}))$ is a prime ideal that contains the prime ideal $I(\tilde{\mathcal{X}})$. Since the dimensions of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are the same, we conclude that $d(\Phi^{-1}(I(\mathcal{X}))) = d(I(\tilde{\mathcal{X}}))$ and, hence,

$$\Phi^{-1}(I(\mathcal{X})) = I(\tilde{\mathcal{X}}). \tag{3.2}$$

Observe that $\mathcal{X} \cap \{\zeta = 0\} = \Sigma_0 \times \{0\}$. By using the specific form of the mapping $\mathcal{H}$, we see that

$$\varphi^{-1}(I(\Sigma_0)) = I(\tilde{\Sigma}_0) \iff \Phi^{-1}(I(\Sigma_0 \times \{0\})) = I(\Sigma_0 \times \{0\}), \tag{3.3}$$

where $\varphi = \varphi_H$. By the Nullstellensatz,

$$I(\Sigma_0 \times \{0\}) = \sqrt{I(\mathcal{X}) + I(\zeta)}. \tag{3.4}$$

Thus, the first identity in statement (i) is equivalent to

$$\sqrt{I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})} = \Phi^{-1}(\sqrt{I(\mathcal{X}) + I(\zeta)}). \tag{3.5}$$

We remark that the inclusion $I(\tilde{\mathcal{X}}) \subset \Phi^{-1}(I(\mathcal{X}))$ implies $I(\tilde{\mathcal{X}}) + I(\tilde{\zeta}) \subset \Phi^{-1}(I(\mathcal{X}) + I(\zeta))$ and hence the inclusion

$$\sqrt{I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})} \subset \Phi^{-1}(\sqrt{I(\mathcal{X}) + I(\zeta)}). \tag{3.6}$$

To prove the opposite inclusion, it suffices to show that $\Phi^{-1}(I(\mathcal{X}) + I(\zeta)) \subset \sqrt{I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})}$. Thus, we suppose that $\Phi(\tilde{f}) \in I(\mathcal{X}) + I(\zeta)$ and must prove $\tilde{f}^k \in I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})$ for some k. Since $\bar{H}$ is a formal finite mapping, there is an integer m such that $I(\zeta)^m \subset I(\Phi(I(\tilde{\zeta})))$. We conclude that, for any $k \geq m$,

$$\Phi(\tilde{f})^k = \Phi(\tilde{f}^k) \in I(\mathcal{X}) + I(\zeta)^k \subset I(\mathcal{X}) + I(\Phi(I(\tilde{\zeta}))). \tag{3.7}$$

Hence, we have

$$\Phi(\bar{f}^k) = g_k(Z,\zeta) + \sum_{j=1}^{N} a_{jk}(Z,\zeta)\bar{H}_j(\zeta), \tag{3.8}$$

where $g_k \in I(\mathcal{X})$ and $a_{jk} \in \mathbb{C}[[Z,\zeta]]$. Let (α_j, β_j), for $j = 0,\dots,p$, be multi-indices such that the images of $Z^{\alpha_j}\zeta^{\beta_j}$ generate the finite dimensional vector space $\mathbb{C}[[Z,\zeta]]/I(\mathcal{H})$. Observe that $(0,0)$ is necessarily one of these; we order the multi-indices in such a way that $(\alpha_0,\beta_0) = (0,0)$. We may then, as is easy to verify, write each $a_j(Z,\zeta)$ in the following way

$$a_{jk}(Z,\zeta) = \sum_{l=0}^{p} b_{jlk}(\mathcal{H}(Z,\zeta))Z^{\alpha_l}\zeta^{\beta_l}, \tag{3.9}$$

where $b_{00k}(0) = 0$ (since $\Phi(\bar{f}^k)(0) = 0$). By substituting (3.9) in (3.8), it follows that

$$\Phi(\bar{f}^k) = g_k(Z,\zeta) + \sum_{l=0}^{p}\sum_{j=1}^{N} b_{jlk}(\mathcal{H}(Z,\zeta))\bar{H}_j(\zeta)Z^{\alpha_l}\zeta^{\beta_l}. \tag{3.10}$$

Let S and $\tilde{S}$ denote rings $\mathbb{C}[[Z,\zeta]]/I(\mathcal{X})$ and $\mathbb{C}[[\tilde{Z},\tilde{\zeta}]]/I(\tilde{\mathcal{X}})$, respectively. We shall denote by f^* and A^* the images of any element $f \in \mathbb{C}[[Z,\zeta]]$ and subset $A \subset \mathbb{C}[[Z,\zeta]]$ in S, and similarly for images in $\tilde{S}$. Since $\Phi(I(\tilde{\mathcal{X}}) \subset I(\mathcal{X})$, the homomorphism Φ induces a homomorphism $\Phi^*: \tilde{S} \to S$. The facts that Φ is injective and $\Phi^{-1}(I(\mathcal{X})) = I(\tilde{\mathcal{X}})$ imply that Φ^* is injective. Let $\tilde{R}$ denote the ring with identity generated by $I(\tilde{\zeta})^* \subset \tilde{S}$, i.e.

$$\tilde{R} := \{\tilde{g}^* + c^* : \tilde{g} \in I(\tilde{\zeta}),\ c \in \mathbb{C}\}.$$

Denote by R the ring $\Phi^*(\tilde{R}) \subset S$ and let $\mathcal{N}$ denote the R-module generated by $(Z_j^{\alpha}\zeta^{\beta_j})^*$ for $j = 0,\dots,p$. Clearly, $\mathcal{N}$ is not annihilated by any elements of S. Let s denote $\Phi(\bar{f}^m)^* \in S$ and observe that

$$s = \Phi(\bar{f}^m)^* = \sum_{l=0}^{p}\sum_{j=1}^{N}\left(b_{jlm}(\mathcal{H}(Z,\zeta))\bar{H}_j(\zeta)Z^{\alpha_l}\zeta^{\beta_l}\right)^*. \tag{3.11}$$

We claim that $s\mathcal{N} \subset \mathcal{N}$. Indeed, if $f^* \in \mathcal{N}$, then

$$f^* = \sum_{i=0}^{p}\left(d_i(\mathcal{H}(Z,\zeta))Z^{\alpha_i}\zeta^{\beta_i}\right)^*,$$

where $d_l = \bar{g} + c$ for some $\bar{g} \in I(\tilde{\zeta})$ and $c \in \mathbb{C}$, and hence

$$sf^* = \sum_{l=0}^{p}\sum_{i=0}^{p}\sum_{j=1}^{N}\left(b_{jlm}(\mathcal{H}(Z,\zeta))d_i(\mathcal{H}(Z,\zeta))\bar{H}_j(\zeta)Z^{\alpha_l+\alpha_i}\zeta^{\beta_l+\beta_i}\right)^*. \tag{3.12}$$

Any monomial $Z^\alpha\zeta^\beta$ can be written

$$Z^\alpha\zeta^\beta = \sum_{q=0}^{p} e_{\alpha\beta q}(\mathcal{H}(Z,\zeta))Z^{\alpha_q}\zeta^{\beta_q}. \tag{3.13}$$

By substituting (3.13) in (3.12), we conclude that $sf^* \in \mathcal{N}$, as claimed. It follows from [Ei95], Corollary 4.6, that s is integral over R, i.e.

$$s^r + \sum_{k=0}^{r-1} (h_k(\mathcal{H}(Z,\zeta)))^* s^k = 0, \tag{3.14}$$

where $h_k = \tilde{g}_k + c_k$ for $\tilde{g}_k \in I(\tilde{\zeta})$ and $c_k \in \mathbb{C}$. Since Φ^* is injective (and $s = \Phi(\tilde{f}^m)^*$), we conclude that

$$(\tilde{f}^{rm}(\tilde{Z},\tilde{\zeta}))^* + \sum_{k=0}^{r-1} (h_k(\tilde{Z},\tilde{\zeta}))^* (\tilde{f}^{km}(\tilde{Z},\tilde{\zeta}))^* = 0. \tag{3.15}$$

Since $h_k = \tilde{g}_k + c_k$, we can rewrite this as

$$(\tilde{f}^{rm}(\tilde{Z},\tilde{\zeta}))^* + \sum_{k=0}^{r-1} (g_k(\tilde{Z},\tilde{\zeta}))^* (\tilde{f}^{km}(\tilde{Z},\tilde{\zeta}))^* + \sum_{k=1}^{r-1} c_k^* (\tilde{f}^{km}(\tilde{Z},\tilde{\zeta}))^* = -c_0^*. \tag{3.16}$$

First, observe that the left hand side of (3.16) is in the maximal ideal of $\tilde{S}$ and, hence, $c_0^* = 0$. Let us write $c_r^* = 1$, and let k_0 be the smallest integer in $\{1, \ldots, r\}$ such that $c_{k_0}^* \neq 0$. We may then rewrite (3.16) as

$$(\tilde{f}^{mk_0}(\tilde{Z},\tilde{\zeta}))^* \left(\sum_{k=k_0}^{r} c_k^* (\tilde{f}^{m(k-k_0)}(\tilde{Z},\tilde{\zeta}))^* \right) = -\sum_{k=0}^{r-1} (g_k(\tilde{Z},\tilde{\zeta}))^* (\tilde{f}^{km}(\tilde{Z},\tilde{\zeta}))^*. \tag{3.17}$$

The right hand side of (3.17) is in $I(\tilde{\zeta})^*$ and, hence, so is then the left hand side. Observe that

$$\tilde{f}^{m(r-k_0)}(\tilde{Z},\tilde{\zeta}))^* + \sum_{k=k_0}^{r-1} c_k^* (\tilde{f}^{m(k-k_0)}(\tilde{Z},\tilde{\zeta}))^*$$

is a unit, since $c_{k_0}^* \neq 0$. We conclude that $(\tilde{f}^{mk_0})^* \in I(\tilde{\zeta})^*$ and, hence, $\tilde{f}^{mk_0} \in I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})$, as desired. This completes the proof of the identity $\varphi_H^{-1}(I(\Sigma_0)) = I(\tilde{\Sigma}_0)$. We observe that if $I(\Sigma_0)$ is prime, then so is $I(\tilde{\Sigma}_0)$. The fact that $d(I(\tilde{\Sigma}_0)) = d(I(\Sigma_0))$ (or equivalently $\dim \Sigma_0 = \dim \tilde{\Sigma}_0$) follows from the assumption that H is a finite mapping (cf. the use of [Ei95], Proposition 9.2 above). This completes the proof of (i) in Lemma 3.1.

4. Proof of (ii) of Lemma 3.1. We now proceed with the proof of statement (ii) in Lemma 3.1. The fact that Σ_0 is assumed to be irreducible at 0 implies that $I(\Sigma_0)$ is a prime ideal. By part (i) of Lemma 3.1, it follows that $I(\tilde{\Sigma}_0) (= \varphi_H^{-1}(I(\Sigma_0)))$ is a prime ideal and $d(\varphi_H^{-1}(I(\Sigma_0))) = d(I(\Sigma_0))$. Equivalently, $\tilde{\Sigma}_0$ is an irreducible complex analytic variety of the same dimension as Σ_0.

We begin by proving that if X is essentially finite at 0, then $\tilde{X}$ is also essentially finite at 0. We shall show the logical negation of this statement. Thus, we shall assume that $\tilde{X}$ is not essentially finite at 0. By Lemma 2.2, there exists a positive dimensional complex analytic variety $\tilde{\Gamma} \subset \mathbb{C}^N$ through 0 such that $\tilde{\Gamma} \times \tilde{\Sigma}_0^* \subset \tilde{\mathcal{X}}$. We may assume that $\tilde{\Gamma}$ is irreducible.

LEMMA 4.1. *Let $\bar{\Gamma}$ be as above and let $\tilde{\mathfrak{P}}$ be the (prime) ideal in $\mathbb{C}[\![\tilde{Z},\tilde{\zeta}]\!]$ generated by the ideal of $\Gamma \times \bar{\Sigma}_0^*$ (in the ring $\mathbb{C}\{\tilde{Z},\tilde{\zeta}\}$ of convergent power series). Let $\Phi := \varphi_{\mathcal{H}}$ be the homomorphism defined in the beginning of the proof of Lemma* 3.1. *If*

$$I(\Phi(\tilde{\mathfrak{P}})) = \bigcap_{j=1}^{m} P_j \tag{4.1}$$

is a primary decomposition of $I(\Phi(\tilde{\mathfrak{P}}))$, then there exists j_0 such that

$$I(\mathcal{X}) \subset \sqrt{P_{j_0}} \text{ and } I(P_{j_0}) = d(\tilde{\mathfrak{P}}). \tag{4.2}$$

Proof. Suppose, in order to reach a contradiction, that this is not the case. Then, each of the ideals $I(\mathcal{X}) + P_j$ must satisfy

$$d(I(\mathcal{X}) + P_j) \leq d(P_j) - 1 \leq d(\tilde{\mathfrak{P}}) - 1.$$

It follows that

$$d\left(\bigcap_{i=1}^{m}(I(\mathcal{X}) + P_j)\right) \leq d(\tilde{\mathfrak{p}}) - 1. \tag{4.3}$$

Consider the induced homomorphism

$$\Phi^*: \mathbb{C}[\![\tilde{Z},\tilde{\zeta}]\!]/I(\tilde{\mathcal{X}}) \to \mathbb{C}[\![Z,\zeta]\!]/I(\mathcal{X}). \tag{4.4}$$

If we, as above, use J^* to denote the image of an ideal $J \subset \mathbb{C}[\![\tilde{Z},\tilde{\zeta}]\!]$ in $\mathbb{C}[\![\tilde{Z},\tilde{\zeta}]\!]/I(\tilde{\mathcal{X}})$ and similarly for images in $\mathbb{C}[\![Z,\zeta]\!]/I(\mathcal{X})$, then we have

$$I(\Phi^*(\tilde{\mathfrak{P}}^*)) = \left(\bigcap_{j=1}^{m} P_j\right)^*. \tag{4.5}$$

Observe that

$$\left(\bigcap_{j=1}^{m} P_j^*\right)^m \subset \left(\bigcap_{j=1}^{m} P_j\right)^* \subset \bigcap_{j=1}^{m} P_j^*.$$

Since $P_j^* = (P_j + I(\mathcal{X}))^*$ and, hence, $d(P_j^*) \leq d(\tilde{\mathfrak{P}}^*) - 1$, we conclude that

$$d(I(\Phi^*(\tilde{\mathfrak{P}}^*))) \leq d(\tilde{\mathfrak{P}}^*) - 1. \tag{4.6}$$

We shall show that this is a contradiction. Recall that the homomorphism Φ^* is injective (see the proof of (i) above). Clearly, $\mathbb{C}[\![Z,\zeta]\!]/I(\mathcal{X})$ is integral over $\Phi^*(\mathbb{C}[\![\tilde{Z},\tilde{\zeta}]\!]/I(\tilde{\mathcal{X}}))$, since $\mathbb{C}[\![Z,\zeta]\!]$ is integral over $\Phi(\mathbb{C}[\![\tilde{Z},\tilde{\zeta}]\!])$. Now, the fact that (4.6) cannot hold is an immediate consequence of Lemma 4.2 below. □

The proof of Lemma 4.1 follows, as mentioned above, from the following commutative algebra lemma. For the reader's convenience, we include a proof; we have been unable to find an exact reference for this statement.

LEMMA 4.2. *Let A and B be (commutative) rings, and $\psi: A \to B$ an injective homomorphism such that B is integral over $\psi(A)$. Then, for any ideal $J \subset A$,*

$$d(I(\psi(J))) = d(J).$$

Proof. [Proof of Lemma 4.2] Since ψ is injective, we may identify A with the subring $\psi(A) \subset B$. Thus, J is identified with its image $\psi(J)$ and $I(J)$ is the ideal in B generated by J. Let

$$I(J) \subset \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \dots \subsetneq \mathfrak{q}_r, \tag{4.7}$$

be a chain of prime ideals. Then $\mathfrak{p}_i = \mathfrak{q}_i \bigcap A$ $(= \psi^{-1}(\mathfrak{q}_i))$ are prime ideals and since $J \subset I(J) \bigcap A$, we obtain

$$J \subset \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_r. \tag{4.8}$$

By Corollary 4.18 in [Ei95], we also have strict inclusions $\mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1}$ for $i = 0, \dots, r-1$. It follows that $d(J) \geq d(I(J))$.

To prove the opposite inequality, let

$$J \subset \mathfrak{p}'_0 \subsetneq \mathfrak{p}'_1 \subsetneq \dots \subsetneq \mathfrak{p}'_s, \tag{4.9}$$

be a chain of prime ideals. By the Going Up Lemma (see [Ei95], Proposition 4.15), there is a prime ideal $\mathfrak{q}'_0$ in B such that $\mathfrak{p}'_0 = \mathfrak{q}'_0 \bigcap A$. Hence, $I(J) \subset \mathfrak{q}'_0$. By inductively applying the Going Up Lemma, we find prime ideals $\mathfrak{q}'_1, \dots, \mathfrak{q}'_s$ such that $\mathfrak{p}'_i = \mathfrak{q}_i \bigcap A$ and

$$I(J) \subset \mathfrak{q}'_0 \subset \mathfrak{q}'_1 \subset \dots \subset \mathfrak{q}'_s. \tag{4.10}$$

Clearly, we have strict inclusions $\mathfrak{q}'_i \subsetneq \mathfrak{q}'_{i+1}$ for $i = 0, \dots, s-1$ (since $\mathfrak{p}'_i = \mathfrak{q}_i \bigcap A$ and the $\mathfrak{p}_i$ are distinct). This proves the opposite inequality $d(I(J)) \geq d(j)$, which completes the proof of the lemma. □

To complete the proof of statement (ii) in Lemma 3.1, we shall need the following lemma.

LEMMA 4.3. *Let $\varphi := \varphi_H \colon \mathbb{C}[\![\tilde{Z}]\!] \to \mathbb{C}[\![Z]\!]$ denote the homomorphism induced by the formal mapping $H \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ and $\psi := \varphi_{\bar{H}} \colon \mathbb{C}[\![\tilde{\zeta}]\!] \to \mathbb{C}[\![\zeta]\!]$ the homomorphism induced by $\bar{H} \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$. Every minimal prime $\sqrt{P} = \sqrt{P_j}$ in the factorization* (4.1) *is of the form*

$$\sqrt{P} = I(\mathfrak{p}) + I(\mathfrak{q}), \tag{4.11}$$

where $\mathfrak{p}$ is a minimal prime of $I(\varphi(I(\tilde{\Gamma}))) \subset \mathbb{C}[\![Z]\!]$ and $\mathfrak{q}$ is a minimal prime of $I(\psi(I(\tilde{\Sigma}_0^*))) \subset \mathbb{C}[\![\zeta]\!]$.

Proof. We first observe that if $\mathfrak{p} \subset \mathbb{C}[\![Z]\!]$ and $\mathfrak{q} \subset \mathbb{C}[\![\zeta]\!]$ are prime ideals, then $I(\mathfrak{p}) + I(\mathfrak{q}) \subset \mathbb{C}[\![Z, \zeta]\!]$ is also prime. For, $\mathbb{C}[\![Z, \zeta]\!] = \mathbb{C}[\![Z]\!] \otimes \mathbb{C}[\![\zeta]\!]$ (where the tensor product is over $\mathbb{C}$) and, by Theorem III.14.35 of [ZS58],

$$\mathbb{C}[\![Z, \zeta]\!]/(I(\mathfrak{p}) + I(\mathfrak{q})) = (\mathbb{C}[\![Z]\!]/\mathfrak{p}) \otimes (\mathbb{C}[\![\zeta]\!]/\mathfrak{q}). \tag{4.12}$$

Since $\mathfrak{p}$ and $\mathfrak{q}$ are prime, $\mathbb{C}[\![Z]\!]/\mathfrak{p}$ and $\mathbb{C}[\![\zeta]\!]/\mathfrak{q}$ are integral domains. It follows that $\mathbb{C}[\![Z]\!]/\mathfrak{p}) \otimes (\mathbb{C}[\![\zeta]\!]/\mathfrak{q})$ is an integral domain, since $(\mathbb{C}[\![Z]\!]/\mathfrak{p}) \otimes (\mathbb{C}[\![\zeta]\!]/\mathfrak{q}) \subset K \otimes K'$, where K and K' denote the quotient fields of $\mathbb{C}[\![Z]\!]/\mathfrak{p}$ and $\mathbb{C}[\![\zeta]\!]/\mathfrak{q}$ respectively, and $K \otimes K'$ is an integral domain by Corollary III.15.1 of [ZS58]. This proves that $I(\mathfrak{p}) + I(\mathfrak{q})$ is prime. By considering maximal chains of prime ideals containing $I(\mathfrak{p}) + I(\mathfrak{q})$ constructed in an obvious way from maximal chains of prime ideals containing $\mathfrak{p}$ and $\mathfrak{q}$, we also deduce that $d(I(\mathfrak{p}) + I(\mathfrak{q})) = d(\mathfrak{p}) + d(\mathfrak{q})$. Now, let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ and

$\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ be the minimal primes of $I(\varphi(I(\tilde{\Gamma})))$ and $I(\psi(I(\tilde{\Sigma}_0^*)))$, respectively, i.e. $d(\mathfrak{p}_i) = d(I(\varphi(I(\tilde{\Gamma}))))$, $d(\mathfrak{q}_j) = d(I(\psi(I(\tilde{\Sigma}_0^*))))$ and

$$\sqrt{I(\varphi(I(\tilde{\Gamma})))} = \bigcap_{i=1}^{k} \mathfrak{p}_i, \quad \sqrt{I(\psi(I(\tilde{\Sigma}_0^*)))} = \bigcap_{i=j}^{l} \mathfrak{q}_j.$$

To prove the decomposition (4.11), we shall show that

$$\sqrt{I(\Phi(\tilde{\mathfrak{P}}))} = \bigcap_{i=1}^{k} \bigcap_{j=1}^{l} (I(\mathfrak{p}_i) + I(\mathfrak{q}_j)). \tag{4.13}$$

The uniqueness of the minimal primes of an ideal then implies that (4.11) must hold. The inclusion

$$\sqrt{I(\Phi(\tilde{\mathfrak{P}}))} \subset \bigcap_{i=1}^{k} \bigcap_{j=1}^{l} (I(\mathfrak{p}_i) + I(\mathfrak{q}_j))$$

is easy to prove and the details of this are left to the reader. Now, let $h \in \cap_{i,j}(I(\mathfrak{p}_i) + I(\mathfrak{q}_j))$. Fix a $j \in \{1, \ldots, l\}$ and let $f_i \in I(\mathfrak{p}_i)$, $g_i \in I(\mathfrak{q}_j)$ such that $h = f_i + g_i$ for $i = 1, \ldots, k$. It follows that

$$h^k = \prod_{i=1}^{k} (f_i + g_i).$$

Since $f_{i_1} \ldots f_{i_{k-r}} g_{i_{k-r+1}} \ldots g_{i_k}$ belongs to $I(\mathfrak{q}_j)$ whenever $r \geq 1$ and $f_1 \ldots f_k$ belongs to $\cap_i I(\mathfrak{p}_i)$, we conclude that

$$h^k = f_j' + g_j', \tag{4.14}$$

where $f_j' \in \cap_i I(\mathfrak{p}_i)$ and $g_j' \in I(\mathfrak{q}_j)$ for every $j = 1, \ldots, l$. A similar argument shows that $h^{kl} = f + g$ with $f \in \cap_i I(\mathfrak{p}_i)$ and $g \in \cap_j I(\mathfrak{q}_j)$. We conclude that

$$h^{kl} \in \cap_i I(\mathfrak{p}_i) + \cap_j I(\mathfrak{q}_j)$$

or, equivalently,

$$h \in \sqrt{\sqrt{I(\varphi(I(\tilde{\Gamma})))} + \sqrt{I(\psi(I(\tilde{\Sigma}_0^*)))}} = \sqrt{I(\Phi(\tilde{\mathfrak{P}}))}. \tag{4.15}$$

This proves (4.13) and, hence, also the lemma. □

We now return to the proof of statement (ii) in Lemma 3.1. Let $P := P_{j_0}$ be a primary ideal in (4.1) as given by Lemma 4.1. Let $\mathfrak{p}$ and $\mathfrak{q}$ be as in Lemma 4.3 such that (4.11) is satisfied. In particular, $d(\mathfrak{p}) \geq 1$. By evaluating at $Z = 0$, we deduce that $I(\Sigma_0^*) \subset \mathfrak{q}$. Since

$$d(\mathfrak{q}) = d(I(\tilde{\Sigma}_0^*)) = d(I(\Sigma_0^*)),$$

where the latter identity follows from statement (i) in Lemma 3.1, and both $\mathfrak{q}$ and $I(\Sigma_0^*)$ are primes, we conclude that, in fact, $\mathfrak{q} = I(\Sigma_0^*)$. Hence, $I(\mathcal{X}) \subset I(\mathfrak{p}) + I(\mathbb{C}^N \times \Sigma_0^*)$. The fact that $\mathcal{X}$ is not essentially finite at 0 now follows from Lemma 2.2, part

(c) with $J = \mathfrak{p}$. This completes the proof of the implication "X is essentially finite at 0" $\implies$ "$\tilde{X}$ is essentially finite at 0".

To finish the proof of (ii), we must show the converse implication, namely "$\tilde{X}$ is essentially finite at 0" $\implies$ "X is essentially finite at 0". Again, we shall prove the logical negation of this statement. Thus, suppose that X is not essentially finite at 0 and let Γ be an irreducible complex analytic variety through 0 in $\mathbb{C}^N$ such that $\Gamma \times \Sigma_0^* \subset I(\mathcal{X})$. Let $\mathfrak{p} := I(\Gamma)$, $\mathfrak{q} := I(\Sigma_0^*)$, and observe, as above, that $\mathfrak{P} := I(\mathfrak{p}) + I(\mathfrak{q})$ is a prime ideal such that $I(\mathcal{X}) \subset \mathfrak{P}$. Let $\tilde{\mathfrak{P}}$ be the prime ideal $\Phi^{-1}(\mathfrak{P})$ and observe that $I(\tilde{\mathcal{X}}) \subset \tilde{\mathfrak{P}}$. Recall the homomorphisms $\varphi\colon \mathbb{C}[[\tilde{Z}]] \to \mathbb{C}[[Z]]$ and $\psi\colon \mathbb{C}[[\tilde{\zeta}]] \to \mathbb{C}[[\zeta]]$ induced by $H(z)$ and $\bar{H}(\zeta)$. We claim that

$$\tilde{\mathfrak{P}} = I(\phi^{-1}(\mathfrak{p})) + I(\psi^{-1}(\mathfrak{q})). \tag{4.16}$$

Indeed, we have $\Phi^{-1}(I(\mathfrak{p})) + \Phi^{-1}(I(\mathfrak{q})) \subset \Phi^{-1}(\mathfrak{P})$, $\Phi^{-1}(I(\mathfrak{p})) = I(\phi^{-1}(\mathfrak{p}))$, and $\Phi^{-1}(I(\mathfrak{q})) = I(\psi^{-1}(\mathfrak{q}))$, which together imply the inclusion

$$I(\phi^{-1}(\mathfrak{p})) + I(\psi^{-1}(\mathfrak{q})) \subset \tilde{\mathfrak{P}}. \tag{4.17}$$

Both sides of (4.17) are prime ideals and the following chain of identities of dimensions follow from the results above

$$d(\tilde{\mathfrak{P}}) = d(\mathfrak{P}) = d(\mathfrak{p}) + d(\mathfrak{q}) = d(\phi^{-1}(\mathfrak{p})) + d(\psi^{-1}(\mathfrak{q})) = d(I(\phi^{-1}(\mathfrak{p})) + I(\psi^{-1}(\mathfrak{q}))).$$

This implies the desired identity (4.16). By (i) of the lemma, we have $\psi^{-1}(\mathfrak{q})) = I(\tilde{\Sigma}_0^*)$. Since $d(\phi^{-1}(\mathfrak{p})) \geq 1$, an argument analogous to that used to complete the proof of the implication "X is essentially finite at 0" $\implies$ "$\tilde{X}$ is essentially finite at 0" above now shows that $\tilde{X}$ is not essentially finite at 0. This completes the proof of Lemma 3.1. □

5. Proofs of Theorems 1.1 and 1.2.

Proof. [Proof of Theorem 1.1] We first prove the implication (i) $\implies$ (ii) in Theorem 1.1. Let $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be the formal holomorphic mapping associated to f sending M into $\tilde{X}$, and $\phi_H\colon \mathbb{C}[[\tilde{Z}]] \to \mathbb{C}[[Z]]$ the homomorphism induced by H. We shall use the following sufficient criterion from [MMZ02] (see Theorem 1.1 and Remark 1.2 in [MMZ02]) for f to be real-analytic at 0 (or, equivalently, extend holomorphically near 0): f is real-analytic near 0 if the germ at 0 of the variety

$$C := \{\tilde{Z}\colon \phi_H(I(\tilde{\Sigma}_{\tilde{Z}})) \subset I(\Sigma_0)\} \tag{5.1}$$

reduces to the single point $\{0\}$. We should perhaps point out that if 0 does not belong to $\tilde{\Sigma}_{\tilde{Z}}$, then $I(\tilde{\Sigma}_{\tilde{Z}})$ is the whole ring $\mathbb{C}[[\tilde{Z}]]$ (and, hence, $\tilde{Z}$ does not belong to C). Observe that C can also be expressed as

$$C := \{\tilde{Z}\colon I(\tilde{\Sigma}_{\tilde{Z}})) \subset \phi_H^{-1}(I(\Sigma_0))\} \tag{5.2}$$

By using Lemma 3.1 (i), we observe that $I(\tilde{\Sigma}_{\tilde{Z}})) \subset \phi_H^{-1}(I(\tilde{\Sigma}_0))$ implies that $I(\tilde{\Sigma}_{\tilde{Z}})) \subset I(\tilde{\Sigma}_0)$, i.e. $\tilde{\Sigma}_0 \subset \tilde{\Sigma}_{\tilde{Z}}$. It follows that $C \subset \tilde{A}$ as germs at 0, where $\tilde{A}$ is given by (2.2) using $\tilde{X}$ instead of X. Since M is assumed to be essentially finite at 0, it follows that $\tilde{X}$ is also essentially finite at 0 by Lemma 3.1 (ii). By Proposition 2.1, the germ of $\tilde{A}$ at 0, and hence that of C, reduces to the point $\{0\}$. The real-analyticity of f near 0 now follows from the above mentioned result from [MMZ02].

To prove (ii) $\Longrightarrow$ (i), we let $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ be the finite holomorphic mapping extending f near 0. We further let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ denote the local complexification of M near 0 and $\mathcal{H}\colon (\mathbb{C}^N \times \mathbb{C}^N,0) \to (\mathbb{C}^N \times \mathbb{C}^N,0)$ the complexification of the mapping H. Since $\mathcal{H}$ is a finite holomorphic mapping and $\mathcal{M}$ a complex submanifold, we conclude, by Remmert's proper mapping theorem, that the image $\tilde{\mathcal{X}} := \mathcal{H}(\mathcal{M})$ is an irreducible complex analytic subvariety of the same (complex) dimension as $\mathcal{M}$. Let $\tilde{X}$ denote the real-analytic subvariety of $\mathbb{C}^N$ obtained by intersecting $\tilde{\mathcal{X}}$ with the anti-diagonal $\{(\tilde{Z},\tilde{\zeta})\colon \tilde{\zeta} = \bar{\tilde{Z}}\}$. It is easy to check that H maps M into $\tilde{X}$. Moreover, since $\det(\partial H/\partial Z)$ does not vanish identically on M, the dimension of $\tilde{X}$ is at least that of M. On the other hand, since $\tilde{X}$ is the intersection between the totally real manifold $\{(\tilde{Z},\tilde{\zeta})\colon \tilde{\zeta} = \bar{\tilde{Z}}\}$ and $\tilde{\mathcal{X}}$, its dimension is at most equal to the complex dimension of $\tilde{\mathcal{X}}$. Since

$$\dim_{\mathbb{C}} \tilde{\mathcal{X}} = \dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{R}} M, \tag{5.3}$$

we conclude that $\dim_{\mathbb{R}} \tilde{X} = \dim_{\mathbb{R}} M$. The fact that $\tilde{X}$ is irreducible now follows from the fact that $\tilde{\mathcal{X}}$ is its complexification and $\tilde{\mathcal{X}}$ is irreducible. This completes the proof of Theorem 1.1. □

Proof. [Proof of Theorem 1.2] The proof of (ii) $\Longrightarrow$ (i) in Theorem 1.2 is exactly the same as the proof of (ii) $\Longrightarrow$ (i) in Theorem 1.1. The proof of (i) $\Longrightarrow$ (ii) in Theorem 1.2 is follows the same reasoning as that of the proof of (i) $\Longrightarrow$ (ii) in Theorem 1.1, except that we apply a result from [Mi02a] instead of the result from [MMZ02] used in the proof above. Indeed, Theorem 9.1 in [Mi02a] combined with the remark preceding it shows that the formal mapping H in Theorem 1.2 is convergent if the variety C given by (5.1) reduces to the single point $\{0\}$. This completes the proof of Theorem 1.2. □

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