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ANALYTICITY OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS ON NILPOTENT LIE GROUPS

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1. Introduction. A differential operator

$$(1.1)\qquad P = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha \qquad D_x^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_N}\right)^{\alpha_N}$$

with $a_\alpha(x)$ real analytic is called analytic hypoelliptic in an open set U if $Pu = f$ with f analytic in an open subset $V \subset U$ implies u must also be analytic in V. We survey here some conditions for analytic hypoellipticity when P is a left invariant differential operator on a nilpotent Lie group.

For constant coefficient differential operators, analytic hypoellipticity is equivalent to ellipticity (see e.g. [7]). Variable coefficient elliptic differential operators are always analytic hypoelliptic, but the converse is false. Here we will be concerned with nonelliptic variable coefficient operators.

2. Homogeneous operators. Now let G be a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is stratified i.e. $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \dots + \mathfrak{g}_r$, vector space direct sum with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ if $i + j \le r$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i + j > r$. We assume that $\mathfrak{g}_1$ generates $\mathfrak{g}$. Then $\mathfrak{g}$ carries a natural family of dilations which are automorphisms: $\delta_t|\mathfrak{g}_i = t^i$. The dilations may be transferred to G via the exponential map, and also extend to the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Thus we may write $\mathfrak{U}(\mathfrak{g}) = \sum_{j=1}^{\infty} \mathfrak{U}_j(\mathfrak{g})$, where each element of $\mathfrak{U}_j$ is homogeneous of degree j under δ_t.

3. Smoothness of solutions. The notion of C^∞ hypoellipticity is defined as for analyticity, but with real analytic replaced by C^∞. For a homogeneous operator $L \in \mathfrak{U}(\mathfrak{g})$ necessary and sufficient conditions for C^∞ hypoellipticity were established by Helffer and Nourrigat [6], who proved the following conjecture of Rockland [15]. Let $\hat{G}$ be the set of irreducible unitary representations of G. For

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$\pi \in \hat{G}$ acting on $L^2(\mathbb{R}^k)$, we denote also by π the corresponding mapping of $\mathfrak{U}(\mathfrak{g})$ into the space of differential operators on $L^2(\mathbb{R}^k)$. Then L is C^∞ hypoelliptic if and only if $\pi(L)$ is injective for all nontrivial $\pi \in \hat{G}$.

4. Nonanalytic hypoelliptic operators. The existence of C^∞ hypoelliptic but not analytic hypoelliptic operators of second order on 2 step groups was suggested by the following example of Baouendi-Goulaouic [1]:

In $\mathbb{R}^{n+2}$ the operator

$$P = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{x_j^2 \partial^2}{\partial y^2} + \frac{\partial^2}{\partial x_{n+1}^2} \tag{4.1}$$

is not analytic hypoelliptic. (It is C^∞ hypoelliptic by a general theorem of Hörmander [8]). P is not a left invariant operator on any group, but it is closely related to

$$L = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \left(\frac{\partial}{\partial t_j} + x_j \frac{\partial}{\partial y}\right)^2 + \frac{\partial^2}{\partial x_{n+1}^2}\,. \tag{4.2}$$

L is of the form

$$L = \sum_{j=1}^{n} (U_j^2 + V_j^2) + W^2\,, \tag{4.3}$$

which is in $\mathfrak{U}^2(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{h}_{2n+1} \oplus \mathbb{R}$, where $\mathfrak{h}_{2n+1}$ is the $2n+1$ dimensional Heisenberg algebra. Here $\{U_j, V_j, W,\ j = 1,2,\dots,n\}$ is a basis of $\mathfrak{g}_1$, and $\dim \mathfrak{g}_2 = 1$. Now it is easy to see that L cannot be analytic hypoelliptic if P is not. Indeed, if Pu vanishes in an open set in $\mathbb{R}^{n+2}$ then Lu vanishes in an open set in $\mathbb{R}^{2n+2}$. If L were analytic hypoelliptic, then u would have to be analytic. Similar reasoning shows that if $\tilde{\mathfrak{g}}$ is any 2-step nilpotent Lie algebra having a quotient algebra of the form $\mathfrak{h}_{2n+1} \oplus \mathbb{R}$, then the operator L pulls back to $\tilde{L} \in \mathfrak{U}(\tilde{\mathfrak{g}})$, where $\tilde{L}$ is C^∞ but not analytic hypoelliptic.

5. H-groups. One is therefore led to consider 2-step algebras of which do not have quotients of the form $\mathfrak{h}_{2n+1} \oplus \mathbb{R}$. These may be characterized as follows. For $\eta \in \mathfrak{g}_2^* \setminus \{0\}$, let $\tilde{\mathfrak{g}}_\eta = \mathfrak{g}/I_\eta$, where $I_\eta = \{Y \in \mathfrak{g}_2 : \eta(Y) = 0\}$. Now let B_η

be the bilinear form on $\mathfrak{g}_1^*$ defined by

$$B_\eta : (X_1,X_2) \longmapsto \eta([X_1,X_2]) .$$

If $[\mathfrak{g}_1,\mathfrak{g}_2] = \mathfrak{g}_2$, then $\det B_\eta \neq 0$, all $\eta \in \mathfrak{g}_2^* - \{0\}$, if and only if none of the quotients $\tilde{\mathfrak{g}}_\eta$ is the Lie algebra direct sum of a Heisenberg algebra with a Euclidean space. In this case the corresponding group G is called a <u>H-group</u>.

6. <u>Analytic regularity of</u> $\Box_b$. Further motivation for positive results on analytic hypoellipticity on H-groups came from the results on analytic hypoellipticity of the boundary Laplacian operator $\Box_b$ on strongly pseudo convex domains. C^∞ regularity for $\Box_b$ had been established much earlier through the work of J. J. Kohn [11], but it was not until the mid '70's that analytic regularity was proved by Trèves [18] and Tartakoff [17]. Their methods were completely different. Tartakoff begins with the well known characterization of analyticity in terms of L^2 norms: a distribution u is analytic near a point x_0 if there is a neighborhood U of x_0 and a constant $C > 0$ such that

$$\| D_x^\alpha u \|_{L^2(U)} \leq C^{|\alpha|+1}|\alpha| ! \tag{6.1}$$

for all multi-indices α. His proof is elementary in the sense that he uses only L^2 estimates with integration by parts.

Trèves methods are microlocal i.e. he works in conic sets in the cotangent space. It is well known that analyticity can be "microlocalized" [9]; a distribution u is <u>analytic</u> near a point $(x_0,\xi_0) \in T^*(U)\backslash 0$ if there is an open cone Γ in $\mathbb{R}^m$ containing ξ_0 and a constant $C > 0$ such that for every integer $N = 0,1,\ldots$ one can find a function $\phi_N \in C_0^\infty(U)$, $\phi_N = 1$ in V, a neighborhood of x_0, $\varphi_N = 0$ outside a fixed compact subset K of U such that

$$|(\varphi_N u)\hat{}\,(\xi)| \leq C^{N+1}N!(1+|\xi|)^{-N} \tag{6.2}$$

for all $\xi \in \Gamma$. It can be shown (see e.g. [19, Chapter V]) that u is analytic in a neighborhood of x_0 if and only if it is analytic at (x_0,ξ_0) all $\xi_0 \in \mathbb{R}^n\backslash\{0\}$. The complement in $T^*(U)\backslash 0$ of $\{(x_0,\xi_0) : u$ is analytic at $x_0,\xi_0\}$

G is not an H-group then there is no $L \in \mathfrak{U}_m(\mathfrak{g})$ which is analytic hypoelliptic. Partial results on necessary conditions for analytic hypoellipticity have been obtained by Métivier [12] and Helffer [5].

(8.2) Theorem (Helffer [5]). If $\mathfrak{g}$ is a 2-step Lie algebra and $L \in \mathfrak{U}_2(\mathfrak{g})$, then L is not analytic hypoelliptic if G is not an H-group.

9. Non-homogeneous operators. We restrict here to the case where G is a H-group. Our result, which is contained in a recent joint paper with Grigis [3], applies to operators $L \in \mathfrak{U}(\mathfrak{g})$ having the property that $\pi(L) \neq 0$ for all non-trivial one dimensional representations $\pi \in \hat{G}$. Such operators will be called transversally elliptic. Another way of describing these operators is by noting that they are elliptic polynomials in the elements of $\mathfrak{g}_1$.

The elements of $\hat{G}$ are parametrized by $\eta \in \mathfrak{g}_2^*\setminus\{0\}$. We now replace L by L^*L and study the family of differential operators $\pi_\eta(L)$ as η varies. Now we introduce spherical coordinates $\eta = (\rho,\omega)$ on $\mathfrak{g}_2^* - \{0\}$, and write $L = L_m + L_{m-1} + \ldots + L_0$, with $L_j \in \mathfrak{U}_j(\mathfrak{g})$. Then π_η may be defined so that $\pi_\eta(L_j)$ is homogeneous in η i.e. $\pi_\eta(L_j) = |\eta|^{j/2}\pi_{(1,\omega)}(L_j)$ (see [13]). Then

$$(9.1) \qquad \pi_\eta(L) = |\eta|^{m/2}(\pi_{(1,\omega)}(L_m) + |\eta|^{-1/2}\,\pi_{(1,\omega)}(L_{m-1}) + \ldots + |\eta|^{-m/2}\pi_{(1,\omega)}(L_0)).$$

Now let $\lambda = |\eta|^{-1/2}$ and define the operator $A(\lambda,\omega)$ by

$$(9.2) \qquad A(\lambda,\omega) = |\eta|^{-m/2}\pi_\eta(L)\ .$$

One can prove that $(\lambda,\omega) \longmapsto A(\lambda,\omega)$ is an analytic family of unbounded operators in the sense of Kato-Rellich [10]. Furthermore, the spectrum of each $\pi_\eta(L)$ is discrete and consists of eigenvalues. For ω_0 fixed, let K_{ω_0} be the multiplicity of 0 as an eigenvalue of $\pi_{(1,\omega_0)}$. Then analytic perturbation theory shows that for $|\lambda|$ small and ω close to ω_0 the product $d(\eta)$ of the K_{ω_0} smallest eigenvalues of $A(\lambda,\omega)$ is analytic and can be expanded

$$(9.3) \qquad d(\eta) = \lambda^{K_{\omega_0}}(a_0(\omega) + a_1(\omega_0)\lambda + a_2(\omega)\lambda^2 + \ldots)\ .$$

In the language of pseudodifferential operators, $d(\eta)$ is a semi-classical analytic symbol on $\mathbb{R}^{n_2}$ which is elliptic near (y_0, η_0) if and only if $a_0(\omega_0) \neq 0$.

Our criterion for analytic hypoellipticity may be stated as follows.

(9.4) Theorem (Grigis-Rothschild [3]). *Let* G *be* n *H-group, and* $L \in U(\mathfrak{g})$ *transversally elliptic. Then* L *is analytic hypoelliptic if and if for any* $\eta \in \mathfrak{g}_2^* - \{0\}$, *the product* $d(\eta)$ *of the small eigenvalues of* $\pi_\eta(L^*L)$, *given by* (9.3), *is an elliptic symbol* i.e. $a_0(\omega_0) \neq 0$.

In the special case where G is a Heisenberg group, the theorem takes a simpler form.

Corollary. *If* G *is a Heisenberg group, then* L *is analytic hypoelliptic if and only if* $\ker L \cap L^2(G) = \phi$.

To see how the corollary follows from Theorem (8.4), we note that for the Heisenberg group $\mathfrak{g}_2^* - \{0\} = \mathbb{R} - \{0\}$ and hence $d(\eta)$ is elliptic if and only if it is not identically zero. On the other hand, if $d(\eta) \equiv 0$, then one can find a non-zero $f \in L^2(G)$ with $Lf \equiv 0$.

(9.5) Example. Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be the 3-dimensional Heisenberg algebra with $\{X_1, X_2\}$ a basis of $\mathfrak{g}_1$. Then for any $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$, the operator

$$L = X_1^2 + X_2^2 + i\alpha[X_1, X_2] + \beta X_1$$

is analytic hypoelliptic. To prove this, one need only check that $\pi_\eta(L)$ has no zero eigenvalue for $|\eta|$ large.

The proof of Theorem (9.4) borrows heavily from techniques of Sjöstrand [16] and those of Métivier [14]. In [16] the question of C^∞ hypoellipticity for a class of transversally elliptic operators more general than ours is reduced to that of determining the C^∞ hypoellipticity of a pseudodifferential operator in fewer variables. In order to carry out this construction in the analytic category, we use the analytic pseudodifferential operators and approximate inverses constructed by Métivier [14].

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