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Automorphisms, Orbits, and Homogeneous Spaces of Non-Connected Lie Groups

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1. Introduction

Let G be a Lie group which is second countable but not necessarily connected with identity component G_0. Let $\mathfrak{B}$ be any subgroup of the group $\mathfrak{A}(G)$ of topological automorphisms such that $\mathfrak{B} \supseteqq \mathfrak{J}(G_0)$, the inner automorphisms obtained from elements of G_0. In this setting we shall generalize earlier results [1] on uniformity properties of unbounded orbits under the action of $\mathfrak{B}$ as well as those of [2] on compactness of certain homogeneous spaces of finite volume.

1. Theorem. *Let G and $\mathfrak{B}$ be as above, and $B = B(G, \mathfrak{B})$ the subgroup of points $x \in G$ with bounded orbit [so $\mathfrak{B}(x)^-$ is compact]. Then B is a closed subgroup in G. Furthermore, there exist closed $\mathfrak{B}$-invariant sets $G = X_m \supseteqq \cdots \supseteqq X_0 = B$ such that each point $x \in X_k \sim X_{k-1}$ has a relative neighborhood in $X_k \sim X_{k-1}$ with infinitely many disjoint $\mathfrak{B}$-transforms.*

Previous results considered $\mathfrak{B} = \mathfrak{J}(G)$ and more importantly applied only to connected groups. This more general setting results in more streamlined proofs as well as valuable viewpoints. While it is not surprising that these results extend to Lie groups with a finite number of components, it is somewhat surprising that they hold for essentially all Lie groups. Theorem 1 directly applies to the study of finite $\mathfrak{B}$-invariant Borel measures on G. If $\mathfrak{B} = \mathfrak{J}(G)$ these are the "central" measures on G.

Next let $\beta \in \mathfrak{A}(G)$ and $G_\beta = \{x \in G : \beta(x) = x\}$. If $\beta = \alpha_y$, an *inner* automorphism of the form $\alpha_y(g) = ygy^{-1}$, then G_β is just the centralizer C_y of a point $y \in G$.

2. Theorem. *Let $\mathfrak{S}$ be any subset of $\mathfrak{A}(G)$ for a second countable Lie group G, and let $G_{\mathfrak{S}} = \cap \{G_\beta : \beta \in \mathfrak{S}\}$. If the homogeneous space $G/G_{\mathfrak{S}}$ has finite volume, then it is compact.*

Theorem 1 will be used to prove Theorem 2, which substantially generalizes [2, Theorem 1]. There $\mathfrak{S}$ was a single inner automorphism and G was connected. Of course, once we allow outer automorphisms β

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then even if G is connected it seems necessary to consider non-connected Lie groups such as $Z \times_\beta G$ and Theorem 1 for such groups, in order to prove Theorem 2. We also give a number of consequences of Theorem 1 which generalize the corresponding results in [1] and [2]. We remark that, by modifications of these methods Claire Sit has proven results analogous to these in the case of linear algebraic groups in [16]. In addition by use of the present results she has recently extended Theorem 1 of [2] to the case of locally compact connected groups.

2. Some Consequences of Theorem 1

Unless otherwise stated G will be a Lie group which is not necessarily connected but with at most countably many components. Let G_0 be the identity component; R = radical of G_0; N = nilradical of G_0; $K(G_0)$ = largest compact normal subgroup of G_0 and $K_0 = K(G_0)_0$ its identity component (the largest compact connected normal subgroup of G_0). The definition of $K(G_0)$ was first discussed by Iwasawa in [7]. Furthermore let $Z(G)$ = center of G and, for any subset $S \subseteqq G$, $Z_G(S)$ be its centralizer. If $\mathfrak{B}$ is a subgroup in $\mathfrak{A}(G)$ and $A \subseteqq G$ a closed $\mathfrak{B}$-invariant subset, a $\mathfrak{B}$-*invariant layering* terminating with A is any family $G = X_m \supseteqq \cdots \supseteqq X_0 = A$ of closed $\mathfrak{B}$-invariant sets such that every point in the k^{th} "layer" $x \in X_k \sim X_{k-1}$ has a relative neighborhood in $X_k \sim X_{k-1}$ with infinitely many disjoint $\mathfrak{B}$-transforms. Our main theorem says that, if $\mathfrak{B} \supseteqq \mathfrak{J}(G_0)$, there is a layering terminating with $B(G, \mathfrak{B})$, the set of elements with bounded $\mathfrak{B}$-orbits. As in [1, Section 1] we immediately get the following result on finite $\mathfrak{B}$-invariant measures.

3. Theorem. *Let G be a second countable Lie group and $\mathfrak{B}$ a subgroup of $\mathfrak{A}(G)$ such that $\mathfrak{B} \supseteqq \mathfrak{J}(G_0)$. Then every finite $\mathfrak{B}$-invariant measure μ has $supp(\mu) \subseteqq B(G, \mathfrak{B})$.*

Taking $\mathfrak{B} = \mathfrak{J}(G)$ we get Corollary 4 concerning central idempotent measures ($\mu * \mu = \mu$). Via [1] the authors studied central idempotent measures on *connected* Lie groups in [3].

4. Corollary. *Let G be a second countable Lie group. For any central idempotent measure μ, the closed subgroup $G(\mu)$ generated by $supp(\mu)$ is compact.*

Proof. By Theorem 2, $supp(\mu) \subseteqq B(G) = B(G, \mathfrak{J}(G))$. Since $B(G)$ is closed for any Lie group, connected or not [17], it is locally compact and is clearly an FC^- group in the sense of [4]. By [15] (for a proof, see [8]), $B(G)$ is an IN-group. It then follows from [10] that $G(\mu)$ is compact.

As another application of Theorem 1 we have the following density theorem which is a generalization of Theorem 3 of [2].

5. Corollary. *Let G be any second countable Lie group and H a closed subgroup such that G/H has finite volume. Then $Z_G(H) \subseteqq B(G)$, and in particular if $B(G) = Z(G)$ we get $Z_G(H) = Z(G)$.*

Proof. Let $x \in Z_G(H)$. Then $H \subseteqq Z_G(x)$ and the G-equivariant surjection $G/H \to G/Z_G(x)$ gives a finite G-invariant measure on $G/Z_G(x)$. Since G is second countable, this gives an $\mathfrak{J}(G)$-invariant finite Borel measure μ on G which lives on the conjugacy class $\mathcal{O} = \mathfrak{J}(G) \cdot x$ in the sense that $\mu(\mathcal{O}) = \|\mu\|$. By Theorem 3, we get $x \in B(G)$.

We remark that if G/H is compact (a case which is not covered in Corollary 5 since G/H might not have an invariant measure) the same conclusion holds. For if G/H is compact, then $G = UH$ where U is a compact set, and so any automorphism $\alpha \in \mathfrak{A}(G)$ which leaves H pointwise fixed is an automorphism of bounded displacement because if $g = uh$, then for all $g, \alpha(g)g^{-1} = \alpha(u)\,\alpha(h)h^{-1}u^{-1} \in \alpha(U) \cdot U^{-1}$ a fixed compact set. Thus if $x \in Z_G(H)$, it follows that $x \in B(G)$; moreover, if G has no nontrivial automorphisms of bounded displacement, then each automorphism having H pointwise fixed is trivial on G.

Our next result extends [2; Corollary 4].

6. Corollary. *Let G be a second countable Lie group such that $B(G) = Z(G)$, and let Γ be a discrete subgroup such that G/Γ has finite volume. (Note: Γ discrete and G/Γ compact $\Rightarrow$ G/Γ has finite volume* [12, p. 21]*.) Then $\Gamma \cap Z(G)$ is a uniform lattice in $Z(G)$.*

Proof. Consider the closed subgroup $N_G(\Gamma) = \{x \in G : x\Gamma x^{-1} \subseteqq \Gamma\}$. Its identity component N_0 must centralize Γ and by Corollary 5 must therefore centralize G. Hence $N_0 \subseteqq \Gamma \cdot Z(G) \subseteqq N_G(\Gamma)$. Thus $\Gamma Z(G)$ is an open and closed subgroup of the closed subgroup $N_G(\Gamma)$ and so $\Gamma Z(G)$ is closed in G. Since G is second countable this implies that $\Gamma \cap Z(G)$ is a (uniform) lattice in $Z(G)$ as in [2].

3. The Proof of Theorem 1

We note that $B(G, \mathfrak{B})$ is a closed subgroup. Tits [17] has shown this in case $\mathfrak{B} = \mathfrak{J}(G)$, but his proof goes through with minor changes for arbitrary $\mathfrak{B}$. In proving that a layering exists we may immediately pass to the case in which $K_0 = K(G_0)_0$ is trivial; just use the following "liftback principle" and notice that $B(G, \mathfrak{B}) = \pi^{-1}(B(G/K_0, \mathfrak{B}^*))$ under the quotient homomorphism $\pi : G \to G/K_0$.

7. Lemma. *Let $G, \mathfrak{B}$ be as above. Let H be a closed $\mathfrak{B}$-invariant subgroup of G, $G^* = G/H$, $\pi : G \to G^*$ the quotient map, and $\mathfrak{B}^*$ the group of automorphisms induced on G^*. Then*

i) *If $x \notin B(G, \mathfrak{B})$ then $x^* = \pi(x) \notin B(G^*, \mathfrak{B}^*)$*

ii) *If $G^* = Y_m \supsetneqq \cdots \supsetneqq Y_0 = A^*$ is a $\mathfrak{B}^*$-invariant layering of G^* terminating at A^*, then $A = \pi^{-1}(A^*)$ is $\mathfrak{B}$-invariant and the sets $X_i = \pi^{-1}(Y_i)$ give a $\mathfrak{B}$-invariant layering in G terminating at A.*

The proof is obvious.

Next consider the semisimple case (i.e., G_0 semisimple).

8. Lemma. *If G is a second countable semisimple Lie group, there is a one-step $\mathfrak{B}$-invariant layering terminating at $B(G, \mathfrak{B})$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra (of G_0), $\mathfrak{g}_1$ the product of the non-compact factors, and define the homomorphism $\mathrm{Ad}_1 : G \to \mathrm{Aut}(\mathfrak{g}_1)$ via $\mathrm{Ad}_1(x) = d(\alpha_x)_e | \mathfrak{g}_1$, where $\alpha_x(g) = xgx^{-1}$. Let $x \notin B(G, \mathfrak{B})$. If $\mathrm{Ad}_1(x) \neq \mathrm{id}$, there exists an inner automorphism $\alpha = \alpha_y \in \mathfrak{B} (y \in G_0)$ and a neighborhood U of x such that $\{\alpha^n(U) : n \in \mathbf{Z}\}$ includes infinitely many disjoint transforms of U. (This follows by an obvious modification of the argument in [14].) If $\mathrm{Ad}_1(x) = \mathrm{id}$, consider the quotient map $\pi : G \to G^* = G/G_1$ where G_1 is the connected Lie subgroup (which is closed, see [11] or [13]) corresponding to $\mathfrak{g}_1$. Now G_1 is characteristic in G so $\mathfrak{B}$ induces a group of automorphisms $\mathfrak{B}^* \subseteq \mathfrak{A}(G^*)$. Now $x \notin B(G, \mathfrak{B})$ insures that $x^* = \pi(x) \notin B(G^*, \mathfrak{B}^*)$, by Lemma 7; thus $\mathfrak{B}^*(x^*)$ meets infinitely many cosets of $G_0^* = \pi(G_0)$ since $G_0^* \cong G_0/G_1$ is compact. Taking $U^* =$ the coset of G_0^* containing x^* we have an open neighborhood with infinitely many disjoint $\mathfrak{B}^*$-transforms. Lifting back to $U = \pi^{-1}(U^*)$ we get a neighborhood of x with similar properties under the action of $\mathfrak{B}$.

Letting $G^* = G/R$ ($R =$ radical of G_0) and $\pi : G \to G^*$ the canonical homomorphism we may apply Lemma 8 to G^*, then upon lifting back as in Lemma 7 we get a $\mathfrak{B}$-invariant layering of G terminating at $\pi^{-1}(B(G/R, \mathfrak{B}^*))$.

The next step is to get a different layering terminating at $Z_G(N)$. As noted in [1; Lemma 5.1], even for non-connected second countable Lie groups there is an $\mathfrak{J}(G_0)$-invariant layering from G to $Z_G(N)$, provided that $K_0 = K(G_0)_0$ is trivial, as we are assuming here. But the layers $G = X_m \supsetneqq \cdots \supsetneqq X_0 = Z_G(N)$ produced in [1] consist of closed characteristic subgroups and are thus $\mathfrak{B}$-invariant. Since $\mathfrak{B} \supsetneqq \mathfrak{J}(G_0)$ we get the desired $\mathfrak{B}$-invariant layering terminating at $Z_G(N)$.

Finally, passing to the discrete group G/G_0 and applying Lemma 7, we get a layering to the group L consisting of those $x \in G$ whose $\mathfrak{B}$-orbit meets only finitely many cosets of G_0. By [1; Lemma 2.2] we may put together the layerings obtained in the last paragraphs to get a $\mathfrak{B}$-invariant layering terminating at $A_{\mathfrak{B}} = L \cap Z_G(N) \cap \pi^{-1}(B(G/R, \mathfrak{B}^*))$. Since we are working in the case where K_0 is trivial, the nilradical N of G_0 is simply connected [1; Lemma 3.1]. We now seek to identify the identity component $A_0 = (A_{\mathfrak{B}})_0$ and show that it is $Z(N)$ = center of the nilradical, independent of the group $\mathfrak{B}$.

9. Lemma. *If G is a semisimple second countable Lie group and $\mathfrak{B} \supseteq \mathfrak{I}(G_0)$ then $G_0 \cap B(G, \mathfrak{B}) = B(G_0, \mathfrak{B} | G_0) = B(G_0, \mathfrak{I}(G_0)) = B(G_0)$.*

Proof. In our terminology, $B(G_0) = B(G_0, \mathfrak{I}(G_0))$. Clearly $B(G, \mathfrak{B}) \cap G_0 \subseteq B(G_0, \mathfrak{B} | G_0) \subseteq B(G_0)$. On the other hand if $x \in B(G_0)$ then since G_0 is characteristic, $\mathfrak{B} | G_0 \supseteq \mathfrak{I}(G_0)$ and $\mathfrak{I}(G_0) \cdot x$ is bounded. But (see [5]) $[\mathfrak{A}(G_0) : \mathfrak{I}(G_0)] < \infty$ since G_0 is semisimple. Therefore $\mathfrak{B}(x) \subseteq \mathfrak{A}(G_0) \cdot x$ is bounded and $x \in B(G, \mathfrak{B}) \cap G_0$.

Now for our identification of $(A_{\mathfrak{B}})_0$.

10. Lemma. $(A_{\mathfrak{B}})_0 = Z(N)$ *for any* $\mathfrak{B} \supseteq \mathfrak{I}(G_0)$.

Proof. By Lemma 9,

$$\begin{aligned}(A_{\mathfrak{B}})_0 &\subseteq G_0 \cap \pi^{-1}(B(G/R, \mathfrak{B}^*)) = \pi_0^{-1}(B(G/R, \mathfrak{B}^*)) \\ &= \pi_0^{-1}(B(G_0/R, \mathfrak{B}^* | G_0/R)) \\ &= \pi_0^{-1}(B(G_0/R))\end{aligned}$$

where $\pi : G \to G/R$ is the canonical map and $\pi_0 : G_0 \to \pi(G_0) = G_0/R$ its restriction to G_0. Moreover, $Z_G(N) \cap G_0 = Z_{G_0}(N)$, so that

$$(A_{\mathfrak{B}})_0 \subseteq A_{\mathfrak{B}} \cap G_0 \subseteq \pi_0^{-1}(B(G_0/R)) \cap Z_{G_0}(N).$$

The latter intersection is precisely the group A studied in [1; Section 6], taking G_0 as the connected Lie group in that discussion. There we showed that $A_0 = Z(N)$; hence in our present situation $(A_{\mathfrak{B}})_0 \subseteq Z(N)$. Clearly the reverse inclusion holds.

Hereafter we simplify notation by writing $A = A_{\mathfrak{B}}$ and $V = A_0 = Z(N)$. Our problem is to layer A down to $B(A, \mathfrak{B}')$ in finitely many steps where $\mathfrak{B}' = \mathfrak{B} | A$. Consider the decomposition of A into open/closed cosets

$$A = \bigcup_{i \in I} a_i V, \quad a_i \in Z_G(N).$$

Since V is characteristic in G, $\mathfrak{B}'(V) \subseteq V$ and $\mathfrak{B}'$ permutes these cosets, $\beta(a_i V) = \beta(a_i) V$. If, for a fixed $j \in I$, $\mathfrak{B}'(a_j V) = \mathfrak{B}'(a_j) V$ is a union of infinitely many cosets, then we can immediately layer (in one step) from A to $A \sim \mathfrak{B}'(a_j V)$ by taking xV for the neighborhood of $x \in a_j V$ with infinitely many $\mathfrak{B}'$-transforms. Thus we may layer, in one step, from A to the open/closed subgroup $A_{\text{fin}} = \cup \{a_i V : \mathfrak{B}'(a_i V)$ is a finite union of V-cosets$\}$; that is, we may assume $A = A_{\text{fin}}$.

Now for each $j \in I$ let $\mathfrak{B}'_j = \mathrm{Stab}_{\mathfrak{B}'}(a_j V) = \{\beta \in \mathfrak{B}' : \beta(a_j V) \subseteq a_j V\}$. This is a subgroup of $\mathfrak{B}'$ with index $[\mathfrak{B}' : \mathfrak{B}'_j]$ = number of V-cosets in $\mathfrak{B}'(a_j V) < \infty$. We next observe that

$$B(A, \mathfrak{B}') = \bigcup_{j \in I} \{x \in A : x \in a_j V \quad \text{and} \quad \mathfrak{B}'_j(x)^- \quad \text{is compact}\}.$$

Clearly the inclusion ($\subseteq$) holds. If $x \in a_j V$ for some $j \in I$ and $\mathfrak{B}'_j(x)^-$ is compact then $\mathfrak{B}' = \bigcup_{k=1}^{m} \beta_k \mathfrak{B}'_j$ where $m = [\mathfrak{B}' : \mathfrak{B}'_j]$, so we get $\mathfrak{B}'(x)^- = \bigcup_{k=1}^{m} \beta_k(\mathfrak{B}_j(x)^-)$, which is compact.

If $A_i = a_i V$ is a coset, write $A_{i,c} = \{x \in A_i : \mathfrak{B}'_i(x)^- \text{ is compact}\}$, so $B(A, \mathfrak{B}') = \bigcup_{i \in I} A_{j,c}$. Divide A into equivalence classes $\{E_1, E_2, \ldots\}$, each a minimal $\mathfrak{B}'$-invariant open/closed finite union of V-cosets. We will now show that each E_k has a $\mathfrak{B}'$-invariant layering of length at most $n = 1 + \dim V$, terminating with $E_{k,c} = E_k \cap B(A, \mathfrak{B}')$. Fix $k \in \boldsymbol{Z}$ and write $E_k = \bigcup_{j=1}^{m} \beta_j(A_i)$, $\beta_j \in \mathfrak{B}'$, for some representative coset A_i in E_k; the $\{\beta_1 \ldots \beta_m\}$ are then coset representatives for $\mathfrak{B}'/\mathfrak{B}'_i$. Suppose we can produce a $\mathfrak{B}'_i$-invariant layering $A_i = X_n \supsetneqq \cdots \supsetneqq X_0 = A_{i,c}$ with at most n steps. Then, clearly, we get a $\mathfrak{B}'$-invariant layering of E_k terminating at $E_{k,c} = \bigcup_{j=1}^{m} \beta_j(A_{i,c})$ by taking $Y_l = \bigcup_{j=1}^{m} \beta_j(X_l)$ for $0 \leqq l \leqq n$. Thus our task of layering A reduces to the following problem.

($*$) In each coset A_i, show that there is a $\mathfrak{B}'_i$-invariant layering with at most $n = 1 + \dim V$ steps, terminating at $A_{i,c} = \{x \in A_i : \mathfrak{B}'_i(x)^- \text{ is compact}\}$.

For this, consider the action of $\mathfrak{B}'_i$ on a single coset A_i. It is easy to see that the action $\mathfrak{B}'_i \times A_i \to A_i$ is equivariant with an action of $\mathfrak{B}'_i$ via affine transformations on the vector space V [1; Section 7]. The latter may be equivariantly identified with the restriction of a *linear* action of $\mathfrak{B}'_i$ on $W = V \oplus \boldsymbol{R}$ to the hyperplane $V + \{1\}$. Proving that there is a $\mathfrak{B}'_i$-invariant layering from A_i to $A_{i,c}$ thus reduces, as in [1; Section 7] to finding a layering consisting of $\mathfrak{B}'_i$-invariant subspaces from $W = V \oplus \boldsymbol{R}$ to the subspace W_c of vectors bounded under the action of the linear group $\mathfrak{B}'_i \subseteq \mathrm{GL}(W)$. Of course such a layering can have length at most $n = \dim W = 1 + \dim V$. Therefore ($*$), and with it Theorem 1, will follow from the following result on linear actions.

11. Theorem. *Let W be a finite dimensional real vector space and H any subgroup of* $\mathrm{GL}(W)$, *not necessarily connected or closed. Let $W_c = \{w \in W : H(w)^-$ is compact$\}$. Then there is an H-invariant layering consisting of H-invariant subspaces, terminating at W_c.*

Proof. Each $T \in H$ has a decomposition

$$T = T_e \cdot T_h \cdot T_u \quad \text{(commuting operators)}$$

where T_u is unipotent, T_h is semisimple with all real eigenvalues, and T_e semisimple with eigenvalues of modulus one. For $T \in H$ let $W_T = \{w \in W : T_h(w) = w = T_u(w)\}$; these are just the points w with bounded orbit under the iterates $\{T^p : p \in \mathbf{Z}\}$ of the single operator T. Let $W_* = \cap\{W_T : T \in H\}$; if $w \in W_*$ it is bounded under the iterated action of each individual operator $T \in H$. It is not immediately clear whether boundedness of w under iterates of individual $T \in H$ implies "uniform" boundedness of w under the action of the full group H. Clearly $W_c \subsetneqq W_*$; but by [9; Lemma 7.1], we get the desired conclusion that $W_* = W_c$.

Now suppose $w \in W \sim W_*$. Then for some $T \in H$, either $T_h(w) \neq w$ or $T_u(w) \neq w$. By trivial modifications of the argument in [1; Section 8] there must be a T-invariant subspace with $w \in W \sim W'$, $W \neq W' \supseteq W_*$, such that every element in $W \sim W'$ has a neighborhood with infinitely many disjoint T-transforms. By [1; Lemma 8.6] we can find a smaller H-invariant subspace W'', $W' \supseteq W'' \supseteq W_*$ such that every point in $W \sim W''$ has a neighborhood with infinitely many disjoint H-transforms. Since W is finite dimensional we may continue this process to get an H-invariant layering by subspaces, terminating at W_*.

This completes the proof of Theorems 1 and 11.

4. Homogeneous Spaces Associated with $\beta \in \mathfrak{A}(G)$

We now prove Theorem 2 which deals with a more general situation than that of Theorem 1 of [2]. While proof follows that of [2] in rough outline in certain respects it is more elaborate. We first recall a preliminary lemma from [12; Lemma 1.7].

12. Lemma. *Let G be a locally compact group, H and M closed subgroups such that H normalizes M. Assume that HM is closed in G. Then HM/H has a finite HM-invariant measure if and only if $M/H \cap M$ has a finite M-invariant measure.*

We first prove Theorem 2 when G is connected. The crucial step here is to use Theorem 1 to show that every $\beta \in \mathfrak{S}$ has "bounded displacement". If $\beta \in \mathfrak{S}$ then $G_\beta \supseteq G_{\mathfrak{S}}$. By [12; Lemma 1.6] G/G_β has finite volume. Let H be the semi-direct product $\mathbf{Z} \times_\beta G$ with $\mathbf{Z}$ acting on G through powers of β; H is a Lie group with identity component $H_0 = G$. Let $b = (1, e) \in H$ and consider its centralizer in H, $C_b = Z_H(b)$. An easy calculation based on the multiplication law $(m, g_1) . (n, g_2) = (m + n, g_1 \cdot \beta^m(g_2))$ shows that $C_b = \mathbf{Z} \times G_\beta$; therefore there is a G-equivariant diffeomorphism

$GC_b/C_b \approx G/G_\beta$. By Lemma 12, GC_b/C_b has finite invariant volume. Now simple calculations show that $GC_b = H$; in fact, $G_\beta = C_b \cap G$ and $C_b = \cup\{(n,e)\cdot G_\beta : n \in Z\}$. We may identify H/C_b with the conjugacy class $\mathcal{O} = \mathfrak{I}(H)\cdot b$ of b in H, and get an $\mathfrak{I}(H)$-invariant finite Borel measure μ on H which lives on $\mathcal{O}$ in the sense that $\mu(\mathcal{O}) = \|\mu\|$. That is μ is a finite central measure on H and, by Theorem 3, $supp(\mu) \subseteqq B(H)$; hence $\mathcal{O} \subseteqq B(H)$ and the automorphism $\alpha_b \in \mathfrak{I}(H)$ is an automorphism of bounded displacement on H, as defined in Tits [17]. Its restriction to G, namely β, is an automorphism of bounded displacement on G. We summarize these observations as follows.

13. Lemma. *Let G be a connected Lie group and $\beta \in \mathfrak{A}(G)$. If G/G_β has finite volume then β is an automorphism of bounded displacement.*

From here on the steps in the proof that ($G/G_\mathfrak{S}$ finite volume)$\Rightarrow$($G/G_\mathfrak{S}$ compact), for connected G, follow the general pattern in [2; Section 2], but are quite different in detail.

Step 1. The case when $K_0 = K(G)_0$ is trivial. Since each $\beta \in \mathfrak{S}$ is of bounded displacement, then, by Tits [17; Theorem 3], $\beta = \alpha_g$ for some $g \in B(G)$ so that $\mathfrak{S} = \{\alpha_g : g \in S\}$ for some set $S \subseteqq B(G)$.

Now, since K_0 is trivial, $B(G) = V \cdot Z(G)$ where $V = B(G)_0$ is a vector group in $Z(N)$ the center of the nilradical. Thus we can write $g \in S$ as $g = zv$; but $\alpha_g = \alpha_v$ since z is central, so we may assume that $S \subseteqq V$. Let $i: G \to G_V = \mathfrak{I}(G)|V \subseteqq \mathrm{GL}(V)$ be the natural homomorphism. Each $v \in V$ has bounded orbit under G_V, so G_V has compact closure. If we give G_V its natural Lie group topology, then the identity map carries the Lie group G_V faithfully, continuously into the compact Lie group G_V^-. By the Freudenthal-Weil theorem [6; p. 145] we can write $G_V = K_1 \times V_1$ where K_1 is compact, V_1 a vector group.

Let $L = \bigcap_{v \in S} \mathrm{Stab}_G(v)$, $L_V = \bigcap_{v \in S} \mathrm{Stab}_{G_V}(v)$ and let $\varphi: G \to G_V$ (G_V with Lie topology) be the natural (continuous, open) surjective homomorphism induced by the maps $G \to \mathfrak{I}(G)|V \to G_V$; since $\ker\varphi \subseteqq \mathrm{Stab}_G(v)$ for every $v \in V$, $L \supseteqq \ker\varphi$ and $\varphi(L) = L_V$. Thus φ induces a homeomorphism between homogeneous spaces $\tilde{\varphi}: G/L \to G_V/L_V$ which intertwines the actions of G and G_V, $\tilde{\varphi}(g \circ \zeta) = \varphi(g) \circ \tilde{\varphi}(\zeta)$ for all $g \in G$. Clearly G/L has finite volume (is compact) iff G_V/L_V has finite volume (is compact). Now suppose G/L has finite volume. Then so do G_V/L_V and $G_V/K_1 \cdot L_V$ (K_1 is normal in G_V). Let $\pi: G_V = K_1 \times V_1 \to \tilde{V}_1 = V_1$ be the projection onto V_1 (continuous, open homomorphism). Clearly $\pi(L_V) = \pi(K_1 \cdot L_V)$ and π induces a V_1-equivariant homeomorphism $\pi^{\sim}: G_V/K_1 L_V \to V_1^{\sim}/\pi(L_V)$. Thus $V_1^{\sim}/\pi(L_V)$ has finite volume, and so is compact since $V_1^{\sim}$ is abelian. Hence, working backward, we obtain compactness for $G_V/K_1 L_V$, G_V/L_V, and $G/L = G/G_\mathfrak{S}$ in the case when K_0 is trivial.

Step 2. Let G have no compact *simple* connected normal subgroups. Let $H_{\mathfrak{S}} = \{g \in G : \beta(g) \in gK_0$ (so $\beta(g) \equiv g \bmod K_0$) for all $\beta \in \mathfrak{S}\}$. Since $G/G_{\mathfrak{S}}$ has finite volume, so do $G/H_{\mathfrak{S}}$ and $H_{\mathfrak{S}}/G_{\mathfrak{S}}$, see [12; Lemma 1.6]. By Step 1, $G/H_{\mathfrak{S}}$ is compact. [Let $\pi : G \to G^* = G/K_0$ be the canonical homomorphism, and $\beta^* \in \mathfrak{A}(G^*)$ the induced automorphism. Let $G^*_{\mathfrak{S}^*} = \{x^* \in G^* : \beta^*(x^*) = x^*$ for all $\beta \in \mathfrak{S}\}$. Now $K(G^*)_0$ is trivial, $\pi(H_{\mathfrak{S}}) = G^*_{\mathfrak{S}^*}$ and $G/H_{\mathfrak{S}} \approx G^*/G^*_{\mathfrak{S}^*}$ via the homeomorphism $\varphi(gH_{\mathfrak{S}}) = \pi(g)G^*_{\mathfrak{S}^*}$ which intertwines the actions of G and G^* : $\varphi(g \circ \zeta) = \pi(g) \circ \varphi(\zeta)$. Thus $G^*/G^*_{\mathfrak{S}^*}$ has finite volume, and is compact by Step 1.] Now K_0 is central in G since its semisimple part is trivial by assumption. For each $\beta \in \mathfrak{S}$ we consider the map $\tau_\beta : H_{\mathfrak{S}} \to K_0$ given by $\tau_\beta(g) = g^{-1}\beta(g)$. Now

$$\tau_\beta(g_1 g_2) = (g_1 g_2)^{-1}\beta(g_1 g_2) = g_2^{-1} g_1^{-1}\beta(g_1)\beta(g_2) = \tau_\beta(g_1)\tau_\beta(g_2)$$

since the range of τ_β is contained in $Z(G)$. Thus τ_β is a continuous homomorphism whose kernel is G_β and $G_{\mathfrak{S}} = \cap\{G_\beta : \beta \in \mathfrak{S}\}$ is normal in $H_{\mathfrak{S}}$. Now the homogeneous space $H_{\mathfrak{S}}/G_{\mathfrak{S}}$ has finite volume and is therefore compact. This implies $G/G_{\mathfrak{S}}$ is compact.

Step 3. Let S_0 be the largest compact connected semisimple subgroup within K_0, and assume its center $Z(S_0)$ is trivial. Now, exactly as in the proof of [2; Theorem 1] we may write $G = S_0 \cdot H \cong S_0 \times H$ where H is a characteristic and closed subgroup without compact connected normal semisimple subgroups. [Triviality of $Z(S_0)$ insures $S_0 \cap H = \{e\}$ and H is the pullback of the simple factors in G/R which do not have compact normal cross-sections in G.] Since S_0 is characteristic $\beta(S_0) \subseteq S_0$, while $\beta(H) \subseteq H$ for all $\beta \in \mathfrak{S}$. Let $(S_0)_{\mathfrak{S}} = \{x \in S_0 : \beta(x) = x$ for all $\beta \in \mathfrak{S}\}$, $H_{\mathfrak{S}} = \{x \in H : \beta(x) = x$ for all $\beta \in \mathfrak{S}\}$; then $G_{\mathfrak{S}} = (S_0)_{\mathfrak{S}} \cdot H_{\mathfrak{S}} \cong (S_0)_{\mathfrak{S}} \times H_{\mathfrak{S}}$, so that $G/G_{\mathfrak{S}} \approx S_0/(S_0)_{\mathfrak{S}} \times H/H_{\mathfrak{S}}$. The first factor is already compact; compactness of $H/H_{\mathfrak{S}}$ follows by Step 2.

Step 4. For arbitrary connected G, without restriction on K_0, let $Z(S_0)$ be the (finite) center of $S_0 \subseteq K_0$. Then, since G is connected, and $Z(S_0)$ characteristic in G, $Z(S_0)$ is *central in G*. Let $\pi : G \to G^* = G/Z(S_0)$ be the canonical homomorphism, and for each $\beta \in \mathfrak{A}(G)$ let β^* be the induced automorphism of G^*, $\beta^*(\pi g) = \pi(\beta(g))$. Then $A_{\mathfrak{S}} = \pi^{-1}(G^*_{\mathfrak{S}^*})$ is readily identified as $A_{\mathfrak{S}} = \{g \in G : \beta(g) \equiv g \bmod Z(S_0)$, all $\beta \in \mathfrak{S}\}$. Clearly $G_{\mathfrak{S}} \subseteq A_{\mathfrak{S}}$, so we get a natural surjective continuous open map $\varphi : G/G_{\mathfrak{S}} \to G^*/G^*_{\mathfrak{S}^*}$ by taking $\varphi(gG_{\mathfrak{S}}) = \pi(g) \cdot G^*_{\mathfrak{S}^*}$; φ is equivariant in the sense that $\varphi(g \circ \zeta) = \pi(g) \cdot \varphi(\zeta)$ for $\zeta \in G/G_{\mathfrak{S}}$, $g \in G$. By Lemma 2 of [2] the finite G-invariant measure μ on $G/G_{\mathfrak{S}}$ induces a finite $\pi(G) = G^*$-invariant measure on $G^*/G^*_{\mathfrak{S}^*}$. But $Z(S_0^*)$ is trivial in G^* so we may apply Step 3 to conclude that $G^*/G^*_{\mathfrak{S}^*}$ is compact. Obviously $G/A_{\mathfrak{S}}$ is compact since $\pi : G \to G^*$ induces a homeomorphism $G/A_{\mathfrak{S}} \approx G^*/G^*_{\mathfrak{S}^*}$ because $A_{\mathfrak{S}} = \pi^{-1}(G^*_{\mathfrak{S}^*})$. Now $G \supseteq A_{\mathfrak{S}} \supseteq G_{\mathfrak{S}}$ so the coset space $A_{\mathfrak{S}}/G_{\mathfrak{S}}$ has finite volume ([12; Lemma 1.6]) Since $Z(S_0) \subseteq Z(G)$, for each $\beta \in \mathfrak{S}$ the map

$\tau_\beta : A_{\mathfrak{S}} \to Z(S_0)$ given by $\tau_\beta(g) = g^{-1}\beta(g)$ is a homomorphism and as in Step 2 above $G_{\mathfrak{S}}$ is normal in $A_{\mathfrak{S}}$. This means that $A_{\mathfrak{S}}/G_{\mathfrak{S}}$ and therefore also $G/G_{\mathfrak{S}}$ is compact. This completes the proof for connected G. We are ready to prove Theorem 2 in general.

We consider the open subgroup $G_{\mathfrak{S}} G_0$ of G and the diagram

$$\begin{tikzcd} & G \arrow[d, dash] & \\ & G_{\mathfrak{S}} G_0 \arrow[dl, dash] \arrow[dr, dash] & \\ G_{\mathfrak{S}} \arrow[dr, dash] & & G_0 \arrow[dl, dash] \\ & G_{\mathfrak{S}} \cap G_0 & \end{tikzcd}$$

Since $G/G_{\mathfrak{S}}$ has finite volume so does $G/G_{\mathfrak{S}} \cdot G_0$ so that this discrete space is finite. On the other hand since $G_{\mathfrak{S}} G_0/G_{\mathfrak{S}}$ has a finite $G_{\mathfrak{S}} \cdot G_0$-invariant volume $G_0/G_{\mathfrak{S}} \cap G_0$ has a finite G_0-invariant volume. By the connected case, (considering the automorphisms gotten by restricting $\beta \in \mathfrak{S}$ to G_0) the latter space is compact and therefore so is $G_{\mathfrak{S}} \cdot G_0/G_{\mathfrak{S}}$. Hence so is $G/G_{\mathfrak{S}}$.

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