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# CR MAPPINGS OF FINITE MULTIPLICITY AND EXTENSION OF PROPER HOLOMORPHIC MAPPINGS

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**1. Introduction.** We shall describe some general theorems about CR mappings between three-dimensional manifolds which, among other results, imply that any proper holomorphic mapping $f : D \to D'$ between pseudoconvex domains in $\mathbf{C}^2$ with real analytic boundaries extends to be holomorphic in a neighborhood of the closure of $D$ (Theorem 8). In case the domain $D$ is strictly pseudoconvex, this result follows from the classical Lewy-Pinčuk reflection principle [8], [10]. In case the proper mapping $f$ is biholomorphic, the extendability has been proved by Baouendi-Jacobowitz-Treves [1]. The general case of a proper holomorphic mapping between weakly pseudoconvex domains which is not biholomorphic is more complicated because branching might occur. We have developed a method in the spirit of [1] which allows us to prove extendability at boundary points even if branching occurs. (Theorems 3 and 6).

The mapping $f(z,w) = (z^2, w)$ which maps the domain $\mathsf{E} = \{(z,w) \in \mathbf{C}^2 : |z|^4 + |w|^2 < 1\}$ onto the unit ball in $\mathbf{C}^2$ has the property that it maps points of type four (in the sense of Kohn [7]) in the boundary of $\mathsf{E}$ to points of type two in the boundary of the ball. Furthermore, the local branching order of $f$ at these points is two. We prove that this phenomenon holds in general. If $M$ and $M'$ are abstract three-dimensional CR manifolds, and $H : M \to M'$ a CR mapping, there is a notion of multiplicity of $H$ at $p_0 \in M$, for which the type of $p_0$ is equal to the multiplicity at $p_0$ times the type of $H(p_0)$. Theorems 1 and 2 state these results more precisely. Theorems 5 and 7 give applications of these results and of the extendability result (Theorem 3) to CR and proper self mappings.

**2. Main Results and Applications.** A real smooth manifold $M$ is called a *CR manifold* if there is a subbundle $\mathcal{V}$ of $\mathbf{C}TM$, the complexified tangent bundle of $M$, satisfying the conditions $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, and $\mathcal{V} \cap \overline{\mathcal{V}} = (0)$; $\mathcal{V}$ is called the CR *bundle* of $M$.

If $M$ and $M'$ are two CR manifolds with CR bundles $\mathcal{V}$ and $\mathcal{V}'$, a CR mapping from $M$ into $M'$ is a smooth mapping $H : M \to M'$, such that at every $p \in M$

$$H'(\theta) \in \mathcal{V}'_{H(p)}, \qquad \forall\, \theta \in \mathcal{V}_p, \tag{1}$$

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where $\mathcal{V}_p$ is the fiber of $\mathcal{V}$ at $p$, $\mathcal{V}'_{H(p)}$ is the fiber of $\mathcal{V}'$ at $H(p)$, and $H' : \mathbb{C}TM \to \mathbb{C}TM'$ the differential map of $H$.

From now on we restrict ourselves to the case: $dim_{\mathbb{R}} M = dim_{\mathbb{R}} M' = 3$, $dim_{\mathbb{C}} \mathcal{V} = dim_{\mathbb{C}} \mathcal{V}' = 1$. Let $p_0 \in M$, $p'_0 = H(p_0) \in M'$, and $L$, $L'$ two nonvanishing smooth sections of $\mathcal{V}$ and $\mathcal{V}'$, defined near $p_0$ and $H(p_0)$ respectively. It follows from (1) that there exists a smooth function $\lambda$ defined in a neighborhood of $p_0$ in $M$ such that, for all $p \in M$ near $p_0$,

$$H'(L_p) = \lambda(p) L'_{H(p)}. \tag{2}$$

DEFINITION. *The CR mapping $H : M \to M'$ is of finite multiplicity at $p_0$ if there exists a differential operator of the form*

$$S = M_1 \dots M_j \tag{3}$$

*(called string of length $j$) with $M_p = L$ or $\overline{L}$, such that*

$$S\lambda(p_0) \neq 0. \tag{4}$$

*The mapping is said to be of multiplicity $k$ if the shortest string $S$ for which (4) holds is of length $k-1$.*

In particular $H$ is of multiplicity 1 if $\lambda(p_0) \neq 0$, of multiplicity 2 if $\lambda(p_0) = 0$, and either $L\lambda(p_0) \neq 0$ or $\overline{L}\lambda(p_0) \neq 0$, etc...

It is clear that for a CR mapping $H$ to be of multiplicity $k$ is independent of the choice of the vector fields $L$ and $L'$.

Following Kohn [7], the CR manifold $M$ is of *finite type* at $p_0$ if and only if any smooth complex vector field defined near $p_0$ is in the Lie algebra spanned by $L$ and $\overline{L}$. It is of *type $m$ at $p_0$* ($m \geq 2$) if the shortest bracket of $L$'s and $\overline{L}$'s not in the span of $L$ and $\overline{L}$ at $p_0$, is of length $m$. (We have used the following convention: the length of $[L, \overline{L}]$ is 2, that of $[L, [L, \overline{L}]]$ is 3 etc....).

We can now state our first result.

THEOREM 1. *Let $H : M \to M'$ be a CR mapping from $M$ into $M'$. If $H$ is of multiplicity $k$ at $p_0 \in M$, ($1 \leq k < \infty$), and $M'$ is of type $m'$ at $p'_0 = H(p_0)$, ($2 \leq m' < \infty$), then the following holds:*

(i) *$M$ is of type $m$ at $p_0$ with $m = km'$.*

(ii) *If $S$ is a string of the form (3) with length $\leq k-1$, and with at least one $M_p = \overline{L}$ then $S\lambda(p_0) = 0$; in particular $L^{k-1}\lambda(p_0) \neq 0$.*

(iii) $H'(\mathbb{C}T_{p_0}M) \not\subset \mathcal{V}'_{H(p_0)} \oplus \overline{\mathcal{V}}'_{H(p_0)}$.

Note that if $H'_{p_0} : T_{p_0}M \to T_{H(p_0)}M'$ is the differential map at $p_0$, then (iii) implies that

$$H'_{p_0} \neq \{0\}.$$

If $k = 1$, it follows from (2), (4) (with $S = 1$) and (iii) that $H'_{p_0}$ is an isomorphism; therefore $H$ is a local diffeomorphism from a neighborhood of $p_0$ onto a neighborhood of $H(p_0)$. This fact, when $H$ is the boundary value of a holomorphic mapping from a pseudoconvex domain in $\mathbb{C}^2$ to another, follows from Derridj [4].

The proof of Theorem 1 uses recursive arguments with repeated applications of the following two observations.

First, suppose $A$ and $B$ are vector fields on $M$ for which there exist smooth functions $\alpha_j(u)$, $\beta_j(u)$ on $M$, $1 \leq j \leq r$, and smooth vector fields $A'_j$ and $B'_j$ on $M'$ such that

$$H'(A_u) = \sum_j \alpha_j(u) A'_{j,H(u)}, \qquad H'(B_u) = \sum_j \beta_j(u) B'_{j,H(u)},$$

then

$$H'([A,B]_u) = \sum_j (A\beta_j)(u) B'_{j,H(u)} - \sum_j (B\alpha_j)(u) A'_{j,H(u)} + \sum_{p,q} (\alpha_p \beta_q)(u) [A'_p, B'_q]_{H(u)}.$$

Second, if $E_1$ and $E_2$ are two commutators of $L$ and $\overline{L}$ of length $n_1$ and $n_2$ respectively then

$$[E_1, E_2] = aL + b\overline{L} + \sum_\alpha c_\alpha C^\alpha,$$

where each $C^\alpha$ is a commutator of $L$ and $\overline{L}$ of length $|\alpha| < n_1 + n_2$, and $a, b, c_\alpha$ are smooth functions on $M$.

One of the crucial steps in the proof of Theorem 1 consists of proving that if $S_1$ and $S_2$ are two strings of the form (3) with length $k-1$, having the same number of $L$'s then

$$S_1\lambda(p_0) = S_2\lambda(p_0).$$

We say that the mapping $H : M \to M'$ is *flat* at $p_0$ if all partial derivatives of $H$ of any order vanish at $p_0$.

The following result is a consequence of Theorem 1.

THEOREM 2. *Let $H : M \to M'$ be a CR mapping and $p_0 \in M$. If $M$ and $M'$ are of finite type at $p_0$ and $p_0' = H(p_0)$ respectively, then the following conditions are equivalent:*

(a) *$H$ is of finite multiplicity at $p_0$.*

(b) $H'_{p_0} \neq \{0\}$.

(c) *$H$ is not flat at $p_0$.*

Examples show that the conclusions of Theorems 1 and 2 are no longer valid if the finite type conditions are dropped in the assumptions.

Our main analyticity result is the following.

THEOREM 3. *Let $M$, $M'$ be two real analytic CR manifolds, $H$ a smooth CR mapping from $M$ into $M'$, and $p_0 \in M$. If $H$ is of finite multiplicity at $p_0$ and $M'$ of finite type at $p_0' = H(p_0)$, then $H$ is real analytic in a neighborhoof of $p_0$.*

The proof of Theorem 3 uses the general approach of [1]. Since $M$ and $M'$ are real analytic, they can be considered as embedded in $\mathbb{C}^2$, where the variables are denoted by $z$, $w$. We assume that $p_0 = H(p_0) = 0$, and that $M$ and $M'$ are respectively given locally by

$$\text{Im } w = \varphi(z, \bar{z}, \Re e\ w), \quad \text{Im } w = \psi(z, \bar{z}, \Re e\ w),$$

with $\varphi(z, 0, \Re e\ w) = \psi(z, 0, \Re e\ w) = 0$. The mapping $H$ is then locally given by a pair of CR functions $(f, g)$ defined on $M$ and satisfying

$$\frac{g - \bar{g}}{2i} = \psi(f, \bar{f}, \frac{g + \bar{g}}{2}). \tag{5}$$

As in [1] it suffices to show that $f$ and $g$ are real analytic with respect to $\Re e\ w$, uniformly in $z$.

We have here $\lambda = L\bar{f}$, where the function $\lambda$ is as in (2). By Theorem 1 we have $L^k\bar{f}(0) \neq 0$ and $L^j\bar{f}(0) = 0,\ 0 \leq j < k$.

An important part of the proof consists of repeatedly applying $L$ to (5). In addition to the arguments used in [1] (where $k = 1$), the following result in one complex variable is crucial.

LEMMA. *Let $a$ be a positive number and $R$ the domain in $\mathbb{C}$ defined by $|\xi| < a$, $0 < \eta < a$, with $\varsigma = \xi + i\eta$. Let $u$, $v$ be two functions defined in $\overline{R}$ and satisfying:*

(i) *$u, v \in C^\infty(\overline{R})$ and $u$, $v$ are holomorphic in $R$,*

(ii) $h(\xi) = \dfrac{u(\xi)}{v(\xi)} \in C^\infty([-a,a])$,

(iii) *there exist a positive integer p and, for* $0 \le j \le p-1$, *functions* $a_j \in C^\infty(\overline{R})$, *holomorphic in* $R$, $a_j(0) = 0$, *such that*

$$(h(\xi))^p + a_{p-1}(\xi)(h(\xi))^{p-1} + \cdots + a_0(\xi) = 0, \qquad \text{in } [-a,a].$$

*Then h extends holomorphically to R as* $\dfrac{u(\varsigma)}{v(\varsigma)}$, and $\dfrac{u}{v} \in C^\infty(R \cup (-a,a))$.

For real analytic CR mappings, we have the following result which gives a justification for the definition of finite multiplicity.

THEOREM 4. *Let* $M$, $M'$ *be two real analytic CR manifolds of finite type at* $p_0$ *and* $p_0'$ *respectively,* $M$ *connected, and* $H : M \to M'$ *a real analytic CR mapping with* $H(p_0) = p_0'$.

(i) *If* $H$ *is not of finite multiplicity at* $p_0$, *then* $H$ *is constant i.e.* $H(M) = p_0'$.

(ii) *If* $H$ *is of multiplicity* $k$ *at* $p_0$ $(1 \le k < \infty)$, *then* $p_0'$ *is an interior point of* $H(M)$. *More precisely, for every* $U$, *a sufficiently small neighborhood of* $p_0$ *in* $M$, *there exists* $V$, *an open neighborhood of* $p_0'$ *in* $M'$ *such that* $V \subset H(U)$. *In addition there is a finite number of real analytic curves* $\gamma_1, \dots, \gamma_r$ *contained in* $V$ *such that, for every* $p' \in V \setminus (\cup_{j=1}^r \gamma_j \cup \{p_0'\})$, *there exist exactly* $k$ *points* $p_1, \dots, p_k \in U$ *satisfying* $H(p_j) = p'$, $1 \le j \le k$, *with* $H$ *of multiplicity* 1 *at each* $p_j$.

Theorems 3 and 4 together with an argument involving the iteration of $H$, and the use of properties of real analytic sets, yield the following global result.

THEOREM 5. *Let* $M$ *be a real analytic compact CR manifold and* $H : M \to M$ *a smooth CR mapping. If* $M$ *is of finite type at each point and* $H$ *of finite multiplicity at each point of* $M$, *then* $H$ *is of multiplicity one at each point. Therefore* $H$ *is a local analytic diffeomorphism.*

Several of the previous results have applications to holomorphic extendability of proper maps in domains in $\mathbb{C}^2$. Indeed it is well known that the boundary value of a holomorphic mapping is a CR mapping, and, when the boundaries are real analytic, the question of holomorphic extendability of a CR mapping reduces to that of its real analyticity. We give here some of these applications.

THEOREM 6. *Let $D$ and $D'$ be two open bounded sets of $\mathbb{C}^2$ with real analytic boundaries. Let $F : D \to D'$ be a proper holomorphic mapping with $F \in C^\infty(\overline{D})$. If $F$ is nowhere flat on $\partial D$ then $F$ extends holomorphically to a neighborhood of $\overline{D}$. More precisely there exist $D_1$ and $D_1'$, two open bounded neighborhoods of $\overline{D}$ and $\overline{D'}$ respectively, with real analytic boundaries, such that $F$ extends as a holomorphic proper mapping from $D_1$ into $D_1'$.*

Using an argument due to Pinčuk [9] and Theorems 5 and 6 we obtain the following, which generalizes a result of Bedford–Bell [2].

THEOREM 7. *Let $D$ be an open bounded set in $\mathbb{C}^2$ with real analytic boundary and $F$ a proper holomorphic self mapping of $D$. If $F \in C^\infty(\overline{D})$ and $F$ is nowhere flat on $\partial D$ then $F$ extends as a biholomorphism from an open neighborhood of $\overline{D}$ onto another.*

In the pseudoconvex case, using a generalized form of an argument due to Fornaess [6] the nowhere flatness can be dropped in Theorem 6. Then making use of Theorem 6 and the result of Bell–Catlin [3], and Diederich–Fornaess [5] we obtain:

THEOREM 8. *Let $D$ and $D'$ be two bounded pseudoconvex domains in $\mathbb{C}^2$ with real analytic boundaries, if $F$ is a proper holomorphic map from $D$ into $D'$ then $F$ extends as a proper holomorphic mapping from a neighborhood of $\overline{D}$ to a neighborhood of $\overline{D}'$.*

Complete proofs will be published elsewhere.

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