Automorphisms, Orbits, and Homogeneous Spaces of Non-Connected Lie Groups

Frederick P. Greenleaf, Martin Moskowitz, and Linda Preiss-Rothschild

1. Introduction

Let $G$ be a Lie group which is second countable but not necessarily connected with identity component $G_0$. Let $\mathcal{B}$ be any subgroup of the group $\mathfrak{A}(G)$ of topological automorphisms such that $\mathcal{B} \supseteq \mathcal{I}(G_0)$, the inner automorphisms obtained from elements of $G_0$. In this setting we shall generalize earlier results [1] on uniformity properties of unbounded orbits under the action of $\mathcal{B}$ as well as those of [2] on compactness of certain homogeneous spaces of finite volume.

1. Theorem. Let $G$ and $\mathcal{B}$ be as above, and $B = B(G, \mathcal{B})$ the subgroup of points $x \in G$ with bounded orbit (so $\mathcal{B}(x)^{-}$ is compact). Then $B$ is a closed subgroup in $G$. Furthermore, there exist closed $\mathcal{B}$-invariant sets $G = X_m \supseteq \ldots \supseteq X_0 = B$ such that each point $x \in X_k \sim X_{k-1}$ has a relative neighborhood in $X_k \sim X_{k-1}$ with infinitely many disjoint $\mathcal{B}$-transforms.

Previous results considered $\mathcal{B} = \mathcal{I}(G)$ and more importantly applied only to connected groups. This more general setting results in more streamlined proofs as well as valuable viewpoints. While it is not surprising that these results extend to Lie groups with a finite number of components, it is somewhat surprising that they hold for essentially all Lie groups. Theorem 1 directly applies to the study of finite $\mathcal{B}$-invariant Borel measures on $G$. If $\mathcal{B} = \mathcal{I}(G)$ these are the “central” measures on $G$.

Next let $\beta \in \mathfrak{A}(G)$ and $G_{\beta} = \{x \in G : \beta(x) = x\}$. If $\beta = \alpha_y$, an inner automorphism of the form $\alpha_y(g) = ygy^{-1}$, then $G_{\beta}$ is just the centralizer $C_y$ of a point $y \in G$.

2. Theorem. Let $\mathcal{S}$ be any subset of $\mathfrak{A}(G)$ for a second countable Lie group $G$, and let $G_\mathcal{S} = \cap \{G_\beta : \beta \in \mathcal{S}\}$. If the homogeneous space $G/G_\mathcal{S}$ has finite volume, then it is compact.

Theorem 1 will be used to prove Theorem 2, which substantially generalizes [2, Theorem 1]. There $\mathcal{S}$ was a single inner automorphism and $G$ was connected. Of course, once we allow outer automorphisms $\beta$.

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then even if $G$ is connected it seems necessary to consider non-connected Lie groups such as $Z \times_\beta G$ and Theorem 1 for such groups, in order to prove Theorem 2. We also give a number of consequences of Theorem 1 which generalize the corresponding results in [1] and [2]. We remark that, by modifications of these methods Claire Sit has proven results analogous to these in the case of linear algebraic groups in [16]. In addition by use of the present results she has recently extended Theorem 1 of [2] to the case of locally compact connected groups.

2. Some Consequences of Theorem 1

Unless otherwise stated $G$ will be a Lie group which is not necessarily connected but with at most countably many components. Let $G_0$ be the identity component; $R = \text{radical of } G_0; N = \text{nilradical of } G_0; K(G_0) = \text{largest compact normal subgroup of } G_0$ and $K_0 = K(G_0)_0$ its identity component (the largest compact connected normal subgroup of $G_0$). The definition of $K(G_0)$ was first discussed by Iwasawa in [7]. Furthermore let $Z(G) = \text{center of } G$ and, for any subset $S \subseteq G, Z_G(S)$ be its centralizer. If $\mathcal{B}$ is a subgroup in $\mathcal{A}(G)$ and $A \subseteq G$ a closed $\mathcal{B}$-invariant subset, a $\mathcal{B}$-invariant layering terminating with $A$ is any family $G = X_m \supseteq \cdots \supseteq X_0 = A$ of closed $\mathcal{B}$-invariant sets such that every point in the $k^{th}$ "layer" $x \in X_k \sim X_{k-1}$ has a relative neighborhood in $X_k \sim X_{k-1}$ with infinitely many disjoint $\mathcal{B}$-transforms. Our main theorem says that, if $\mathcal{B} \supseteq \mathcal{I}(G_0)$, there is a layering terminating with $B(G, \mathcal{B})$, the set of elements with bounded $\mathcal{B}$-orbits. As in [1, Section 1] we immediately get the following result on finite $\mathcal{B}$-invariant measures.

3. Theorem. Let $G$ be a second countable Lie group and $\mathcal{B}$ a subgroup of $\mathcal{A}(G)$ such that $\mathcal{B} \supseteq \mathcal{I}(G_0)$. Then every finite $\mathcal{B}$-invariant measure $\mu$ has $\text{supp}(\mu) \subseteq B(G, \mathcal{B})$.

Taking $\mathcal{B} = \mathcal{I}(G)$ we get Corollary 4 concerning central idempotent measures ($\mu \ast \mu = \mu$). Via [1] the authors studied central idempotent measures on connected Lie groups in [3].

4. Corollary. Let $G$ be a second countable Lie group. For any central idempotent measure $\mu$, the closed subgroup $G(\mu)$ generated by $\text{supp}(\mu)$ is compact.

Proof. By Theorem 2, $\text{supp}(\mu) \subseteq B(G) = B(G, \mathcal{I}(G))$. Since $B(G)$ is closed for any Lie group, connected or not [17], it is locally compact and is clearly an $FC^-$ group in the sense of [4]. By [15] (for a proof, see [8]), $B(G)$ is an $IN$-group. It then follows from [10] that $G(\mu)$ is compact.
As another application of Theorem 1 we have the following density theorem which is a generalization of Theorem 3 of [2].

5. Corollary. Let $G$ be any second countable Lie group and $H$ a closed subgroup such that $G/H$ has finite volume. Then $Z_G(H) \subseteq B(G)$, and in particular if $B(G) = Z(G)$ we get $Z_G(H) = Z(G)$.

Proof. Let $x \in Z_G(H)$. Then $H \subseteq Z_G(x)$ and the $G$-equivariant surjection $G/H \to G/Z_G(x)$ gives a finite $G$-invariant measure on $G/Z_G(x)$. Since $G$ is second countable, this gives an $\mathfrak{S}(G)$-invariant finite Borel measure $\mu$ on $G$ which lives on the conjugacy class $\mathfrak{C} = \mathfrak{S}(G) \cdot x$ in the sense that $\mu(\mathfrak{C}) = \|\mu\|$. By Theorem 3, we get $x \in B(G)$.

We remark that if $G/H$ is compact (a case which is not covered in Corollary 5 since $G/H$ might not have an invariant measure) the same conclusion holds. For if $G/H$ is compact, then $G = UH$ where $U$ is a compact set, and so any automorphism $\alpha \in \mathfrak{U}(G)$ which leaves $H$ pointwise fixed is an automorphism of bounded displacement because if $g = uh$, then for all $g, \alpha(g)g^{-1} = \alpha(u)\alpha(h)h^{-1}u^{-1} \in \alpha(U) \cdot U^{-1}$ a fixed compact set. Thus if $x \in Z_G(H)$, it follows that $x \in B(G)$; moreover, if $G$ has no nontrivial automorphisms of bounded displacement, then each automorphism having $H$ pointwise fixed is trivial on $G$.

Our next result extends [2; Corollary 4].

6. Corollary. Let $G$ be a second countable Lie group such that $B(G) = Z(G)$, and let $\Gamma$ be a discrete subgroup such that $G/\Gamma$ has finite volume. (Note: $\Gamma$ discrete and $G/\Gamma$ compact $\Rightarrow$ $G/\Gamma$ has finite volume [12, p. 21].) Then $\Gamma \cap Z(G)$ is a uniform lattice in $Z(G)$.

Proof. Consider the closed subgroup $N_0(\Gamma) = \{x \in G : x\Gamma x^{-1} \subseteq \Gamma\}$. Its identity component $N_0$ must centralize $\Gamma$ and by Corollary 5 must therefore centralize $G$. Hence $N_0 \subseteq \Gamma \cdot Z(G) \subseteq N_0(\Gamma)$. Thus $\Gamma Z(G)$ is an open and closed subgroup of the closed subgroup $N_0(\Gamma)$ and so $\Gamma Z(G)$ is closed in $G$. Since $G$ is second countable this implies that $\Gamma \cap Z(G)$ is a (uniform) lattice in $Z(G)$ as in [2].

3. The Proof of Theorem 1

We note that $B(G, \mathfrak{B})$ is a closed subgroup. Tits [17] has shown this in case $\mathfrak{B} = \mathfrak{S}(G)$, but his proof goes through with minor changes for arbitrary $\mathfrak{B}$. In proving that a layering exists we may immediately pass to the case in which $K_0 = K(G)_0$ is trivial; just use the following "liftback principle" and notice that $B(G, \mathfrak{B}) = \pi^{-1}(B(G/K_0, \mathfrak{B}^*))$ under the quotient homomorphism $\pi : G \to G/K_0$. 
7. Lemma. Let $G$, $\mathfrak{B}$ be as above. Let $H$ be a closed $\mathfrak{B}$-invariant subgroup of $G$, $G^* = G/H$, $\pi: G \to G^*$ the quotient map, and $\mathfrak{B}^*$ the group of automorphisms induced on $G^*$. Then

i) If $x \notin B(G, \mathfrak{B})$ then $x^* = \pi(x) \notin B(G^*, \mathfrak{B}^*)$

ii) If $G^* = Y_1 \supseteq \cdots \supseteq Y_0 = A^*$ is a $\mathfrak{B}^*$-invariant layering of $G^*$ terminating at $A^*$, then $A = \pi^{-1}(A^*)$ is $\mathfrak{B}$-invariant and the sets $X_i = \pi^{-1}(Y_i)$ give a $\mathfrak{B}$-invariant layering in $G$ terminating at $A$.

The proof is obvious.

Next consider the semisimple case (i.e., $G_0$ semisimple).

8. Lemma. If $G$ is a second countable semisimple Lie group, there is a one-step $\mathfrak{B}$-invariant layering terminating at $B(G, \mathfrak{B})$.

Proof. Let $g$ be the Lie algebra (of $G_0$), $g_1$ the product of the non-compact factors, and define the homomorphism $\text{Ad}_1: G \to \text{Aut}(g_1)$ via $\text{Ad}_1(x) = d(\alpha_x)|_{g_1}$, where $\alpha_x(g) = xgx^{-1}$. Let $x \notin B(G, \mathfrak{B})$. If $\text{Ad}_1(x) \neq \text{id}$, there exists an inner automorphism $\alpha = \alpha_y \in \mathfrak{B}(y \in G_0)$ and a neighborhood $U$ of $x$ such that $\{\alpha^n(U) : n \in \mathbb{Z}\}$ includes infinitely many disjoint transforms of $U$. (This follows by an obvious modification of the argument in [14].) If $\text{Ad}_1(x) = \text{id}$, consider the quotient map $\pi: G \to G^* = G/G_1$ where $G_1$ is the connected Lie subgroup (which is closed, see [11] or [13]) corresponding to $g_1$. Now $G_1$ is characteristic in $G$ so $\mathfrak{B}$ induces a group of automorphisms $\mathfrak{B}^* \subseteq \mathfrak{U}(G^*)$. Now $x \notin B(G, \mathfrak{B})$ insures that $x^* = \pi(x) \notin B(G^*, \mathfrak{B}^*)$, by Lemma 7; thus $\mathfrak{B}^*(x^*)$ meets infinitely many cosets of $G_0^* = \pi(G_0)$ since $G_0^* \cong G_0/G_1$ is compact. Taking $U^* = \mathfrak{U}(x^*)$ containing $x^*$ we have an open neighborhood with infinitely many disjoint $\mathfrak{B}^*$-transforms. Lifting back to $U = \pi^{-1}(U^*)$ we get a neighborhood of $x$ with similar properties under the action of $\mathfrak{B}$.

Letting $G^* = G/R$ ($R$ = radical of $G_0$) and $\pi: G \to G^*$ the canonical homomorphism we may apply Lemma 8 to $G^*$, then upon lifting back as in Lemma 7 we get a $\mathfrak{B}$-invariant layering of $G$ terminating at $\pi^{-1}(B(G/R, \mathfrak{B}^*))$.

The next step is to get a different layering terminating at $Z_G(N)$. As noted in [1; Lemma 5.1], even for non-connected second countable Lie groups there is an $\mathfrak{I}(G_0)$-invariant layering from $G$ to $Z_G(N)$, provided that $K_0 = K(G_0)_0$ is trivial, as we are assuming here. But the layers $G = X_m \supseteq \cdots \supseteq X_0 = Z_G(N)$ produced in [1] consist of closed characteristic subgroups and are thus $\mathfrak{B}$-invariant. Since $\mathfrak{B} \supseteq \mathfrak{I}(G_0)$ we get the desired $\mathfrak{B}$-invariant layering terminating at $Z_G(N)$.

Finally, passing to the discrete group $G/G_0$ and applying Lemma 7, we get a layering to the group $L$ consisting of those $x \in G$ whose $\mathfrak{B}$-orbit meets only finitely many cosets of $G_0$. By [1; Lemma 2.2] we may put together the layerings obtained in the last paragraphs to get a $\mathfrak{B}$-invariant layering terminating at $A_\mathfrak{B} = L \cap Z_G(N) \cap \pi^{-1}(B(G/R, \mathfrak{B}^*))$. Since
we are working in the case where \( K_0 \) is trivial, the nilradical \( N \) of \( G_0 \) is simply connected [1; Lemma 3.1]. We now seek to identify the identity component \( A_0 = (A_{B_0})_0 \) and show that it is \( Z(N) = \text{center of the nilradical, independent of the group} \ B \).

9. Lemma. If \( G \) is a semisimple second countable Lie group and \( B \supseteq \mathfrak{I}(G_0) \) then \( G_0 \cap B(G, B) = B(G_0, B | G_0) = B(G_0, \mathfrak{I}(G_0)) = B(G_0) \).

Proof. In our terminology, \( B(G_0) = B(G_0, \mathfrak{I}(G_0)) \). Clearly \( B(G, B) \cap G_0 \subseteq B(G_0, B | G_0) \subseteq B(G_0) \). On the other hand if \( x \in B(G_0) \) then since \( G_0 \) is characteristic, \( B | G_0 \supseteq \mathfrak{I}(G_0) \) and \( \mathfrak{I}(G_0) \cdot x \) is bounded. But (see [5]) \( [\mathfrak{A}(G_0, \mathfrak{I}(G_0)] < \infty \) since \( G_0 \) is semisimple. Therefore \( B(x) \subseteq \mathfrak{I}(G_0) \cdot x \) is bounded and \( x \in B(G, B) \cap G_0 \).

Now for our identification of \( (A_{B_0})_0 \).

10. Lemma. \( (A_{B_0})_0 = Z(N) \) for any \( B \supseteq \mathfrak{I}(G_0) \).

Proof. By Lemma 9,

\[
(A_{B_0})_0 \subseteq G_0 \cap \pi^{-1}(B(G/R, B^*)) = \pi_0^{-1}(B(G/R, B^*))
\]

\[
= \pi_0^{-1}(B(G_0/R, B^* | G_0/R))
\]

\[
= \pi_0^{-1}(B(G_0/R))
\]

where \( \pi : G \to G/R \) is the canonical map and \( \pi_0 : G_0 \to \pi(G_0) = G_0/R \) its restriction to \( G_0 \). Moreover, \( Z_G(N) \cap G_0 = Z_{G_0}(N) \), so that

\[
(A_{B_0})_0 \subseteq A_{B} \cap G_0 \subseteq \pi_0^{-1}(B(G_0/R)) \cap Z_{G_0}(N) .
\]

The latter intersection is precisely the group \( A_0 \) studied in [1; Section 6], taking \( G_0 \) as the connected Lie group in that discussion. There we showed that \( A_0 = Z(N) \); hence in our present situation \( (A_{B_0})_0 \subseteq Z(N) \). Clearly the reverse inclusion holds.

Hereafter we simplify notation by writing \( A = A_{B_0} \) and \( V = A_0 = Z(N) \). Our problem is to layer \( A \) down to \( B(A, B') \) in finitely many steps where \( B' = B | A \). Consider the decomposition of \( A \) into open/closed cosets

\[
A = \bigcup_{i \in I} a_i V, \quad a_i \in Z_G(N).
\]

Since \( V \) is characteristic in \( G \), \( B'(V) \subseteq V \) and \( B' \) permutes these cosets, \( \beta(a_i V) = \beta(a_i) V \). If, for a fixed \( j \in I \), \( B'(a_j V) = B'(a_j) V \) is a union of infinitely many cosets, then we can immediately layer (in one step) from \( A \) to \( A \sim B'(a_j V) \) by taking \( x V \) for the neighborhood of \( x \in a_j V \) with infinitely many \( B' \)-transforms. Thus we may layer, in one step, from \( A \) to the open/closed subgroup \( A_{\text{fin}} = \cup \{ a_i V : B'(a_i V) \text{ is a finite union of } V\text{-cosets} \} \); that is, we may assume \( A = A_{\text{fin}} \).
Now for each \( j \in I \) let \( \mathcal{B}_j = \text{Stab}_{\mathcal{B}'}(a_j V) = \{ \beta \in \mathcal{B}' : \beta(a_j V) \subseteq a_j V \} \). This is a subgroup of \( \mathcal{B}' \) with index \([\mathcal{B}' : \mathcal{B}_j]\) = number of \( V \)-cosets in \( \mathcal{B}'(a_j V) < \infty \). We next observe that
\[
B(A, \mathcal{B}') = \bigcup_{j \in I} \{ x \in A : x \in a_j V \text{ and } \mathcal{B}_j(x)^{-} \text{ is compact} \}.
\]
Clearly the inclusion (\( \subseteq \)) holds. If \( x \in a_j V \) for some \( j \in I \) and \( \mathcal{B}_j(x)^{-} \) is compact then \( \mathcal{B}' = \bigcup_{k=1}^{m} \beta_k \mathcal{B}_j \) where \( m = [\mathcal{B}' : \mathcal{B}_j] \), so we get \( \mathcal{B}'(x)^{-} = \bigcup_{k=1}^{m} \beta_k \mathcal{B}_j(x)^{-} \), which is compact.

If \( A_{i} = a_{i} V \) is a coset, write \( A_{i,c} = \{ x \in A_{i} : \mathcal{B}_j(x)^{-} \text{ is compact} \} \), so \( B(A, \mathcal{B}') = \bigcup_{i \in I} A_{i,c} \). Divide \( A \) into equivalence classes \( \{ E_1, E_2, \ldots \} \), each a minimal \( \mathcal{B}' \)-invariant open/closed finite union of \( V \)-cosets. We will now show that each \( E_k \) has a \( \mathcal{B}' \)-invariant layering of length at most \( n = 1 + \dim V \), terminating with \( E_{k,c} = E_k \cap B(A, \mathcal{B}') \). Fix \( k \in \mathbb{Z} \) and write \( E_k = \bigcup_{j=1}^{m} \beta_j(A_i) \), \( \beta_j \in \mathcal{B}' \), for some representative coset \( A_i \) in \( E_k \); the \( \{ \beta_1, \ldots, \beta_m \} \) are then coset representatives for \( \mathcal{B}'/\mathcal{B}_i \). Suppose we can produce a \( \mathcal{B}_i \)-invariant layering \( A_i = X_n \supseteq \cdots \supseteq X_0 = A_{i,c} \) with at most \( n \) steps. Then, clearly, we get a \( \mathcal{B}' \)-invariant layering of \( E_k \) terminating at \( E_{k,c} = \bigcup_{j=1}^{m} \beta_j(A_{i,c}) \) by taking \( Y_i = \bigcup_{j=1}^{m} \beta_j(X_i) \) for \( 0 \leq i \leq n \). Thus our task of layering \( A \) reduces to the following problem.

\((\ast)\) In each coset \( A_i \), show that there is a \( \mathcal{B}_i \)-invariant layering with at most \( n = 1 + \dim V \) steps, terminating at \( A_{i,c} = \{ x \in A_i : \mathcal{B}_i(x)^{-} \text{ is compact} \} \).

For this, consider the action of \( \mathcal{B}_i \) on a single coset \( A_i \). It is easy to see that the action \( \mathcal{B}_i \times A_i \rightarrow A_i \) is equivariant with an action of \( \mathcal{B}_i \) via affine transformations on the vector space \( V \) [1; Section 7]. The latter may be equivariantly identified with the restriction of a linear action of \( \mathcal{B}_i \) on \( W = V \oplus \mathbb{R} \) to the hyperplane \( V + \{1\} \). Proving that there is a \( \mathcal{B}_i \)-invariant layering from \( A_i \) to \( A_{i,c} \) thus reduces, as in [1; Section 7] to finding a layering consisting of \( \mathcal{B}_i \)-invariant subspaces from \( W = V \oplus \mathbb{R} \) to the subspace \( W_c \) of vectors bounded under the action of the linear group \( \mathcal{B}_i \subseteq \text{GL}(W) \). Of course such a layering can have length at most \( n = \dim W = 1 + \dim V \). Therefore \((\ast)\), and with it Theorem 1, will follow from the following result on linear actions.

11. Theorem. Let \( W \) be a finite dimensional real vector space and \( H \) any subgroup of \( \text{GL}(W) \), not necessarily connected or closed. Let \( W_c = \{ w \in W : H(w)^{-} \text{ is compact} \} \). Then there is an \( H \)-invariant layering consisting of \( H \)-invariant subspaces, terminating at \( W_c \).
Proof. Each $T \in H$ has a decomposition

$$T = T_v \cdot T_h \cdot T_u$$

(commuting operators)

where $T_v$ is unipotent, $T_h$ is semisimple with all real eigenvalues, and $T_u$ is semisimple with eigenvalues of modulus one. For $T \in H$ let

$$W_T = \{w \in W : T_h(w) = w = T_u(w)\}$$

these are just the points $w$ with bounded orbit under the iterates $\{T^p : p \in \mathbb{Z}\}$ of the single operator $T$. Let $W_* = \cap \{W_T : T \in H\}$; if $w \in W_*$ it is bounded under the iterated action of each individual operator $T \in H$. It is not immediately clear whether boundedness of $w$ under iterates of individual $T \in H$ implies "uniform" boundedness of $w$ under the action of the full group $H$. Clearly $W_c \subseteq W_*$; but by [9; Lemma 7.1], we get the desired conclusion that $W_* = W_c$.

Now suppose $w \in W \sim W_*$. Then for some $T \in H$, either $T_h(w) \neq w$ or $T_u(w) \neq w$. By trivial modifications of the argument in [1; Section 8] there must be a $T$-invariant subspace with $w \in W \leadsto W'$, $w \perp W' \supseteq W_*$, such that every element in $W \leadsto W'$ has a neighborhood with infinitely many disjoint $T$-transforms. By [1; Lemma 8.6] we can find a smaller $H$-invariant subspace $W''$, $W' \supseteq W'' \supseteq W_*$ such that every point in $W \sim W''$ has a neighborhood with infinitely many disjoint $H$-transforms. Since $W$ is finite dimensional we may continue this process to get an $H$-invariant layering by subspaces, terminating at $W_*$.

This completes the proof of Theorems 1 and 11.

4. Homogeneous Spaces Associated with $\beta \in \mathfrak{g} \mathfrak{l} (G)$

We now prove Theorem 2 which deals with a more general situation than that of Theorem 1 of [2]. While proof follows that of [2] in rough outline in certain respects it is more elaborate. We first recall a preliminary lemma from [12; Lemma 1.7].

12. Lemma. Let $G$ be a locally compact group, $H$ and $M$ closed subgroups such that $H$ normalizes $M$. Assume that $HM$ is closed in $G$. Then $HM/H$ has a finite $HM$-invariant measure if and only if $M/H \cap M$ has a finite $M$-invariant measure.

We first prove Theorem 2 when $G$ is connected. The crucial step here is to use Theorem 1 to show that every $\beta \in \mathfrak{g}$ has "bounded displacement". If $\beta \in \mathfrak{g}$ then $G_\beta \supseteq G_\alpha$. By [12; Lemma 1.6] $G/G_\beta$ has finite volume. Let $H$ be the semi-direct product $Z \times \rho G$ with $Z$ acting on $G$ through powers of $\beta$; $H$ is a Lie group with identity component $H_0 = G$. Let $b = (1, e) \in H$ and consider its centralizer in $H$, $C_b = Z_H(b)$. An easy calculation based on the multiplication law $(m, g_1) \cdot (n, g_2) = (m + n, g_1 \cdot \beta^m(g_2))$ shows that $C_b = Z \times G_\beta$; therefore there is a $G$-equivariant diffeomorphism
GC_b/C_b \approx G/G_\beta. By Lemma 12, GC_b/C_b has finite invariant volume. Now simple calculations show that GC_b = H; in fact, G_\beta = C_b \cap G and C_b = \cup\{m(e) \cdot G_\beta : n \in \mathbb{Z}\}. We may identify H/C_b with the conjugacy class \(0 = \mathcal{I}(H) \cdot b\) of \(b\) in \(H\), and get an \(\mathcal{I}(H)\)-invariant finite Borel measure \(\mu\) on \(H\) which lives on \(\mathcal{I}\) in the sense that \(\mu(\mathcal{I}) = \|\mu\|. That is \(\mu\) is a finite central measure on \(H\) and, by Theorem 3, supp(\(\mu\)) \subseteq B(H); hence \(\mathcal{I} \subseteq B(H)\) and the automorphism \(\alpha_b \in \mathcal{I}(H)\) is an automorphism of bounded displacement on \(H\), as defined in Tits [17]. Its restriction to \(G\), namely \(\beta\), is an automorphism of bounded displacement on \(G\). We summarize these observations as follows.

**13. Lemma.** Let \(G\) be a connected Lie group and \(\beta \in \mathcal{A}(G)\). If \(G/G_\beta\) has finite volume then \(\beta\) is an automorphism of bounded displacement.

From here on the steps in the proof that \((G/G_\infty\) finite volume) \(\Rightarrow\) \((G/G_\infty\) compact), for connected \(G\), follow the general pattern in [2; Section 2], but are quite different in detail.

**Step 1. The case when \(K_0 = K(G)_0\) is trivial.** Since each \(\beta \in \mathcal{S}\) is of bounded displacement, then, by Tits [17; Theorem 3], \(\beta = \alpha_g\) for some \(g \in B(G)\) so that \(\mathcal{S} = \{\alpha_g : g \in S\}\) for some set \(S \subseteq B(G)\).

Now, since \(K_0\) is trivial, \(B(G) = V \cdot Z(G)\) where \(V = B(G)_0\) is a vector group in \(Z(N)\) the center of the nilradical. Thus we can write \(g \in S\) as \(g = zv\); but \(\alpha_g = \alpha_v\) since \(z\) is central, so we may assume that \(S \subseteq V\). Let \(i : G \to G_v = \mathcal{I}(G) \cdot V \subseteq GL(V)\) be the natural homomorphism. Each \(v \in V\) has bounded orbit under \(G_v\), so \(G_v\) has compact closure. If we give \(G_v\) its natural Lie group topology, then the identity map carries the Lie group \(G_v\) faithfully, continuously into the compact Lie group \(G_v^{-}\). By the Freudenthal-Weil theorem [6; p. 145] we can write \(G_v = K_1 \times V_1\) where \(K_1\) is compact, \(V_1\) a vector group.

Let \(L = \bigcap\text{Stab}_G(v), L_v = \bigcap\text{Stab}_{G_v}(v)\) and let \(\varphi : G \to G_v\) (\(G_v\) with Lie topology) be the natural (continuous, open) surjective homomorphism induced by the maps \(G \to \mathcal{I}(G) \cdot V \to G_v\); since ker \(\varphi \subseteq \text{Stab}_G(v)\) for every \(v \in V\), \(L \supseteq \text{ker} \varphi\) and \(\varphi(L) = L_v\). Thus \(\varphi\) induces a homeomorphism between homogeneous spaces \(\tilde{\varphi} : G/L \to G_v/L_v\) which intertwines the actions of \(G\) and \(G_v\), \(\tilde{\varphi}(g \cdot \zeta) = \varphi(g) \cdot \tilde{\varphi}(\zeta)\) for all \(g \in G\). Clearly \(G/L\) has finite volume (is compact) iff \(G_v/L_v\) has finite volume (is compact).

Now suppose \(G/L\) has finite volume. Then so do \(G_v/L_v\) and \(G_v/K_1 \cdot L_v\) (\(K_1\) is normal in \(G_v\)). Let \(\pi : G_v = K_1 \times V_1 \to \tilde{V}_1 = V_1\) be the projection onto \(V_1\) (continuous, open homomorphism). Clearly \(\pi(L_v) = \pi(K_1 \cdot L_v)\) and \(\pi\) induces a \(V_1\)-equivariant homeomorphism \(\pi^* : G_v/K_1 \cdot L_v \to V_1/\pi(L_v)\). Thus \(V_1/\pi(L_v)\) has finite volume, and so is compact since \(V_1\) is abelian. Hence, working backward, we obtain compactness for \(G_v/K_1 \cdot L_v, G_v/L_v,\) and \(G/L = G/G_\infty\) in the case when \(K_0\) is trivial.
Step 2. Let $G$ have no compact simple connected normal subgroups. Let $H_{\beta} = \{ g \in G : \beta(g) \equiv g \mod K_0 \}$ for all $\beta \in \mathfrak{S}$. Since $G/G_{\mathfrak{S}}$ has finite volume, so do $G/H_{\mathfrak{S}}$ and $H_{\mathfrak{S}}/G_{\mathfrak{S}}$, see [12; Lemma 1.6]. By Step 1, $G/H_{\mathfrak{S}}$ is compact. [Let $\pi : G \to G^* = G/K_0$ be the canonical homomorphism, and $\beta^* \in \mathfrak{U}(G^*)$ the induced automorphism. Let $G^*_{\mathfrak{S}} = \{ x^* \in G^* : \beta^*(x^*) = x^* \}$ for all $\beta \in \mathfrak{S}$. Now $K(G^*)_0$ is trivial, $\pi(H_{\mathfrak{S}}) = G^*_{\mathfrak{S}}$, and $G/H_{\mathfrak{S}} \approx G^*/G^*_{\mathfrak{S}}$, via the homeomorphism $\varphi(gH_{\mathfrak{S}}) = \pi(g)G^*_{\mathfrak{S}}$, which intertwines the actions of $G$ and $G^* : \varphi(g \cdot \zeta) = \pi(g) \cdot \varphi(\zeta)$. Thus $G^*/G^*_{\mathfrak{S}}$ has finite volume, and is compact by Step 1.] Now $K_0$ is central in $G$ since its semisimple part is trivial by assumption. For each $\beta \in \mathfrak{S}$ we consider the map $\tau_\beta : H_{\mathfrak{S}} \to K_0$ given by $\tau_\beta(g) = g^{-1} \beta(g)$. Now $\tau_\beta(g_1 g_2) = (g_1 g_2)^{-1} \beta(g_1) \beta(g_2) = g_2^{-1} g_1^{-1} \beta(g_1) \beta(g_2) = \tau_\beta(g_1) \tau_\beta(g_2)$ since the range of $\tau_\beta$ is contained in $Z(G)$. Thus $\tau_\beta$ is a continuous homomorphism whose kernel is $G_\beta$ and $G_{\mathfrak{S}} = \cap \{ G_\beta : \beta \in \mathfrak{S} \}$ is normal in $H_{\mathfrak{S}}$. Now the homogeneous space $H_{\mathfrak{S}}/G_{\mathfrak{S}}$ has finite volume and is therefore compact. This implies $G/G_{\mathfrak{S}}$ is compact.

Step 3. Let $S_0$ be the largest compact connected semisimple subgroup within $K_0$, and assume its center $Z(S_0)$ is trivial. Now, exactly as in the proof of [2; Theorem 1] we may write $G = S_0 \cdot H \cong S_0 \times H$ where $H$ is a characteristic and closed subgroup without compact connected normal semisimple subgroups. [Triviality of $Z(S_0)$ insures $S_0 \cap H = \{ e \}$ and $H$ is the pullback of the simple factors in $G/R$ which do not have compact normal cross-sections in $G$.] Since $S_0$ is characteristic $\beta(S_0) \subseteq S_0$, while $\beta(H) \subseteq H$ for all $\beta \in \mathfrak{S}$. Let $(S_0)_{\mathfrak{S}} = \{ x \in S_0 : \beta(x) = x \}$ for all $\beta \in \mathfrak{S}$, $H_{\mathfrak{S}} = \{ x \in H : \beta(x) = x \}$ for all $\beta \in \mathfrak{S}$; then $G_{\mathfrak{S}} = (S_0)_{\mathfrak{S}} \cdot H_{\mathfrak{S}} \cong (S_0)_{\mathfrak{S}} \times H_{\mathfrak{S}}$, so that $G/G_{\mathfrak{S}} \approx (S_0)_{\mathfrak{S}} \times H/H_{\mathfrak{S}}$. The first factor is already compact; compactness of $H/H_{\mathfrak{S}}$ follows by Step 2.

Step 4. For arbitrary connected $G$, without restriction on $K_0$, let $Z(S_0)$ be the (finite) center of $S_0 \subseteq K_0$. Then, since $G$ is connected, and $Z(S_0)$ characteristic in $G$, $Z(S_0)$ is central in $G$. Let $\pi : G \to G^* = G/Z(S_0)$ be the canonical homomorphism, and for each $\beta \in \mathfrak{U}(G)$ let $\beta^*$ be the induced automorphism of $G^*$, $\beta^*(\pi(g)) = \pi(\beta(g))$. Then $A_{\mathfrak{S}} = \pi^{-1}(G^*_{\mathfrak{S}})$ is readily identified as $A_{\mathfrak{S}} = \{ g \in G : \beta(g) \equiv g \mod Z(S_0), \text{ all } \beta \in \mathfrak{S} \}$. Clearly $G_{\mathfrak{S}} \subseteq A_{\mathfrak{S}}$, so we get a natural surjective continuous open map $\varphi : G/G_{\mathfrak{S}} \to G^*/G^*_{\mathfrak{S}}$ by taking $\varphi(gG_{\mathfrak{S}}) = \pi(g) \cdot G^*_{\mathfrak{S}}$; $\varphi$ is equivariant in the sense that $\varphi(g \cdot \zeta) = \pi(g) \cdot \varphi(\zeta)$ for $\zeta \in G/G_{\mathfrak{S}}, g \in G$. By Lemma 2 of [2] the finite $G$-invariant measure $\mu$ on $G/G_{\mathfrak{S}}$ induces a finite $\pi(G) = G^*$-invariant measure on $G^*/G^*_{\mathfrak{S}}$. But $Z(S_0)_{\mathfrak{S}}$ is trivial in $G^*$ so we may apply Step 3 to conclude that $G^*/G^*_{\mathfrak{S}}$ is compact. Obviously $G/A_{\mathfrak{S}}$ is compact since $\pi : G \to G^*$ induces a homeomorphism $G/A_{\mathfrak{S}} \approx G^*/G^*_{\mathfrak{S}}$, because $A_{\mathfrak{S}} = \pi^{-1}(G^*_{\mathfrak{S}})$. Now $G \geq A_{\mathfrak{S}} \geq G_{\mathfrak{S}}$ so the coset space $A_{\mathfrak{S}}/G_{\mathfrak{S}}$ has finite volume ([12; Lemma 1.6]) Since $Z(S_0) \subseteq Z(G)$, for each $\beta \in \mathfrak{S}$ the map...
\( \tau_\beta : A_\circ \to Z(S_0) \) given by \( \tau_\beta(g) = g^{-1} \beta(g) \) is a homomorphism and as in Step 2 above \( G_\circ \) is normal in \( A_\circ \). This means that \( A_\circ / G_\circ \) and therefore also \( G/G_\circ \) is compact. This completes the proof for connected \( G \). We are ready to prove Theorem 2 in general.

We consider the open subgroup \( G_\circ G_0 \) of \( G \) and the diagram

\[
\begin{array}{c}
G \\
\downarrow \\
G_\circ G_0 \\
\downarrow \\
G_\circ \cap G_0 \\
\end{array}
\]

Since \( G/G_\circ \) has finite volume so does \( G/G_\circ \cdot G_0 \) so that this discrete space is finite. On the other hand since \( G_\circ G_0 / G_\circ \) has a finite \( G_\circ \cdot G_0 \)-invariant volume \( G_0 / G_\circ \cap G_0 \) has a finite \( G_0 \)-invariant volume. By the connected case, (considering the automorphisms gotten by restricting \( \beta \in \mathcal{g} \) to \( G_0 \)) the latter space is compact and therefore so is \( G_\circ \cdot G_0 / G_\circ \). Hence so is \( G/G_\circ \).

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F. P. Greenleaf
Courant Institute
New York University
251 Mercer Street
New York, N.Y. 10012, USA

M. Moskowitz
Department of Mathematics
City University of New York
New York, N.Y. 10036, USA

L. P. Rothschild
Institute for Advanced Study
Princeton, N.J. 08540, USA

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