Invent. math. 82, 359-396 (1985)

Inventiones mathematicae

CR structures with group action and extendability of CR functions*

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0. Introduction

This paper consists of three parts:

Part I presents results on local embedding of CR structures. We consider an abstract CR manifold whose structure is invariant under a transversal Lie group action. We show that such a manifold can always be locally embedded in complex space as a generic submanifold. The proof is based on selection of canonical coordinates and repeated use of the Newlander-Nirenberg theorem [13]. When the Lie group is abelian the embedding can be given a particularly simple form. Let $l \geq 1$ be the codimension of our submanifold (called M throughout the paper); it is then convenient to denote by $n+l$ the dimension of the ambient complex space and by $z_1, \ldots, z_n, w_1, \ldots, w_l$ the complex coordinates; we shall systematically write

$$z=(z_1, \ldots, z_n), \qquad w=(w_1, \ldots, w_l).$$

One can then arrange that an equation of the embedded submanifold M be given by an equation

$$\operatorname{Im} w = \phi(z, \bar{z}). \tag{0.1}$$

Our viewpoint will be strictly local, about a central point of M which we take to be the origin. Thus $\phi = 0$ at 0. It is also convenient to assume that the tangent space to M at 0 is the (real) vector subspace $\operatorname{Im} w = 0$, which means that $d\phi = 0$ at 0. We have chosen to call *rigid* any CR structure that admits an embedding of the kind (0.1). Let us underline the fact that the codimension l can be arbitrary.

Parts II & III are devoted to the study of local properties of CR functions or distributions on a rigid CR manifold M. Our first step is to define an adapted FBI (Fourier-Bros-Iagolnitzer) transform of such functions. Our definition is a

* During the completion of this work, M.S. Baouendi was Visiting Professor at Rutgers University and partly supported by NSF grant MCS 8401588. L.P. Rothschild was partly supported by NSF grant MCS 8319819. F. Treves was a visiting member at the Institute for Advanced Study and partly supported by NSF grant MCS 8102436

simplification of that in Baouendi-Chang-Treves [2] made possible by the fact that we are not dealing with general hypo-analytic structures, only with CR ones. Actually the definition extends to arbitrary embedded CR manifolds as will be shown in a forthcoming paper. Here we restrict its use to rigid ones for which it is particularly simple. In this respect the present paper is essentially self-contained. Thus we prove, in the present context, that the exponential decay of the FBI transform along certain conic subsets of the cotangent bundle is equivalent to the microlocal hypo-analyticity of the CR distribution – the microlocal hypo-analyticity being understood in the standard Sato sense.

By using the FBI transform we decompose any CR distribution as a finite sum of CR distributions, each of which extends holomorphically to what we call a *wedge*. If M is defined near 0 by Eq. (0.1), a wedge is a set of the kind

$$\mathscr{W}=\{(z,w)\in\mathcal{O};\ \operatorname{Im} w-\phi(z,\bar{z})\in\Gamma\}, \tag{0.2}$$

where $\mathcal{O}$ is an open neighborhood of the origin in $\mathbb{C}^{n+l}$ and Γ is an open cone in $\mathbb{R}^{n+l}$ with vertex at the origin. This can be viewed as an extension to rigid CR manifolds of a result of Andreotti and Hill [1] about general (not necessarily rigid) hypersurfaces (that any CR distribution is the sum of one that extends holomorphically to one side and one that extends to the opposite side). In codimension $l=2$ there have recently been results of Henkin [10] under certain special hypotheses.

Part III presents sufficient conditions for microlocal hypo-analyticity and local holomorphic extendability of CR distributions. In particular, if the *sector property* (Def. III.1; cf. Baouendi-Treves [5]) is valid at a characteristic point, then any CR distribution is hypo-analytic at that point (Th. III.1). An application of this fact is that, when the rigid CR structure has finite type, then every CR distribution extends holomorphically to a single wedge of the form (0.2) (Th. III.3). The latter result is a generalization to rigid structures of arbitrary codimension of a result of Baouendi-Treves [5] for hypersurfaces (not necessarily rigid). Finally Theorem III.4 shows how extendability results for a hypersurface in $\mathbb{C}^2$ can be used to yield microlocal hypo-analyticity of CR distributions on rigid CR manifolds of any codimension.

I. Integrability of abstract CR structures. Rigid and tube CR structures

In this section we prove that if a smooth manifold is equipped with an abstract CR structure invariant under the action of a finite dimensional Lie group, then it is locally realizable as an embedded CR manifold of a complex space. Special attention will be given to the case where the Lie group is abelian.

We introduce first some notation. Since our results are local, we may assume that the given manifold is an open set Ω of $\mathbb{R}^N$. Let $\mathbb{C}T\Omega$ be the complexified tangent bundle to Ω, and $\mathscr{V}$ be a subbundle of $\mathbb{C}T\Omega$. For $\omega\in\Omega$, we denote by $\mathscr{V}_\omega$ the fiber at ω and assume that

$$\dim_{\mathbb{C}}\mathscr{V}_\omega=n,\quad \forall\omega\in\Omega. \tag{I.1}$$

Let

$$\mathbb{L} = C^\infty(\Omega, \mathscr{V})$$

be the space of smooth sections of $\mathscr{V}$ defined in Ω. We assume that $\mathscr{V}$ satisfies the formal Frobenius integrability condition

$$[\mathscr{V}, \mathscr{V}] \subset \mathscr{V}. \tag{I.2}$$

This means that for every L, $M \in \mathbb{L}$, the vector field $[L, M] = LM - ML$ is also in $\mathbb{L}$.

It is customary to say that Ω is equipped with an *abstract* CR *structure* if, in addition to (I.1) and (I.2) we also have

$$\mathscr{V}_\omega \cap \bar{\mathscr{V}}_\omega = \{0\}, \quad \forall \omega \in \Omega. \tag{I.3}$$

We say that $\mathscr{V}$ (or $\mathbb{L}$) is *locally integrable* if for every $\omega_0 \in \Omega$, there exist an open neighborhood Ω', $\omega_0 \in \Omega' \subset \Omega$, and $Z_1, \ldots, Z_{n+l} \in C^\infty(\Omega')$ ($\dim_{\mathbb{R}} \Omega = N = 2n + l$) satisfying in Ω',

$$LZ_j = 0, \quad \forall j = 1, \ldots, n+l, \quad \forall L \in \mathbb{L}, \tag{I.4}$$

and

$$dZ_j(\omega_0), \quad j = 1, \ldots, n+l \textit{ are } \mathbb{C} \textit{ linearly independent.} \tag{I.5}$$

A system of functions satisfying (I.4) and (I.5) will be called *first integrals* of $\mathbb{L}$ or $\mathscr{V}$. Possibly after shrinking Ω', the map $Z\colon \Omega' \to \mathbb{C}^{n+l}$ defined by $Z = (Z_1, \ldots, Z_{n+l})$ is a diffeomorphism from Ω' onto $Z(\Omega')$.

We shall introduce the notion of an abstract CR structure with a transversal Lie group action. A vector subspace $\mathfrak{g}$ of $C^\infty(\Omega, T\Omega)$ is said to be a *finite dimensional Lie subalgebra*, if

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g},$$

and

$$\dim_{\mathbb{R}} \mathfrak{g}_\omega = \dim_{\mathbb{R}} \mathfrak{g}, \quad \forall \omega \in \Omega.$$

Definition I.1. If $\Omega \subset \mathbb{R}^{2n+l}$ has an abstract CR structure $\mathscr{V}$ (i.e. (I.1), (I.2) and (I.3) hold), we shall say that $\mathscr{V}$ is *invariant under a transversal Lie group action* if there is a finite dimensional Lie subalgebra

$$\mathfrak{g} \subset C^\infty(\Omega, T\Omega), \tag{I.6}$$

such that

$$\mathscr{V}_\omega \oplus \bar{\mathscr{V}}_\omega \oplus (\mathfrak{g}_\omega \otimes \mathbb{C}) = \mathbb{C}T_\omega \Omega, \quad \forall \omega \in \Omega, \tag{I.7}$$

and

$$[\mathbb{L}, \mathfrak{g}] \subset \mathbb{L}. \tag{I.8}$$

Note that it follows from (I.1), (I.3), and (I.7) that

$$\dim_{\mathbb{R}} \mathfrak{g} = l. \tag{I.9}$$

Definition II.2. An abstract CR structure is called *rigid* if it is invariant under an abelian group action, i.e. if the Lie algebra $\mathfrak{g}$ of Def. I.1 satisfies

$$[\mathfrak{g}, \mathfrak{g}] = 0.$$

We have the following local integrability results.

Theorem II.1. *Any abstract* CR *structure in Ω invariant under a transversal Lie group action is locally integrable.*

In addition, if the CR *structure is rigid* (Def. I.2), *then around every $\omega_0 \in \Omega$, there exist coordinates x, y, s defined in a neighborhood Ω' of ω_0, vanishing at ω_0, and first integrals $Z_1, \ldots, Z_{n+l}$ defined in Ω' of the form*

$$\begin{aligned} Z_j &= x_j + i y_j, \quad j = 1, \ldots, n, \\ Z_k &= s_{k-n} + i\phi_{k-n}(x, y), \quad k = n+1, \ldots, n+l, \end{aligned} \tag{I.10}$$

where $\phi_j \in C^\infty(\mathbb{R}^{2n})$, $j = 1, \ldots, l$, is real valued.

It should be noted that rigid hypersurfaces (i.e. $l = 1$) were considered in Tanaka [15].

The following lemma is standard; it will be needed in the proof of Theorem I.1.

Lemma I.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra satisfying* (I.6) *and* (I.7) *and $T_1, \ldots, T_l$ a basis of (the real vector space) $\mathfrak{g}$. Then for every $\omega_0 \in \Omega$, one may find local coordinates (s, u) around ω_0 so that, in these coordinates,*

$$T_j = \sum_{p=1}^{l} c_j^p(s) \frac{\partial}{\partial s_p}, \quad j = 1, \ldots, l,$$

with $c_j^p(s)$ analytic. If $\mathfrak{g}$ is abelian, then the c_j^p are constants.

Proof. Let $U_1, U_2, \ldots, U_{2n}$ be a local real basis for (the $C^\infty(\Omega)$-module) $\mathbb{L} \oplus \bar{\mathbb{L}}$ around ω_0. Then define local coordinates around ω_0 by

$$(s, u) \leftrightarrow \exp\left(\sum_{j=1}^{l} s_j T_j\right) \exp\left(\sum_{k=1}^{2n} u_k U_k\right) \cdot \omega_0 . \tag{I.11}$$

If $f = f(s, u)$ is a smooth function defined near ω_0, then in this coordinate system one has for any $T \in \mathfrak{g}$,

$$Tf(s, u) = \frac{d}{dt} f\left[(\exp tT) \exp\left(\sum_{j=1}^{l} s_j T_j\right) \exp\left(\sum_{k=1}^{2n} u_k U_k\right) \omega_0\right]_{t=0} .$$

Since $\mathfrak{g}$ is a Lie algebra, for any $T \in \mathfrak{g}$,

$$\exp tT \exp\left(\sum_{j=1}^{l} s_j T_j\right) = \exp\left(\sum_{j=1}^{l} e_j(t, s) T_j\right),$$

by the Baker-Campbell-Hausdorff formula, where the functions $e_j(t,s)$ are real analytic. Now the lemma follows by differentiation. In addition, if $\mathfrak{g}$ is abelian then

$$T_j = \frac{\partial}{\partial s_j}, \quad 1 \leqq j \leqq l. \quad \square$$

Proof of Theorem I.1. Let $\omega_0 \in \Omega$ and $T_1, \ldots, T_l$ a basis of $\mathfrak{g}$. It follows from Lemma I.1 and its proof, that using the coordinates (s, u) defined by (I.11) around ω_0, every $L \in \mathbb{L}$ can be uniquely written

$$L = M + T, \tag{I.12}$$

with

$$M = \sum_{p=1}^{2n} a_p(s,u) \frac{\partial}{\partial u_p}, \qquad T = \sum_{j=1}^{l} b_j(s,u) T_j,$$

where a_p and b_j are smooth functions defined in a neighborhood Ω' of ω_0.

Denote by $\mathbb{T}$ the $C^\infty(\Omega')$-module spanned by $T_1, \ldots, T_l$. Let $\mathbb{M}$ be the set of all vector fields M in the decomposition (I.12), when L varies in $\mathbb{L}$. It is clear that

$$\mathbb{M} \subset C^\infty(\Omega', \mathbb{C}T\Omega'),$$

is a $C^\infty(\Omega')$-module. We claim that $\mathbb{M}$ satisfies the Frobenius condition

$$[\mathbb{M}, \mathbb{M}] \subset \mathbb{M}. \tag{I.13}$$

Indeed it suffices to take $L, L' \in \mathbb{L}$ of the form (I.12), $L = M + T$, $L' = M' + T'$, and to show that

$$[M, M'] \in \mathbb{M}. \tag{I.14}$$

For this note that

$$[L, L'] = [M, M'] + [T, M'] + [M, T'] + [T, T'].$$

Hence

$$[M, M'] = [L, L'] - [T, M'] - [M, T'] + X, \tag{I.15}$$

where $X \in \mathbb{T}$. We have

$$[T, M'] = \sum_{j=1}^{l} [b_j T_j, M'] = \sum b_j [T_j, M'] - (M' b_j) T_j. \tag{I.16}$$

By assumption (I.8) above,

$$\sum b_j [T_j, M'] = \sum b_j [T_j, L'] - \sum b_j [T_j, T'] = L'' + X' \tag{I.17}$$

with $L'' \in \mathbb{L}$ and $X' \in \mathbb{T}$.

Since $[M, T']$ has the same form as $[T, M']$ we obtain from (I.15), (I.16), and (I.17) that

$$[M, M'] = [L, L'] + L''' + X''$$

with $L'''\in\mathbb{L}$, $X''\in\mathbb{T}$. Since $[L,L']\in\mathbb{L}$, we conclude that

$$[M,M']-X''\in\mathbb{L}$$

which proves our claim (I.14).

Note that the definition of $\mathbb{M}$ and (I.7) imply

$$\mathbb{M}\cap\bar{\mathbb{M}}=\{0\}. \tag{I.18}$$

Denote by $\mathbb{M}^0$ the set of vector fields obtained by setting $s=0$ in the coefficients of the vector fields in $\mathbb{M}$. Since those vector fields do not differentiate with respect to s (only with respect to u), (I.13) and (I.18) imply

$$[\mathbb{M}^0,\mathbb{M}^0]\subset\mathbb{M}^0, \quad \mathbb{M}^0\cap\bar{\mathbb{M}}^0=\{0\}. \tag{I.19}$$

Making use of (II.19) and the fact that

$$\dim_{\mathbb{C}}\mathbb{M}^0_{\omega_0}=2n,$$

we can apply the Newlander-Nirenberg theorem [13] in the u variables. We may find coordinates

$$z_j=x_j+iy_j=z_j(u_1,\dots,u_{2n}), \quad j=1,\dots,n,$$

smooth in u, and a basis of the $C^\infty(U)$-module $\mathbb{M}^0$ (U is an open neighborhood of the origin in $\mathbb{R}^{2n}$) of the form

$$M_j^0=\frac{\partial}{\partial\bar{z}_j}=\frac{1}{2}\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right), \quad j=1,\dots,n.$$

Therefore, possibly after shrinking Ω' about ω_0, we can find a basis of the $C^\infty(\Omega')$-module $\mathbb{M}$ of the form

$$M_j=\frac{\partial}{\partial\bar{z}_j}+\sum_{p=1}^{n}d_j^p(z,\bar{z},s)\frac{\partial}{\partial z_p}, \quad j=1,\dots,n \tag{I.20}$$

with

$$d_j^p(z,\bar{z},0)=0.$$

We regard $(T_1,\dots,T_l)$ as a basis for the left invariant vector fields on the Lie group G associated to g. Let $(S_1,\dots,S_l)$ be a basis for the right invariant vector fields. We may choose the S_j so that in the coordinates (I.11) we have

$$S_j=\sum_{p=1}^{l}\tilde{c}_j^p(s)\frac{\partial}{\partial s_p}, \quad 1\leqq j\leqq l \tag{I.21}$$

with $\tilde{c}_j^p(s)$ again real analytic.

Since the S_j are right invariant and the T_j are left invariant, we have

$$[S_j,T_k]=0 \quad \forall j,k=1,\dots,l. \tag{I.22}$$

If M_j is given by (I.20), we may find a basis of (the $C^\infty(\Omega')$-module) $\mathbb{L}$ of the form

$$L_j = M_j + \sum_{p=1}^{l} h_j^p(z, \bar{z}, s) S_p, \qquad 1 \leqq j \leqq n, \tag{I.23}$$

where the h_j^p are smooth functions in Ω'.

Since the coefficients of T_j depend only on s, using (I.20), (I.22), and (I.23), we conclude that, for $q = 1, \ldots, l$, $j = 1, \ldots, n$.

$$[T_q, L_j] = \sum_{p=1}^{n} T_q d_j^p(z, \bar{z}, s) \frac{\partial}{\partial z_p} + \sum_{k=1}^{l} T_q h_j^k(z, \bar{z}, s) S_k. \tag{I.24}$$

Since $[\mathbb{T}, \mathbb{L}] \subset \mathbb{L}$ (Condition (I.8)), the right hand side of (I.24) must vanish i.e.

$$T_q d_j^p \equiv 0, \qquad T_q h_j^k \equiv 0$$

which yields

$$d_j^p(z, \bar{z}, s) = d_j^p(z, \bar{z}, 0) = 0,$$
$$h_j^k(z, \bar{z}, s) = h_j^k(z, \bar{z}, 0).$$

Now putting $h_j^k(z, \bar{z}, 0) = a_j^k(z, \bar{z})$ we get, for $j = 1, \ldots, n$,

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{p=1}^{l} a_j^p(z, \bar{z}) S_p, \tag{I.25}$$

and by (I.2),

$$[L_j, L_k] = 0, \qquad j = k = 1, \ldots, n. \tag{I.26}$$

Let us first assume $\mathfrak{g}$ is abelian. In that case $S_p = T_p = \dfrac{\partial}{\partial s_p}$, so that

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{p=1}^{l} a_j^p(z, \bar{z}) \frac{\partial}{\partial s_p}, \tag{I.27}$$

and (I.26) reads,

$$\frac{\partial}{\partial \bar{z}_j} a_k^p(z, \bar{z}) = \frac{\partial}{\partial \bar{z}_k} a_j^p(z, \bar{z}),$$

for $j, k = 1, \ldots, n$, $p = 1, \ldots, l$.

Hence we may find smooth functions $\phi_k(z, \bar{z})$, $1 \leqq k \leqq l$, so that

$$Z_{n+k} = s_k + i\phi_k(z, \bar{z}),$$

is a solution of $L_j Z_{n+k} = 0$, $1 \leqq j \leqq n$, $1 \leqq k \leqq l$. After making the change of coordinates

$$s_k' = s_k - \operatorname{Im} \phi_k(x, y), \qquad 1 \leqq k \leqq n,$$

we can assume the ϕ_k are real valued, which proves (I.4).

To complete the proof of Theorem I.1, we return to (I.25), (I.26) and drop the assumption that $\mathfrak{g}$ is abelian. Using (I.21) we get

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{s=1}^{l} e_j^p(z,\bar{z},s)\frac{\partial}{\partial s_p}, \quad 1 \leqq j \leqq n, \tag{I.28}$$

with $e_j^p(z,\bar{z},s)$ smooth and real analytic in s.

The proof of Theorem II.1 is completed by the following, which is a corollary of the Newlander-Nirenberg theorem.

Proposition I.1. *If* (I.28) *and* (I.26), *hold in an open neighborhood of the origin* $\Omega \subset \mathbb{R}^{2n+l}$, *then the* $C^\infty(\Omega)$*-module* $\mathbb{L}$ *spanned by* $L_1, \ldots, L_n$ *is locally integrable.*

Proof. We first complexify s and write

$$s = u + iv, \quad u, v \in \mathbb{R}^l,$$

$$\frac{\partial}{\partial s_k} = \frac{1}{2}\left(\frac{\partial}{\partial u_k} - i\frac{\partial}{\partial v_k}\right), \quad \frac{\partial}{\partial \bar{s}_k} = \frac{1}{2}\left(\frac{\partial}{\partial u_k} + i\frac{\partial}{\partial v_k}\right).$$

Since the coefficients in (I.28) can be holomorphically extended for s complex, we may think of them as defined in an open set $\tilde{\Omega} \subset \mathbb{R}^{2(n+l)}$. In this open set we consider the vector fields $L'_1, \ldots, L'_{n+l}$ defined by $L'_j = L_j$ $\left(\text{where } \frac{\partial}{\partial s_p} \text{ is the complex differentiation}\right)$, $1 \leqq j \leqq n$, $L'_{n+k} = \frac{\partial}{\partial \bar{s}_k}$, $1 \leqq k \leqq l$.

It follows from (I.26) and the holomorphy of $e_j^p(z,\bar{z},s)$ with respect to s that

$$[L'_j, L'_k] = 0, \quad j,k = 1, \ldots, n+l.$$

On the other hand, it is clear that $L_1, \ldots, L_{n+l}, \bar{L}_1, \ldots, \bar{L}_{n+l}$ are linearly independent in $\tilde{\Omega}$. Therefore we can use again the Newlander-Nirenberg theorem and find smooth functions with independent differentials, $Z_j(z,\bar{z},s,\bar{s})$, $j = 1, \ldots, n+l$, solutions of

$$L'_j Z_k = 0, \quad j = 1, \ldots, n+1, k = 1, \ldots, n+l,$$

but this implies that the Z_k are holomorphic in s. Restricting the Z_k to s real, gives the desired set of first integrals of $\mathbb{L}$ around ω_0. □

Let us introduce the following definition.

Definition I.3. Let $\mathscr{V}$ be a rigid CR structure defined in Ω (Def. I.2). A basis $L_1, \ldots, L_n$ of (the $C^\infty(\Omega)$-module) $\mathbb{L}$ is called *canonical* if

$$[L_j, L_k] = 0, \quad \forall j,k = 1, \ldots, n, \tag{I.29a}$$

$$[L_j, \mathfrak{g}] = 0, \quad j = 1, \ldots, n, \tag{I.29b}$$

$$[L_j, \bar{L}_k]_\omega \in \mathfrak{g}_\omega \otimes \mathbb{C}, \quad \forall \omega \in \Omega, j,k = 1, \ldots, n. \tag{I.29c}$$

We have the following result:

Proposition I.2. *Assume that* $\mathscr{V}$ *is a rigid* CR *structure defined in* Ω *and let* $\omega_0 \in \Omega$. *Possibly after contracting* Ω *about* ω_0, *in the coordinates of Theorem* II.1, *the vector* *fields,*

$$L_j=\frac{\partial}{\partial\bar z_j}-i\sum_{k=1}^{l}\phi_{k,\bar z_j}(z,\bar z)\frac{\partial}{\partial s_k},\quad j=1,\dots,n,\tag{I.30}$$

where the ϕ_k are given by (I.10), *form a canonical basis of* $\mathbb{L}$.

In addition, if $\tilde L_1,\dots,\tilde L_n$ is another canonical basis defined in an open neighborhood of ω_0, then there exists a holomorphism H defined in a neighborhood of 0 in $\mathbb{C}^n$ such that if

$$z=H(\tilde z)$$

then

$$\tilde L_j=\frac{\partial}{\partial\bar{\tilde z}_j}-i\sum_{k=1}^{l}\tilde\phi_{k,\bar{\tilde z}_j}(\tilde z,\bar{\tilde z})\frac{\partial}{\partial s_k}\tag{I.31}$$

with

$$\tilde\phi_k(\tilde z,\bar{\tilde z})=\phi_k(H(\tilde z),\overline{H(\tilde z)}).$$

Proof. In the course of proving Theorem I.1 we have also shown the first part of the proposition. It remains only to show the second part.

Let $L_1,\dots,L_n$ be the canonical basis defined by (I.30) and $\tilde L_1,\dots,\tilde L_n$ another canonical basis defined near ω_0. We have

$$\tilde L_j=\sum_{k=1}^{n}a_{j,k}(z,\bar z,s)L_k,\quad j=1,\dots,n\tag{I.32}$$

where the matrix $(a_{j,k})$ is invertible

Since $\frac{\partial}{\partial s_1},\dots,\frac{\partial}{\partial s_l}$ form a basis of (the real vector space) $\mathbb{T}$ and since

$$[L_j,\mathfrak{g}]=0,\quad[\tilde L_j,\mathfrak{g}]=0,\quad j=1,\dots,n,$$

we conclude from (I.32) that the $a_{j,k}$ are independent of s.

On the other hand, using (I.32) we have for each (i,j),

$$\begin{aligned}[\tilde L_i,\bar{\tilde L}_j]=\sum_{k,p}a_{i,k}(L_k\bar a_{j,p})\bar L_p-\sum_{k,p}\bar a_{j,k}(\bar L_k a_{i,p})L_p\\+\sum_{k,p}a_{i,k}\bar a_{j,k}[L_k,\bar L_p].\end{aligned}\tag{I.33}$$

Using (I.29c) for the L_j and the $\tilde L_j$ we conclude from (I.33) that

$$\sum_{k=1}^{n}a_{i,k}(z,\bar z)(L_k\overline{a_{j,p}(z,\bar z)})=0,\tag{I.34}$$

for $i,j,p\in\{1,\dots,n\}$.

Since the matrix $(a_{i,k})$ is invertible we conclude from (I.34) that

$$L_k\overline{a_{j,p}(z,\bar z)}=0\quad\forall k,j,p\in\{1,\dots,n\}.\tag{I.35}$$

Thus, for each $j, p \in \{1, \ldots, n\}$, the function $a_{j,p}$ is anti-holomorphic (i.e. independent of z). For $j = 1, \ldots, n$, define

$$M_j = \sum_{k=1}^{n} \overline{a_{j,k}(\bar{z})} \frac{\partial}{\partial z_k}.$$

Since $[L_j, L_k] = [\tilde{L}_j, \tilde{L}_k] = 0$, it follows that

$$[M_j, M_k] = 0, \quad j, k = 1, \ldots, n.$$

Since the coefficients of the M_j are holomorphic, we may apply the classical (complex) Frobenius theorem to find coordinates $\tilde{z}$, and a holomorphism $z = H(\tilde{z})$, such that

$$\frac{\partial}{\partial \tilde{z}_j} = M_j, \quad j = 1, \ldots, n. \tag{I.36}$$

(I.36) easily implies (I.31), which is the desired conclusion of Proposition I.2. □

For our final result of this section we shall loosen the constraint in (I.7),

$$(\mathscr{V}_\omega \oplus \bar{\mathscr{V}}_\omega) \cap (\mathfrak{g}_\omega \otimes \mathbb{C}) = \{0\},$$

in the case where $\mathfrak{g}$ is abelian, allowing us to obtain an invariant description of "tube" CR structures.

We have the following result

Theorem I.2. *Let $\mathscr{V}$ be a* CR *structure defined in $\Omega \subset \mathbb{R}^{2n+l}$. Assume* (I.1) *and suppose there is an abelian algebra $\mathfrak{g}$ satisfying* (I.6), (I.8) *and the following conditions*

$$\dim_{\mathbb{R}} \mathfrak{g} = l + k, \quad 0 \leqq k \leqq n. \tag{I.37}$$

$$(\mathscr{V}_\omega \oplus \bar{\mathscr{V}}_\omega) + (\mathfrak{g}_\omega \otimes \mathbb{C}) = \mathbb{C}T_\omega \Omega, \quad \forall \omega \in \Omega, \tag{I.38}$$

$$\mathscr{V}_\omega \cap (\mathfrak{g}_\omega \otimes \mathbb{C}) = \{0\}, \quad \forall \omega \in \Omega. \tag{I.39}$$

Then around every $\omega_0 \in \Omega$, there exist coordinates x, y, s vanishing at ω_0, and first integrals $Z_1, \ldots, Z_{n+l}$ of the form

$$\begin{aligned} Z_j &= x_j + i y_j, & 1 \leqq j \leqq n \\ Z_{n+p} &= s_p + i\phi_p(x, y), & 1 \leqq p \leqq l, \end{aligned} \tag{I.40}$$

where the real valued smooth functions ϕ_p are independent of $x_1, x_2, \ldots, x_k$.

Note that if $k = 0$ in (I.37) then the last summand in (I.38) is necessarily direct, and (I.39) is automatically satisfied. We have a rigid CR structure and Theorem I.2 is then a repetition of the second part of Theorem I.1.

If $k = n$, then the functions ϕ_p in (I.40) depend only on $y_1, \ldots, y_n$ (and are independent of $x_1, \ldots, x_n$). It is customary to say in this case that we have a *tube* CR structure.

Proof of Theorem I.2. Assume $0 < k \leqq n$ in (I.37). Under assumptions (I.1), (I.37), and (I.38), and by elementary linear algebra, we can decompose

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1,$$

where $\mathfrak{g}^0$ and $\mathfrak{g}^1$ are two abelian Lie algebras satisfying

$$\dim_{\mathbb{R}}\mathfrak{g}^0 = l, \quad \dim_{\mathbb{R}}\mathfrak{g}^1 = k,$$

and so that

$$\mathscr{V}_\omega \oplus \bar{\mathscr{V}}_\omega \oplus (\mathfrak{g}^0_\omega \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{C}T_\omega \Omega, \quad \forall \omega \in \Omega.$$

We can now apply part 2 of Theorem I.1, with $\mathfrak{g}$ replaced by $\mathfrak{g}^0$. Then we may choose coordinates (x, y, s) and first integrals satisfying (I.10). We can choose a basis of $\mathbb{L}$, $L_1, \ldots, L_n$, of the form (I.30), and a basis of $\mathfrak{g}^0$ of the form $S_j = \frac{\partial}{\partial s_j}$, $1 \leqq j \leqq l$. With these coordinates we may write a basis $T_1, \ldots, T_k$ of $\mathfrak{g}^1$ as

$$T_j = \mathscr{Re}\left(\sum_{k=1}^{n} a_j^k(z, \bar{z}, s)\frac{\partial}{\partial z_k}\right) + \sum_{p=1}^{l} b_j^p(z, \bar{z}, s)\frac{\partial}{\partial s_p}, \tag{I.41}$$

where the coefficients are smooth and the b_j^p are real. Since $\mathfrak{g}$ is abelian, writing $[S_k, T_j] = 0$, implies that the functions a_j^k and b_j^p are independent of s. We claim that the a_j^k are holomorphic in z i.e.

$$\frac{\partial}{\partial \bar{z}_p} a_j^k = 0, \quad \text{for all} \quad j, k, p. \tag{I.42}$$

Indeed, since $[\mathfrak{g}^1, \mathbb{L}] \subset \mathbb{L}$ (condition (I.8)), (I.42) follows immediately from writing that $[T_j, L_p]$ is a linear combination of $L_1, \ldots, L_n$. We write $a_j^k(z)$ instead of $a_j^k(z, \bar{z}, s)$.

Now let

$$M_j = \sum_{k=1}^{n} a_j^k(z)\frac{\partial}{\partial z_k}, \quad 1 \leqq j \leqq k,$$

so that rewriting (I.41), we have

$$T_j = \frac{1}{2}(M_j + \bar{M}_j) + \sum_{p=1}^{l} b_j^p(z, \bar{z})\frac{\partial}{\partial s_p}, \quad 1 \leqq j \leqq n. \tag{I.43}$$

The commutation relations $[T_j, T_k] = 0$ imply

$$[M_j, M_k] = 0, \quad 1 \leqq j, k \leqq n. \tag{I.44}$$

We claim that $M_1, \ldots, M_n$ are $\mathbb{C}$-linearly independent in an open neighborhood of ω_0. Indeed reasoning by contradiction, assume that, in each such neighborhood, we can find ω such that

$$\sum_{j=1}^{n} \lambda_j M_j|_\omega = 0, \quad \lambda_j \in \mathbb{C}, \sum_j |\lambda_j| \neq 0. \tag{I.45}$$

Since the $T_{j,\omega}$ and the $S_{p,\omega}$ form a basis of $\mathfrak{g}_\omega$, we conclude from (I.43) that

$$\sum \lambda_j \bar{M}_j|_\omega \in \mathfrak{g}_\omega \otimes \mathbb{C} \setminus \{0\}. \tag{I.46}$$

It is clear that, by using the basis (I.30) of $\mathbb{L}$, we can find a linear combination

$$N=\sum_{p=1}^{l} c_p(z,\bar{z})\frac{\partial}{\partial s_p},$$

such that

$$\sum_j \lambda_j \bar{M}_j + N \in \mathbb{L}.$$

In particular

$$\sum_j \lambda_j \bar{M}_j|_\omega + N_\omega \in \mathscr{V}_\omega.$$

On the other hand, it follows from (I.46) that the latter is a nonzero vector of $\mathfrak{g}_\omega \otimes \mathbb{C}$. This contradicts (I.39) and proves the linear independence of the M_j.

The preceding result and (I.44) allow us to use the complex Frobenius theorem. There is a holomorphic change of variables $z'=H(z)$ so that

$$M_j=\frac{\partial}{\partial z_j'}, \quad j=1,\dots,k.$$

Then (with different coefficients) we have

$$T_j=\frac{\partial}{\partial x_j'}+\sum_{p=1}^{l} b_j^p(z',\bar{z}')\frac{\partial}{\partial s_p}, \quad 1\leqq j\leqq k. \tag{I.47}$$

By choosing a new basis L_j' of $\mathbb{L}$ if necessary, we may still assume that

$$L_j'=\frac{\partial}{\partial \bar{z}_j'}-i\sum_{k=1}^{l} \phi_{k,\bar{z}_j}'(z',\bar{z}')\frac{\partial}{\partial s_k}, \quad 1\leqq j\leqq n, \tag{I.48}$$

with ϕ_k' real.

Now, for simplicity of notation we drop the primes. Since $[\mathfrak{g},\mathbb{L}]\subset\mathbb{L}$, making use of (I.47) and (I.48) yields

$$[L_j,T_q]=\sum_{p=1}^{l}(b^p_{q,\bar{z}_j}+i\phi_{p,\bar{z}_j,x_q})\frac{\partial}{\partial s_p}=0.$$

Hence

$$b^p_{q,\bar{z}_j}+i\phi_{p,x_q,\bar{z}_j}=0,$$

for $p=1,\dots,l$, $q=1,\dots,k$, $j=1,\dots,n$, which implies that the functions $b_q^p+i\phi_{p,x_q}$ are holomorphic.

We easily conclude from this, that there are holomorphic functions $F_p(z_1,\dots,z_n)$, $1\leqq p\leqq l$, and real functions $\phi_p^0(x_{k+1},\dots,x_n,y_1,\dots,y_n)$ (independent of $x_1,\dots,x_k$) such that

$$\phi_p(x,y)=\operatorname{Im}F_p(z_1,\dots,z_n)+\phi_p^0(x_{k+1},\dots,x_n,y_1,\dots,y_n).$$

Now putting

$$\tilde{Z}_j = Z_j, \quad 1 \leqq j \leqq n,$$

$$\tilde{Z}_{n+p} = Z_{n+p} - F_p(Z_1, \dots, Z_n), \quad 1 \leqq p \leqq l,$$

and

$$\tilde{s}_p = s_p - \mathscr{R}e\, F_p(z_1, \dots, z_n), \quad 1 \leqq p \leqq l,$$

yields the desired result (I.40) with ϕ_p replaced by ϕ_p^0 and s_p by $\tilde{s}_p$. □

II. Microlocal hypo-analyticity and decomposition of CR distributions

In this section we restrict ourselves to rigid CR structures. Let U be an open neighborhood of 0 in $\mathbb{R}^{2n}$ and V an open neighborhood of 0 in $\mathbb{R}^l$. Let $(x, y) \in U \mapsto \phi(x, y) \in \mathbb{R}^l$ be a smooth real vector valued function defined in U, (we also write $\phi(z, \bar{z})$, $z = x + iy$). We shall assume

$$\phi(0) = 0, \quad \phi'(0) = 0. \tag{II.1}$$

We denote by (z, w) the complex variable in $\mathbb{C}^{n+l}$ $(z = x + iy,\ w = s + it)$, and by M the submanifold of $\mathbb{C}^{n+l}$ defined by

$$\operatorname{Im} w = \phi(x, y), \quad (x, y) \in U, \quad \mathscr{R}e\, w \in V. \tag{II.2}$$

The map $Z: (x, y, s) \mapsto (z, s + i\phi(z, \bar{z}))$ is a diffeomorphism from $\Omega = U \times V$ onto M. Note that M is a generic CR manifold of (real) codimension l.

A distribution $\tilde{h}$ defined on the manifold M is called a CR distribution if it is annihilated by the induced Cauchy operator on M. This is equivalent to say that its pullback to Ω, $h = \tilde{h} \circ Z$, satisfies the equations

$$L_j h = 0, \quad 1 \leqq j \leqq n, \tag{II.3}$$

with

$$L_j = \frac{\partial}{\partial \bar{z}_j} - i \sum_{k=1}^{l} \frac{\partial \phi_k}{\partial \bar{z}_j}(z, \bar{z}) \frac{\partial}{\partial s_k}. \tag{II.4}$$

We shall often identify $\tilde{h}$ and its pullback h. Since we are interested here only in local results, we shall deal with germs of CR distributions at $\omega_0 \in \Omega$ (or $Z(\omega_0) \in M$). The central point ω_0 will be often taken to be the origin.

Definition II.1. Let h be a CR distribution defined in an open neighborhood of $\omega_0 \in \Omega$. We say that h *extends holomorphically at* ω_0 (or at $Z(\omega_0)$) if there exists a holomorphic function $H(z, w)$ defined in an open neighborhood of $Z(\omega_0)$ in $\mathbb{C}^{n+l}$ such that

$$h(x, y, s) = H(z, s + i\phi(z, \bar{z}))$$

in some neighborhood of ω_0 in Ω.

Very often it suffices to consider only CR functions (say C^1) instead of distributions. Indeed we have the following proposition which is a variant of a result in [3].

Proposition II.1. *Let h be a* CR *distribution defined in Ω. There exist $r \in \mathbb{Z}_+$ and a* CR *function $f \in C^1(\Omega')$, $0 \in \Omega' \subset \Omega$, such that*

$$h = \Delta_s^r f \quad \text{in} \quad \Omega',$$

with

$$\Delta_s = \sum_{j=1}^{l} \frac{\partial^2}{\partial s_j^2}.$$

Proof. First observe that the l-dimensional manifolds in Ω, $\{x = x_0, y = y_0\}$, are non-characteristic with respect to the vector fields L_j, $1 \leqq j \leqq n$, defined by (II.4). Trading off regularity with respect to s with regularity with respect to (x, y), it is standard to see that (after shrinking Ω if needed) we have

$$h \in C^\infty(U, \mathcal{D}'(V)).$$

After further shrinking of Ω it is then also standard to write

$$h = \Delta_s^r u, \tag{II.5}$$

with $u \in C^\infty(U, C^1(V))$.

Since the vector fields L_j's commute with Δ_s, we get from (II.3) and (II.5),

$$\Delta_s^r(L_j u) = 0. \tag{II.6}$$

Set

$$g_j(x, y, s) = (L_j u)(x, y, s).$$

It follows from the analytic hypoellipticity of the Laplacian and Eq. (II.6) that the functions g_j, $1 \leqq j \leqq n$, are analytic with respect to s (uniformly with respect to (x, y)).

For $1 \leqq j \leqq n$, define

$$H_j(x, y, w) = g_j(x, y, w - i\phi(x, y)).$$

It is easy to see that

$$H_j \in C^1(U', \mathcal{H}(\tilde{V}))$$

where $U' \subset U$ is an open neighborhood of 0 in $\mathbb{R}^{2n}$, $\tilde{V}$ an open neighborhood of 0 in $\mathbb{C}^l$, and $\mathcal{H}(\tilde{V})$ is the space of holomorphic functions in $\tilde{V}$.

Note that we have

$$\begin{aligned} g_j(x, y, s) &= H_j(x, y, w)|_{w = s + i\phi(x,y)}, \\ L_k g_j(x, y, s) &= \left(\frac{\partial}{\partial \bar{z}_k} H_j(x, y, w) \right) \Bigg|_{w = s + i\phi(x,y)}. \end{aligned} \tag{II.7}$$

Since $L_j g_k = L_k g_j$, $1 \leqq j, k \leqq n$, we conclude that we have

$$\frac{\partial H_j}{\partial \bar{z}_k} = \frac{\partial H_k}{\partial \bar{z}_j}. \tag{II.8}$$

Also (II.6) and (II.7) imply

$$\Delta^r_w H_j = 0, \quad 1 \leqq j \leqq r. \tag{II.9}$$

It is quite standard to see that, using (II.8) and (II.9), and after further shrinking of U' and $\tilde{V}$, we can find $H \in C^1(U', \mathscr{H}(\tilde{V}))$ satisfying

$$\frac{\partial H}{\partial \bar{z}_j} = H_j, \quad 1 \leqq j \leqq n, \quad \Delta^r_w H = 0.$$

The reader can now easily check that we reach the desired conclusion if we set

$$f(x, y, s) = u(x, y, s) - H(x, y, s + i\phi(x, y)). \quad \square$$

For CR distributions we shall need the notion of microlocal hypo-analyticity as introduced, in a more general set up, in Baouendi-Chang-Treves [2]. In fact we shall give here a simpler definition (valid for rigid CR structures). The reader can check that it coincides with the one in [2]. However this fact is not used in this paper.

For distributions in $C^0(U, \mathscr{E}'(V))$, we need to introduce the concept of microlocal analyticity at $(s_0, \sigma^0) \in T^*V\backslash 0$, uniformly with respect to (x, y) near 0 in U. Let $u \in C^0(U, \mathscr{E}'(V))$; following Bros-Iagolnitzer [8] (see Sjöstrand [14]) we introduce its FBI transform defined by

$$I(z, w, \sigma) = \int e^{i(w-\tilde{s})\sigma - |\sigma|(w-\tilde{s})^2} \Delta(w - \tilde{s}, \sigma) u(x, y, \tilde{s}) d\tilde{s}, \tag{II.10}$$

where $z = x + iy$, $w \in \mathbb{C}^l$, $\sigma \in \mathbb{R}^l$ and $\Delta(s, \sigma) = \left(1 + i \sum_{j=1}^{l} s_j \sigma_j |\sigma|^{-1}\right)$.

We shall say that u is *analytic at* $(s_0, \sigma^0) \in T^*V\backslash 0$ *uniformly in* (x, y) *near* 0 in $\mathbb{R}^{2n}$ if the following inequality holds with $C > 0$,

$$|I(z, w, \sigma)| \leqq Ce^{-|\sigma|/C}, \tag{II.11}$$

for (x, y) in an open neighborhood of 0 in U, w in a *complex* neighborhood of s_0 in $\mathbb{C}^l$ and σ in an open cone $\Gamma \subset R^l$, $\sigma^0 \in \Gamma$.

A similar definition can be given for $h \in C^0(U, \mathscr{D}'(V))$ by replacing u by $\chi(\tilde{s}) h(x, y, \tilde{s})$ in (II.10), with $\chi \in C_0^\infty(V)$, $\chi \equiv 1$ near s_0.

It is easy to check (see Sjöstrand [14] for example) that the previous definition is equivalent to the existence of open convex cones $\Gamma_j \subset \mathbb{R}^l$, $1 \leqq j \leqq r$, open neighborhoods of 0 and s_0 respectively, $U' \subset U$, $V' \subset V$, and functions

$$f_j \in C^0(U', \mathscr{H}((V' + i\Gamma_j) \cap \mathscr{O}))$$

($\mathscr{O}$ is an open neighborhood of s_0 in $\mathbb{C}^l$), with tempered growth uniformly in (x, y) (i.e. $|f_j(x, y, s + it)| \leqq Ct^{-N}$) satisfying

$$h = \sum_{j=1}^{r} bf_j \quad \text{near} \quad (x, y) = 0, s = s_0, \tag{II.12}$$

and

$$\Gamma_j \cdot \sigma^0 < 0,$$

where bf_j is the boundary value of f_j at $t=0$.

For CR distributions we shall need the following definition.

Definition II.2. Let h be a CR distribution defined in $\Omega = U \times V$. We say that h is *hypo-analytic* at $(s_0, \sigma^0) \in T^*V\backslash\{0\}$ if h is analytic at (s_0, σ^0) uniformly in (x, y) near 0 in $\mathbb{R}^{2n}$.

We first prove the following (local) equivalence.

Proposition II.2. *A* CR *distribution h defined in Ω extends holomorphically at* $(0, s_0)$, $s_0 \in V$, *(see* Def. II.1*) if and only if h is hypo-analytic at* (s_0, σ^0) *for every* $\sigma^0 \in \mathbb{R}^l\backslash\{0\}$.

Proof. It is clear that if h extends holomorphically at $(0, s_0)$ then it is analytic with respect to s, uniformly in (x, y). It is therefore hypo-analytic at (s_0, σ^0) for every $\sigma^0 \in \mathbb{R}^l\backslash\{0\}$.

Conversely assume that h is hypo-analytic at (s_0, σ^0) for every $\sigma^0 \in \mathbb{R}^l\backslash\{0\}$. This implies that h is analytic in s uniformly in (x, y). For (x, y) near 0 in $\mathbb{R}^{2n}$ and w near s_0 in $\mathbb{C}^l$ define

$$H(x, y, w) = h(x, y, w - i\phi(x, y)).$$

We have

$$H \in C^0(U', \mathscr{H}(\tilde{V}))$$

with $0 \in U' \subset U$, $s_0 \in \tilde{V} \subset \mathbb{C}^l$, and

$$h(x, y, s) = H(x, y, s + i\phi(x, y)).$$

Since h satisfies (II.3) we also have

$$\frac{\partial H}{\partial \bar{z}_j}(x, y, w)\bigg|_{w = s + i\phi(x, y)} = 0, \qquad 1 \leqq j \leqq n. \tag{II.13}$$

It is easy to see that the holomorphy of H with respect to w together with (II.13) imply

$$\frac{\partial H}{\partial \bar{z}_j}(x, y, w) \equiv 0.$$

This shows that H is the desired holomorphic extension of h. □

Next we will give a microlocal version of Proposition II.2 where local holomorphic extendability is replaced by extendability to certain open sets in $\mathbb{C}^{n+l}$, called wedges, whose boundaries contain the manifold M defined by (II.2).

More precisely, if Γ is an open convex cone of $\mathbb{R}^l$ and $\mathcal{O}$ an open neighborhood of the origin in $\mathbb{C}^{n+l}$, a *wedge* of $\mathbb{C}^{n+l}$ is an open set of the form

$$\mathscr{W} = \mathscr{W}(\mathcal{O}, \Gamma) = \{(z, w) \in \mathcal{O};\ \mathrm{Im}\, w - \phi(z, \bar{z}) \in \Gamma\}. \tag{II.14}$$

Clearly $M\cap\mathcal{O}$ is contained in the boundary of $\mathcal{W}$. We say that $M\cap\mathcal{O}$ is the *edge* of $\mathcal{W}(\mathcal{O},\Gamma)$.

If H is a holomorphic function in $\mathcal{W}$ we say that it has *slow growth* at M if there exist C and N such that

$$|H(z,w)|\leqq C[\operatorname{dist}((z,w),M)]^{-N},\qquad \forall(z,w)\in\mathcal{W}. \tag{II.15}$$

If H is such a function the reader can check that it has a boundary value on M, $h=b_M H$, which is a CR distribution on $M\cap\mathcal{O}$. (See similar proof in [2].) We say that the CR distribution h *extends holomorphically to the wedge* $\mathcal{W}$.

If h is a CR distribution defined in $\Omega=U\times V$ let $I(z,w,\sigma)$ be the FBI transform of χh ($\chi\in C_0^\infty(V)$, $\chi\equiv 1$ near 0) defined by (II.10) with $u=\chi h$. It is convenient to introduce a slightly modified transform. Set

$$F(z,w,\sigma)=I(z,w-i\phi(x,y),\sigma). \tag{II.16}$$

Note that (II.10) and (II.16) are well defined for CR distributions ($u=\chi h$; see proof of Proposition II.1). However the reader can easily check that, making use of Proposition II.1, it suffices to consider only CR functions of class C^1 in the rest of this section.

We are now ready to state a microlocal version of Proposition II.2.

Theorem II.1. *Let h be a* CR *distribution defined in $\Omega=U\times V$ and σ^0 a unit vector of R^l. The following conditions are equivalent:*

(i) *h is hypo-analytic at $(0,\sigma^0)$ (see* Def. II.2*).*

(ii) *If F is defined by* (II.16) *then there exists $C>0$ such that*

$$|F(z,w,\sigma)|\leqq Ce^{-|\sigma|/C} \tag{II.17}$$

uniformly for (x,y) near 0 in $\mathbb{R}^{2n}$, w near 0 in $\mathbb{C}^l$ and σ in a conic neighborhood of σ^0 in $\mathbb{R}^l$.

(iii) *There exist open convex cones $\Gamma_j\subset\mathbb{R}^l$, $1\leqq j\leqq r$, an open neighborhood $\mathcal{O}$ of 0 in $\mathbb{C}^{n+l}$, and* CR *distributions h_j on $M\cap\mathcal{O}$ extending holomorphically to the wedge $\mathcal{W}(\mathcal{O},\Gamma_j)$ (see* (II.14)*) such that*

$$h=\sum_{j=1}^{r} h_j \tag{II.18}$$

and

$$\Gamma_j\cdot\sigma^0<0,\qquad 1\leqq j\leqq r. \tag{II.19}$$

The modified FBI transform $F(z,w,\sigma)$ of $u=\chi h$ (where h is a CR distribution) is holomorphic with respect to w, but not with respect to z. The following lemma and its proof will be used several times in the rest of this paper to overcome this difficulty.

Lemma II.1. *There exist open sets U' and $\tilde{V}$, $0\in U'\subset\mathbb{R}^{2n}$, $0\in\tilde{V}\subset\mathbb{C}^l$, and a holomorphic function $G(z,w,\sigma)$ defined in the domain*

$$z\in U',\qquad w\in\tilde{V},\qquad \sigma\in\mathbb{C}^l,\qquad |\operatorname{Im}\sigma|<\frac{|\mathscr{Re}\,\sigma|}{2}, \tag{II.20}$$

such that

$$|F(z,w,\sigma)-G(z,w,\sigma)|<Ce^{-|\sigma|/C} \tag{II.21}$$

uniformly for (z,w,σ) *in the domain* (II.20).

Proof of Lemma II.1. Using (II.3) and (II.4), and integrating by parts we have for $1\leqq j\leqq n$,

$$\frac{\partial F}{\partial \bar z_j}(z,w,\sigma)=-i\int e^{i(w-\tilde s-i\phi(x,y))\sigma-|\sigma|(w-\tilde s-i\phi(x,y))^2}$$

$$\cdot\Delta(w-\tilde s-i\phi(x,y),\sigma)\sum_{k=1}^{l}\frac{\partial\phi_k}{\partial\bar z_j}(x,y)$$

$$\cdot\frac{\partial\chi}{\partial\tilde s_k}(\tilde s)h(x,y,\tilde s)d\tilde s\,.$$

Since $\chi(\tilde s)\equiv 1$ for $\tilde s$ in a neighborhood of the origin in $\mathbb{R}^l$, we conclude that $\frac{\partial F}{\partial \bar z_j}$ is exponentially decaying as $|\sigma|\to\infty$ (i.e. satisfying an inequality similar to (II.17)) uniformly in (x,y) in an open neighborhood of 0 in $\mathbb{R}^{2n}$, w in an open neighborhood of 0 in $\mathbb{C}^l$, and $\sigma\in\mathbb{C}^l$, $|\mathrm{Im}\,\sigma|<\frac{1}{2}|\mathscr{R}e\sigma|$).

We can make use of a standard inverse of the differential operator $\partial_{\bar z}$ in a small enough neighborhood of 0 in $\mathbb{R}^{2n}$, and solve the equation

$$\partial_{\bar z}Q(z,w,\sigma)=\partial_{\bar z}F(z,w,\sigma) \tag{II.22}$$

in such a way that Q is smooth in (x,y,w,σ), holomorphic with respect to (w,σ), and moreover such that Q satisfies (with $C>0$):

$$|Q(z,w,\sigma)|\leqq Ce^{-|\sigma|/C}, \tag{II.23}$$

for (z,w,σ) in a domain of the form (II.20).

It follows from (II.22) and (II.23) that we reach the conclusion of the lemma by taking

$$G(z,w,\sigma)=F(z,w,\sigma)-Q(z,w,\sigma)\,. \quad \square$$

Proof of Theorem II.1. The equivalence of (i) and (ii) follows immediately from Definition II.2 (i.e. (II.11)) and (II.16).

It is easy to see that (iii) implies (i). Indeed if (iii) holds then

$$h=\sum_{j=1}^{r}b_M H_j$$

with $H_j\in\mathscr{H}(\mathscr{W}(\mathscr{O},\Gamma_j))$, having slow growth at M, and $\Gamma_j\cdot\sigma^0<0$. This implies in particular that h is analytic at $(0,\sigma^0)$ uniformly in (x,y) (Sato's definition, see (II.12)); therefore (i) holds.

It only remains to prove that (i) implies (iii).

Let h be our CR distribution (or function) defined in Ω. We set

$$u(x,y,s)=\chi(s)h(x,y,s) \tag{II.24}$$

with $\chi \in C_0^\infty(V)$, $\chi \equiv 1$ near 0. Recall that we have the following inversion formula

$$u(x,y,s)=\frac{1}{(2\pi)^l}\iint_{\substack{s\in\mathbb{R}^l\\ \sigma\in\mathbb{R}^l}} e^{i(s-\tilde{s})\sigma-|\sigma|(s-\tilde{s})^2}\Delta(s-\tilde{s},\sigma)u(x,y,\tilde{s})d\tilde{s}d\sigma\,, \tag{II.25}$$

where Δ is defined as in (II.10), and the integral (II.25) is defined by introducing a convergent factor $e^{-\varepsilon|\sigma|^2}$ and passing to the limit as $\varepsilon\to 0$ (see Sjöstrand [14] or Baouendi-Chang-Treves [2] for example).

Note that (II.25) can also be written

$$\begin{aligned} u(x,y,s)&=\frac{1}{(2\pi)^l}\int_{\sigma\in\mathbb{R}^l} I(z,s,\sigma)d\sigma \\ &=\frac{1}{(2\pi)^l}\int_{\sigma\in\mathbb{R}^l} F(z,s+i\phi(x,y),\sigma)d\sigma \end{aligned} \tag{II.26}$$

where I and F are defined by (II.10) and (II.16).

Let Γ_j, $0\leqq j\leqq r$, be closed strictly convex cones in $\mathbb{R}^l$ such that

$$\mathbb{R}^l=\bigcup_{j=0}^{r}\Gamma_j\,,\quad \text{measure }(\Gamma_j\cap\Gamma_k)=0 \tag{II.27}$$

for $j\neq k$. Assume that Γ_0 is a neighborhood of σ^0 and that (II.11) holds for $\sigma\in\Gamma_0$. Also $\sigma^0\notin\Gamma_j$ for $1\leqq j\leqq r$.

For $j=0,\ldots,r$ define

$$u_j(x,y,s)=\frac{1}{(2\pi)^l}\int_{\sigma\in\Gamma_j} F(z,s+i\phi(x,y),\sigma)d\sigma\,. \tag{II.28}$$

If Γ is a closed strictly convex cone $\subset\mathbb{R}^l$ denote by $\mathring{\Gamma}$ its polar (or dual cone),

$$\mathring{\Gamma}=\{v\in\mathbb{R}^l;\, v\cdot\sigma>0\ \forall\sigma\in\Gamma\backslash 0\}\,.$$

It is an open convex cone.

If $\mathscr{C}$ and $\mathscr{C}'$ are two open cones we write $\mathscr{C}\subset\subset\mathscr{C}'$ if $\mathscr{C}\cap S^{l-1}$ is relatively compact in $\mathscr{C}'\cap S^{l-1}$.

For $j=1,\ldots,r$ let $\mathscr{C}_j$ be an open convex cone of $\mathbb{R}^l$ satisfying:

$$\mathscr{C}_j\subset\subset\mathring{\Gamma}_j,\quad \mathscr{C}_j\cdot\sigma^0<0\,.$$

Note that for $\sigma\in\Gamma_j$ and $v\in\mathscr{C}_j$ we have

$$v\cdot\sigma\geqq\alpha|v|\,|\sigma|\,,\quad\text{with}\quad \alpha>0\,. \tag{II.29}$$

For $j=0,\ldots,r$, define

$$K_j(x,y,w)=\frac{1}{(2\pi)^l}\int_{\sigma\in\Gamma_j} F(z,w,\sigma)d\sigma\,. \tag{II.30}$$

It follows from (II.11) (or (II.17)) that there exist an open neighborhood U' of 0 in $\mathbb{R}^{2n}$ and an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^l$ such that

$$K_0\in C^0(U',\mathscr{H}(\tilde{V}))\,. \tag{II.31}$$

For $j=1, \ldots, r$ the reader can easily check that (II.29) implies that $K_j(x, y, w)$ is continuous in the wedge $\mathscr{W}(U' \times V, \mathscr{C}_j)$ (it is even C^∞), holomorphic with respect to w, and with slow growth at M. In addition we have

$$u_j = b_M K_j, \quad \text{near the origin,} \tag{II.32}$$

where u_j is defined by (II.28). It should be emphasized that K_j is holomorphic in w but *not* in z. We shall overcome this difficulty by modifying K_j.

As in the proof of Lemma II.1, an integration by parts shows that (possibly after shrinking U' and $\tilde{V}$), we have for $j=0, \ldots, r$,

$$\partial_{\bar{z}} K_j(x, y, w) \in C^0(U', \mathscr{H}(\tilde{V}))\,.$$

(K_j is defined only in the wedge $\mathscr{W}(U' \times \tilde{V}, \mathscr{C}_j)$, whereas $\partial_{\bar{z}} K_j$ extends to $U' \times \tilde{V}$, holomorphically in w).

After further shrinking of U' and $\tilde{V}$ we can solve the differential equations

$$\partial_{\bar{z}} R_j(x, y, w) = \partial_{\bar{z}} K_j(x, y, w) \tag{II.33}$$

with

$$R_j \in C^0(U', \mathscr{H}(\tilde{V}))\,,$$

(of course we can take $R_0 = K_0$).

For $j=0, 1, \ldots, r$ define

$$H_j = K_j - R_j + \frac{1}{r+1} \sum_{p=0}^{r} R_p\,. \tag{II.34}$$

It is clear that H_j is continuous in the wedge $\mathscr{W}(U' \times \tilde{V}, \mathscr{C}_j)$, holomorphic with respect to w, with tempered growth at M ($H_0 \in C^0(U', \mathscr{H}(\tilde{V}))$). The desired decomposition (II.18) is then an immediate consequence of the following

Lemma II.2. *If H_j is defined by* (II.34) *then*

(a) *H_j is holomorphic in $\mathscr{W}(U' \times \tilde{V}, \mathscr{C}_j)$.*

(b) $u = \sum_{j=0}^{r} b_M H_j$, *near* 0.

Proof of Lemma II.2. (a) We have only to prove that H_j is holomorphic with respect to z. Let $Z^*(L_k)$, $1 \leq k \leq n$, be the push forward of the vector field L_k from Ω into $\mathbb{C}^{n+l}$. We have

$$Z^*(L_k) = \frac{\partial}{\partial \bar{z}_k} - 2i \sum_{p=1}^{l} \frac{\partial \phi_p}{\partial \bar{z}_k}(z, \bar{z}) \frac{\partial}{\partial \bar{w}_p}\,.$$

Since K_j is holomorphic with respect to w, using (II.30) we have, $1 \leq k \leq n$, $0 \leq j \leq r$,

$$L_k u_j = b_M \frac{\partial}{\partial \bar{z}_k} K_j\,.$$

By (II.33) we also have

$$L_k u_j = \left. \frac{\partial}{\partial \bar{z}_k} R_j \right|_{w = s + i\phi(x,y)}\,.$$

Since $\sum_{j=0}^{r} u_j = u$ and $L_k u = 0$ in a neighborhood of 0, we conclude that for $1 \leqq k \leqq n$,

$$\sum_{j=0}^{r} \frac{\partial}{\partial \bar{z}_k} R_j \bigg|_{w=s+i\phi(x,y)} = 0,$$

and therefore it is easy to see that

$$\sum_{j=0}^{r} \frac{\partial}{\partial \bar{z}_k} R_j \equiv 0. \tag{II.35}$$

(II.33), (II.34), and (II.35) imply at once, after further shrinking of U' and $\tilde{V}$,

$$\partial_{\bar{z}} H_j \equiv 0 \quad \text{in} \quad \mathscr{W}(U' \times \tilde{V}, \mathscr{C}_j)$$

($\partial_{\bar{z}} H_0 \equiv 0$ in $U' \times \tilde{V}$).

(b) Claim (b) is an immediate consequence of (II.32) and (II.34).

This completes the proof of Lemma II.2 and hence of Theorem II.1. □

If h is a CR distribution defined in $\Omega = U \times V$, its *hypo-analytic wave front set* (denoted $WF_{ha}h$) consists of the points $(s, \sigma) \in T^*V \backslash 0$ such that h is not hypo-analytic at (s, σ) (Def. II.2). $WF_{ha}h$ is a closed conic subset of $T^*V\backslash 0$. Its canonical projection on V consists of the points $s \in V$ such that h does not extend holomorphically at $(0, s)$; (see Prop. II.2). If $s_0 \in V$ we shall use the following notation

$$WF_{ha,s_0}h = \{\sigma \in \mathbb{R}^l \backslash \{0\}; (s_0, \sigma) \in WF_{ha}h\}.$$

For holomorphic extendability to a single wedge, we have the following result.

Theorem II.2. *Let Γ be a strictly convex closed cone contained in* $\mathbb{R}^l$, *and h a* CR *distribution defined in Ω. The following properties are equivalent.*

(a) $Wf_{ha,0}h \subset \Gamma$.

(b) *For every open cone $\mathscr{C} \subset \mathbb{R}^l$, with $\mathscr{C} \subset\subset \mathring{\Gamma}$, there exists an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^{n+l}$ such that h extends holomorphically to the wedge $\mathscr{W}(\mathcal{O}, \mathscr{C})$.*

Proof of (b)$\Rightarrow$(a). Let $\sigma^0 \in \mathbb{R}^l \backslash \{0\}$, $\sigma^0 \notin \Gamma$. Let $\mathscr{C}$ be an open convex cone of $\mathbb{R}^l$ satisfying

$$\sigma^0 \cdot \mathscr{C} < 0 \quad \text{and} \quad \mathscr{C} \subset\subset \mathring{\Gamma}.$$

Property (b) implies that h extends holomorphically to a wedge $\mathscr{W}(\mathcal{O}, \mathscr{C})$. Theorem II.1 ((iii)$\Rightarrow$(i)) implies that

$$\sigma^0 \notin WF_{ha,0}h,$$

which proves (a). □

Proof of (a)$\Rightarrow$(b). Assume (a) and let $\mathscr{C}$ be an open cone, $\mathscr{C} \subset\subset \mathring{\Gamma}$. Let Γ' be a strictly convex closed cone satisfying

$$\Gamma \subset \text{Int.}\,\Gamma', \qquad \mathscr{C} \subset\subset \mathring{\Gamma}'. \tag{II.36}$$

The proof follows quite closely the part of the proof of Theorem III.1 showing that (i) implies (iii).

Set $\Gamma_0 = \mathbb{R}^l \setminus \Gamma'$ and $\Gamma_1 = \Gamma'$, and with $u = \chi h$, define u_j and K_j, $j = 0, 1$, by (II.29) and (II.30). By assumption (a) and (II.36) we see that (II.17) holds for $\sigma \in \Gamma_0$; therefore we have (II.31).

Using (II.36) we can take $\mathscr{C}_1 = \mathscr{C}$ and (II.32) holds for $j = 1$.

The rest of the proof is identical to the one in Theorem II.1: For $j = 0, 1$ define H_j by (II.34). We finally get

$$u = H_0|_{w = s + i\phi(z, \bar{z})} + b_M H_1$$

where $H_0(z, w)$ is holomorphic near 0 in $\mathbb{C}^{n+l}$, and H_1 holomorphic in a wedge $\mathscr{W}(\mathscr{O}, \mathscr{C})$. It suffices to write

$$u = b_M(H_0 + H_1)$$

in order to complete the proof of Theorem II.2. □

The following decomposition theorem is reminiscent of a similar decomposition result for distributions in real space, and of the edge of the wedge theorem (cf. Hörmander [11]).

Theorem II.3. *Let $\Gamma_1, \ldots, \Gamma_r$ be strictly convex closed cones contained in $\mathbb{R}^l$ and satisfying:*

$$\mathbb{R}^l = \bigcup_{j=1}^{r} \Gamma_j, \quad \text{meas}(\Gamma_j \cap \Gamma_k) = 0, j \neq k.$$

If h is a CR *distribution defined in Ω there exists an open neighborhood of* 0, *$\Omega' \subset \Omega$, such that*

$$h = \sum_{j=1}^{r} h_j \quad \text{in} \quad \Omega', \tag{II.37}$$

where h_j is a CR *distribution in Ω' satisfying*

$$WF_{ha,0} h_j \subset \Gamma_j \cap (WF_{ha,0} h). \tag{II.38}$$

In addition if $h = \sum h'_j$ is another decomposition satisfying (II.37) *and* (II.38), *then*

$$h'_j = h_j + \sum_{k=1}^{r} h_{jk}, \tag{II.39}$$

where h_{jk} is a CR *distribution satisfying*

$$WF_{ha,0} h_{jk} \subset \Gamma_j \cap \Gamma_k \cap WF_{ha,0} h, \tag{II.40}$$

and $h_{jk} = -h_{kj}$.

Proof. The proof of this theorem also follows closely the part of the proof of Theorem II.1 showing that (i)⇒(iii)

With $u = \chi h$ define u_j and K_j, for $j = 1, \ldots, r$, by (II.28) and (II.30). Then we have

$$u = \sum_{j=1}^{r} u_j,$$

and (II.32) holds for $j = 1, \ldots, r$.

If we define H_j by (II.34) we obtain the decomposition (II.37) with

$$h_j = b_M H_j, \quad 1 \leq j \leq r.$$

It is clear that h_j is independent of the choice of the open cone $\mathscr{C}_j$ satisfying

$$\mathscr{C}_j \subset\subset \hat{\Gamma}_j,$$

and that for each such choice there exists an open neighborhood $\mathcal{O}$ of 0 in $\mathbb{C}^{n+l}$ such that h_j extends holomorphically to $\mathscr{W}(\mathcal{O}, \mathscr{C}_j)$. By Theorem II.2 we conclude that

$$WF_{ha,0} h_j \subset \Gamma_j. \tag{II.41}$$

In order to complete the proof of (II.37), taking (II.41) into account, we must show that if $\sigma^0 \in \Gamma_j \setminus \{0\}$ and $\sigma^0 \notin WF_{ha,0} h$, then $\sigma^0 \notin WF_{ha,0} h_j$.

By Theorem II.1 given such a σ^0 we can write

$$h = \sum_{k=1}^{r'} \tilde{h}_k \tag{II.42}$$

where $\tilde{h}_k$ is a CR distribution and $WF_{ha,0} \tilde{h}_k \subset \tilde{\Gamma}_k$, $\tilde{\Gamma}_k$ being a strictly convex closed cone, $\sigma^0 \notin \tilde{\Gamma}_k$.

Since for each $j = 1, \ldots, r$ the map $h \mapsto h_j$ is linear in h, in view of (II.42) we may assume that h is one of the $\tilde{h}_k$. For the sake of simplicity of notation we shall assume that

$$WF_{ha,0} h \subset \Gamma$$

where Γ is a strictly convex closed cone, $\sigma^0 \notin \Gamma$. Let Γ' be a strictly convex closed conic neighborhood of Γ, $\sigma^0 \notin \Gamma'$, and $\mathscr{C}'_j$ an open cone satisfying

$$\mathscr{C}'_j \subset\subset \Gamma' \subset \Gamma_j, \qquad \sigma^0 \cdot \mathscr{C}'_j < 0.$$

Since (II.17) holds for $\sigma \in \mathbb{R}^l \setminus \Gamma'$, we conclude that K_j extends holomorphically in w, for $w \in \mathscr{C}'_j$. This in turn implies that H_j extends holomorphically to a wedge of the form $\mathscr{W}(\mathcal{O}, \mathscr{C}'_j)$. The hypo-analyticity of h_j at $(0, \sigma^0)$ follows at once. [Note that if $\Gamma' \cap \Gamma_j = \emptyset$ then h_j extends holomorphically at 0].

To prove (II.39) and (II.40), let $h''_j = h_j - h'_j$. Then $\sum h''_j = 0$. Hence we may assume the h_j satisfy (II.38) and

$$\sum_{j=1}^{r} h_j = 0, \tag{II.43}$$

and find CR distributions h_{jk} satisfying

$$h_j = \sum_{k=1}^{r} h_{jk}, \qquad h_{jk} = -h_{kj}, \tag{II.44}$$

with

$$WF_{ha,0} h_{jk} \subset \Gamma_j \cap \Gamma_k \cap WF_{ha,0} h. \tag{II.45}$$

The desired conclusion (II.39) and (II.40) would follow at once.

Let $F_j(z, w, \sigma)$ be the modified FBI transform of $u = \chi h_j$ defined by (II.16). As in the proof of Lemma II.1, we may find $Q_j(z, w, \sigma)$ satisfying (II.23), $j = 1, \ldots, r$, with

$$\partial_{\bar{z}} Q_j(z, w, \sigma) = \partial_{\bar{z}} F_j(z, w, \sigma). \tag{II.46}$$

Now, with $w=s+i\phi(x,y)$, let

$$h_{jk}=\left\{\int_{\Gamma_k}[F_j(z,w,\sigma)-Q_j(z,w,\sigma)]d\sigma+\frac{1}{r}\int_{\mathbb{R}^l}Q_j(z,w,\sigma)d\sigma\right\}$$
$$-\left\{\int_{\Gamma_j}[F_k(z,w,\sigma)-Q_k(z,w,\sigma)]d\sigma+\frac{1}{r}\int_{\mathbb{R}^l}Q_k(z,w,\sigma)d\sigma\right\}$$
$$=h_{jk1}-h_{jk2}\,. \tag{II.47}$$

Then for j fixed

$$\sum_k h_{jk}=\left\{\int_{\mathbb{R}^l}[F_j(z,w,\sigma)-Q_j(z,w,\sigma)]d\sigma+\int_{\mathbb{R}^l}Q_j(z,w,\sigma)d\sigma\right\}$$
$$-\left\{\int_{\Gamma_j}\sum_k[F_k(z,w,\sigma)-Q_k(z,w,\sigma)]d\sigma+\frac{1}{r}\int_{\mathbb{R}^l}\sum_k Q_k(z,w,\sigma)d\sigma\right\}. \tag{II.48}$$

We claim first that the second term on the right hand side in (II.48) is zero. Indeed, $\sum_k F_k(z,w,\sigma)=0$ since $\sum h_k=0$. By construction, this implies $\sum_k Q_k=0$, which proves the claim. Since the first term on the right hand side in (II.48) equals h_j, this proves (II.44). The same reasoning used in the proof of (II.38) shows that we have

$$WF_{ha,0}h_{jk1}\subset\Gamma_k\cap WF_{ha,0}h_j$$

and

$$WF_{ha,0}h_{jk2}\subset\Gamma_j\cap WF_{ha,0}h_k\,,$$

which, together with (II.38), proves (II.45). □

A minor change in the previous proof left to the reader yields the following result.

Theorem II.4. *Let h be a* CR *distribution defined in Ω, and $\Gamma_1,\ldots,\Gamma_r$ be strictly convex closed cones contained in $\mathbb{R}^l$ and satisfying:*

$$WF_{ha,0}h\subset\text{Int.}\left(\bigcup_{j=1}^{r}\Gamma_j\right),$$
$$\text{meas}(\Gamma_j\cap\Gamma_k)=0\quad \textit{for}\quad j\neq k\,.$$

Then there exists an open neighborhood of 0, $\Omega'\subset\Omega$, *and* CR *distributions in Ω', h_j, satisfying*

$$h=\sum_{j=1}^{r}h_j\,,\qquad WF_{ha,0}h_j\subset\Gamma_j\cap WF_{ha,0}h\,.$$

Moreover, if $\mathscr{C}_j$ is a nonempty open cone satisfying

$$\mathscr{C}_j\subset\subset\mathring{\Gamma}_j^{\circ},\qquad 1\leqq j\leqq r,$$

then there exists an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^{n+l}$ such that h_j extends holomorphically to the wedge $\mathscr{W}(\mathcal{O},\mathscr{C}_j)$.

Remark II.1. If $l=1$, the CR manifold M is a hypersurface of $\mathbb{C}^{n+1}$. A wedge determines one side of M. The decomposition theorem states that any CR distribution is the sum of two CR distributions, each extending holomorphically to one side of M. This result (without the rigidity condition) can be found in Andreotti-Hill [1]. In addition, if a CR distribution is hypo-analytic at $(0,\sigma^0)$, $\sigma^0 \in \mathbb{R}\backslash 0$, then it has a holomorphic extension to one side of M. This result also holds without the rigidity condition (see Baouendi-Treves [5]).

We believe that the decomposition results stated here are new when $l>1$. For example when $l=2$ (i.e. codimension $M=2$) we obtain that any CR distribution is locally the sum of *three* CR distributions, each extending holomorphically to a wedge. (See Henkin [10] where non rigid manifolds are considered but some extra conditions on the Levi form are imposed.) In addition if a CR distribution is hypo-analytic at $(0,\sigma^0)$, $\sigma^0 \in \mathbb{R}\backslash\{0\}$, then it is the sum of *two* CR distributions extending holomorphically to wedges.

III. Criteria for microlocal hypo-analyticity in rigid CR structures of finite type

In this section Theorem II.1 is used to give sufficient conditions for microlocal hypo-analyticity of CR distributions. Theorem II.2 is then applied to show that, in the finite type case, any CR distribution extends holomorphically to a single wedge. We follow here the notation of Sect. II.

If f is a real vector valued smooth function (or a formal power series) defined in a neighborhood of the origin in $\mathbb{R}^{2n}$, then the power series of f has a unique decomposition of the form

$$f(z,\bar{z}) \sim f_{(p)}(z,\bar{z}) + f_{(n)}(z,\bar{z}) \tag{III.1}$$

with

$$f_{(p)}(z,\bar{z}) = \sum_{|\alpha|\geqq 0} \mathcal{R}e(a_\alpha z^\alpha)\,, \quad \text{(pure terms)}$$

$$f_{(n)}(z,\bar{z}) = \sum_{\substack{|\alpha|\geqq 1\\ |\beta|\geqq 1}} a_{\alpha,\beta} z^\alpha \bar{z}^\beta\,, \quad \text{(nonpure terms)},$$

a_α and $a_{\alpha\beta}$ are complex vectors satisfying

$$a_{\alpha,\beta} = \bar{a}_{\beta,\alpha}\,.$$

We shall need the following definition:

Definition III.1. We say that a vector $\sigma^0 \in \mathbb{R}^l\backslash\{0\}$ satisfies the *sector property* if there exists a holomorphic curve $\zeta \mapsto \gamma(\zeta) \in \mathbb{C}^n$ defined in a neighborhood of 0 in $\mathbb{C}$, $\gamma(0)=0$, such that

$$\sigma^0 \cdot \phi_{(n)}(\gamma(\zeta), \overline{\gamma(\zeta)}) = P_m(\zeta,\bar{\zeta}) + O(|\zeta|^{m+1})\,, \tag{III.2}$$

where $\phi_{(n)}$ is the series of nonpure terms of ϕ defined in (III.1), P_m is a real valued homogeneous polynomial of degree $m \in \mathbb{Z}_+$, and moreover such that there exists a sector $\mathcal{S}$ in the plane and a complex number $\mu \in \mathbb{C}$ satisfying

$$P_m(\zeta,\bar{\zeta})+\mathcal{R}e(\mu\zeta^m)|_{\mathcal{S}}<0\,, \tag{III.3}$$

$$\text{angle}\,\mathcal{S}>\frac{\pi}{m}\,. \tag{III.4}$$

We can now state a criterion for microlocal hypo-analyticity.

Theorem III.1. *Assume that* $\sigma^0 \in \mathbb{R}^l\backslash\{0\}$ *satisfies the sector property of Definition III.1. Then any* CR *distribution in* Ω *is hypo-analytic at* $(0,\sigma^0)$.

The following lemma will be needed in the proof of Theorem III.1. It is related to, and replaces, the argument given in [5], Sect. III. One of the ideas used in the present proof was suggested to us in a referee report for that paper.

Lemma III.1. *Let* $P(x,y)$ *be a real valued* C^1 *function homogeneous of degree* m, $(m\in\mathbb{R},\ m\geqq 1)$ *defined in* $\mathbb{R}^2\backslash\{0\}$. *Assume there exists a sector* $\mathcal{S}$ *in the plane such that*

$$P|_{\mathcal{S}}>0 \quad \text{and} \quad \text{angle}\,\mathcal{S}>\frac{\pi}{m}\,.$$

Then there exist a bounded domain D, $0\in D\subset\mathbb{R}^2$, *and a holomorphic function* f *defined in* D, *continuous on* $\bar{D}$, *satisfying*

$$(P+\mathcal{R}e\,f)|_{\partial D}>0\,, \qquad f(0)=0\,,$$

where ∂D *is the boundary of* D.

Proof. We shall write $P(z)$ instead of $P(x,y)$. After a rotation we may assume that

$$P(z)>0 \quad \text{for} \quad |\arg z|\leqq \frac{\pi}{2m'}\,, \qquad z\neq 0 \tag{III.5}$$

for some m', $0<m'<m$.

Let $\lambda>0$ be small enough so that

$$P(z)-\lambda\mathcal{R}e\,z^{m'}>0 \quad \text{for} \quad |z|=1\,, \qquad |\arg z|\leqq\frac{\pi}{2m'}\,. \tag{III.6}$$

Here we take the principal determination of $z^{m'}$. Since $\mathcal{R}e\,z^{m'}=|z|^{m'}\cos m'\theta$ vanishes for $|\arg z|=\frac{\pi}{2m'}$, we conclude from (III.5) and the homogeneity of P that

$$P(z)-\lambda\mathcal{R}e\,z^{m'}\geqq\alpha|z|^m\,, \quad \text{for} \quad |\arg z|=\frac{\pi}{2m'}\,, \qquad (\alpha>0)\,. \tag{III.7}$$

For $\varepsilon>0$ let D_ε be the domain defined by

$$|\arg(z+\varepsilon)|<\frac{\pi}{2m'}\,, \qquad 0<|z+\varepsilon|<1\,.$$

(By (III.6) and (III.7) we have

$$P(z+\varepsilon)-\lambda\mathcal{R}e(z+\varepsilon)^{m'}+\lambda\varepsilon^{m'}>0 \quad \text{for} \quad z\in\partial D_{\varepsilon}.$$

We must show that if $\varepsilon>0$ is small enough then

$$P(z)-\lambda\mathcal{R}e(z+\varepsilon)^{m'}+\lambda\varepsilon^{m'}>0 \quad \text{for} \quad z\in\partial D_{\varepsilon}. \tag{III.8}$$

Therefore we need to estimate the difference $P(z+\varepsilon)-P(z)$. We have

$$P(z+\varepsilon)-P(z)=\varepsilon\int_0^1\frac{\partial P}{\partial x}(z+t\varepsilon)dt,$$

thus

$$|P(z+\varepsilon)-P(z)|\leqq M\varepsilon(|z+\varepsilon|^{m-1}+\varepsilon^{m-1}), \tag{III.9}$$

with M independent of ε.

It follows from (III.6) that on the arc $|z+\varepsilon|=1$, $|\arg(z+\varepsilon)|\leqq\frac{\pi}{2m'}$ we have

$$P(z+\varepsilon)-\lambda\mathcal{R}e(z+\varepsilon)^{m'}\geqq C_1, \tag{III.10}$$

with $C_1>0$ independent of ε. If $\varepsilon>0$ is sufficiently small (III.9) and (III.10) imply that we have, for z on the same arc,

$$P(z)-\lambda\mathcal{R}e(z+\varepsilon)^{m'}\geqq C_1/2.$$

In order to show the desired positivity (III.8) on the segments $|\arg(z+\varepsilon)|=\frac{\pi}{2m'}$, $|z+\varepsilon|\leqq 1$, we avail ourselves of (III.7) and (III.9). By (III.9) we have

$$|P(z+\varepsilon)-P(z)|\leqq\alpha|z+\varepsilon|^m+M'\varepsilon^m,$$

where $M'>0$ is independent of ε. Therefore we conclude by using (III.7) that, for $|\arg(z+\varepsilon)|=\frac{\pi}{2m'}$, we have

$$P(z)-\lambda\mathcal{R}e(z+\varepsilon)^{m'}+\lambda\varepsilon^{m'}\geqq\lambda\varepsilon^{m'}-M'\varepsilon^m.$$

The right hand side of the latter is >0, if $\varepsilon>0$ is sufficiently small. □

Proof of Theorem III.1.

Part I. In this part we prove that if Theorem III.1 holds when the holomorphic curve γ in Definition III.1 is given by $\gamma(\zeta)=(\zeta,0,\dots,0)$, then it also holds for an arbitrary γ (satisfying (III.2)–(III.4)). Indeed let $\gamma_1(\zeta),\dots,\gamma_n(\zeta)$ be the components of the vector $\gamma(\zeta)$. We may assume

$$\gamma_1(\zeta)\not\equiv 0. \tag{III.11}$$

Consider the map $z=\theta(\tilde{z})$ defined in a neighborhood of 0 in $\mathbb{C}^n$ by

$$\begin{aligned} z_1 &= \theta_1(\tilde{z}) = \gamma_1(\tilde{z}_1), \\ z_j &= \theta_j(\tilde{z}) = \tilde{z}_j + \gamma_j(\tilde{z}_1), \quad 2 \leqq j \leqq n. \end{aligned} \tag{III.12}$$

Setting $\tilde{z}=\tilde{x}+i\tilde{y}$, we have

$$x = x(\tilde{x},\tilde{y}) = \mathcal{R}e\,\theta(\tilde{x}+i\tilde{y}),$$

$$y = y(\tilde{x},\tilde{y}) = \operatorname{Im}\theta(\tilde{x}+i\tilde{y}).$$

Using the chain rule the reader can easily check that the pull back of h defined by

$$\tilde{h}(\tilde{x},\tilde{y},s) = h(x(\tilde{x},\tilde{y}), y(\tilde{x},\tilde{y}), s),$$

is a CR distribution on the submanifold $\tilde{M}$ of $\mathbb{C}^{n+l}$ defined by

$$\operatorname{Im} w = \tilde{\phi}(\tilde{z},\bar{\tilde{z}}) = \phi(\theta(\tilde{z}), \overline{\theta(\tilde{z})}).$$

In the new rigid CR structure (defined by $\tilde{\phi}$ instead of ϕ), the vector σ^0 satisfies the sector property of Definition III.1 with a new curve γ defined by $\gamma(\zeta) = (\zeta, 0, \ldots, 0)$. Assuming the theorem holds in this case, we conclude that $\tilde{h}$ is hypo-analytic at $(0,\sigma^0)$.

Let $F(z,w,\sigma)$ be the modified FBI transform of $u=\chi h$ given by (II.16), and $\tilde{F}(\tilde{z},w,\sigma)$ the one of $\tilde{u}=\chi\tilde{h}$. We clearly have

$$\tilde{F}(\tilde{z},w,\sigma) = F(\theta(\tilde{z}),w,\sigma). \tag{III.13}$$

Using Theorem II.1, the hypo-analyticity of $\tilde{h}$ at $(0,\sigma^0)$ implies that (II.17) holds for $\tilde{F}$. Since the map θ is open (by using (III.11)), we conclude from (III.13) that (II.17) also holds for F. Applying again Theorem II.1 yields the hypo-analyticity of h at $(0,\sigma^0)$.

Part 2. Assume that σ^0 satisfies the sector property of Definition III.1 with γ given by

$$\gamma(\zeta) = (\zeta, 0, \ldots, 0). \tag{III.14}$$

We shall prove in this part that (II.17) holds.

After an $\mathbb{R}$-linear change of coordinates in the w space, we may assume

$$\sigma^0 = (1, 0, \ldots, 0). \tag{III.15}$$

It follows from (III.2)–(III.4) and (III.14), (III.15) that we have

$$\begin{aligned} \phi_1(z,\bar{z}) = \operatorname{Im}\left(\sum_{j=2}^{m} a_j z_1^j\right) + P_m(z_1,\bar{z}_1) \\ + O(|z_1|^{m+1} + |z_1|\,|z'| + |z'|^2) \end{aligned} \tag{III.16}$$

with $a_j \in \mathbb{C}$, $z'=(z_2,\ldots,z_n)$, $m \in \mathbb{Z}_+$, $m \geqq 2$, and P_m is a real homogeneous polynomial of degree m satisfying,

$$P_m|_{\mathcal{S}} < 0, \quad \text{angle}\,\mathcal{S} > \frac{\pi}{m}. \tag{III.17}$$

In order to eliminate the first sum in the right hand side of (III.16) we make a holomorphic change of coordinates in (z, w) space defined by

$$\tilde{z}=z\,,$$
$$\tilde{w}_1=w_1-\sum_{j=2}^{m} a_j z_1^j\,, \qquad \text{(III.18)}$$
$$\tilde{w}_j=w_j\,,\quad 2\leqq j\leqq l\,.$$

After replacing s_1 by $s_1-\mathscr{R}e\sum_{j=2}^{m} a_j z_1^j$, we can now drop the "tildas" and assume that (III.16) holds with $a_j=0$, $2\leqq j\leqq m$. (Note that Theorem II.1 (see (iii)) shows that the hypo-analyticity of h at $(0,\sigma^0)$ is not affected by a change of coordinates of the form (III.18).)

We shall need a final change of coordinates in $\mathbb{C}^{n+l}$. Let δ be a positive small real number to be determined at the end of the proof. Consider the dilation

$$z_1=\delta\tilde{z}_1\,,$$
$$z_j=\delta^m\tilde{z}_j\,,\quad 2\leqq j\leqq n\,,$$
$$w_1=\delta^m\tilde{w}_1\,, \qquad \text{(III.19)}$$
$$w_j=\delta\tilde{w}_j\,,\quad 2\leqq j\leqq l\,.$$

For the sake of simplicity of notation we now drop the "tildas" and we assume that

$$\phi_1(z,\bar{z})=P_m(z_1,\bar{z}_1)+O(\delta)\,,$$
$$\phi_j(z,\bar{z})=O(\delta)\,,\quad 2\leqq j\leqq l\,. \qquad \text{(III.20)}$$

with P_m satisfying (III.17).

We apply now Lemma III.1 with $P=-P_m$, and P_m satisfying (III.17). If f and D are as in the statement of Lemma III.1, there exists $\tau_0>0$ such that for $0<\tau\leqq\tau_0$, we have

$$-P_m(z_1,\bar{z}_1)+\mathscr{R}e\,f(z_1)-\tau^m(P_m(z_1,\bar{z}_1))^2|_{\partial D}>0\,. \qquad \text{(III.21)}$$

Define

$$f_\tau(z_1)=\tau^m f\left(\frac{z_1}{\tau}\right),$$
$$D_\tau=\{z_1\in\mathbb{C}\,;\,z_1/\tau\in D\}\,.$$

It follows from (III.21) that we have

$$-P_m(z_1,\bar{z}_1)+\mathscr{R}e\,f_\tau(z_1)-(P_m(z_1,\bar{z}_1))^2|_{\partial D_\tau}>0\,. \qquad \text{(III.22)}$$

Let h be a CR function. We wish to estimate the modified FBI transform of $u=\chi h$, $F(z,w,\sigma)$, defined by (II.16) and (II.10). Note that after the change of variables (III.19), h depends on the dilation parameter δ. We can choose the cut-off function χ to be independent of δ.

We shall make use of Lemma II.1. The reader can easily check that U', $\tilde{V}$ and C in (II.20) and (II.21) can be chosen independent of the parameter δ, $0<\delta\leqq 1$. Assume that

$$U'=\{(z_1,z')\in\mathbb{C}^n;\, |z_1|<r,\, |z'|<r\}, \qquad \tilde{V}=\{w\in\mathbb{C}^l,\, |w|<r\}$$

with $r>0$.

Since $f_\tau(z_1)=O(\tau^{m-m'})$ for $|z|\leqq 1$, and $0<m'<m$ (see proof of Lemma III.1) we can choose $\tau>0$ satisfying

(i) $0<\tau\leqq\tau_0$, so that (II.22) holds,

(ii) $D_\tau\subset\{z_1;\, |z_1|<r\}$

(iii) $|f_\tau(z_1)|\leqq 1/2C$ for $|z|\leqq 1$, where C is the constant in (II.21).

A straightforward estimate of the integral defining F shows that there exists $C'>0$ (independent of δ) such that, for $|z|<r$, $|w|<1$, and $\sigma\in\mathbb{R}^l\backslash\{0\}$, we have

$$|e^{-\sigma_1 f_\tau(z_1)}F(z,w,\sigma)|\leqq C'\int_{\mathbb{R}^l} e^{-|\sigma|Q(z,w,\tilde{s},\sigma/|\sigma|)}|\chi(\tilde{s})|d\tilde{s}, \tag{III.23}$$

with

$$Q(z,w,\tilde{s},\sigma/|\sigma|)=\frac{\sigma_1}{|\sigma|}\mathscr{R}e f_\tau(z_1)+\operatorname{Im}w\cdot\frac{\sigma}{|\sigma|}-\phi(z,\bar{z})\cdot\frac{\sigma}{|\sigma|}$$
$$+(\mathscr{R}e w-\tilde{s})^2-(\operatorname{Im}w-\phi(z,\bar{z}))^2.$$

If we put $w=0$, $\sigma=\sigma^0=(1,0,\ldots,0)$, and use (III.20) we get

$$Q(z,0,\tilde{s},\sigma^0)=-P_m(z_1,\bar{z}_1)+\mathscr{R}e f_\tau(z_1)-(P_m(z_1,\bar{z}_1))^2+\tilde{s}^2+0(\delta). \tag{III.24}$$

Using (III.22) we can choose $\delta>0$ (and fix it from now on) so that the right hand of (III.24) is >0 for $z_1\in\partial D_\tau$, $|z'|\leqq\frac{r}{2}$, and $\tilde{s}\in\operatorname{supp}\chi$.

Therefore we conclude from (III.23) that there exist positive numbers r_1 ($r_1<r$), ϱ and C'' such that for

$$z_1\in\partial D_\tau, \quad |z'|\leqq\frac{r}{2}, \quad \left|\frac{\sigma}{|\sigma|}-\sigma^0\right|\leqq r_1, \quad |w|\leqq r_1 \tag{III.25}$$

we have,

$$|e^{-\sigma_1 f_\tau(z_1)}F(z,w,\sigma)|\leqq C''e^{-\varrho|\sigma|}. \tag{III.26}$$

On the other hand, making use of (II.21) and (iii) above we get

$$|e^{-\sigma_1 f_\tau(z_1)}(F(z,w,\sigma)-G(z,w,\sigma))|\leqq Ce^{-|\sigma|/2C}, \tag{III.27}$$

for $|z|<r$, $|w|<r$, and $\sigma\in\mathbb{R}^l$.

We conclude that for (z,w,σ) in the set defined by (III.25) the following holds

$$|e^{-\sigma_1 f_\tau(z_1)}G(z,w,\sigma)|\leqq C'''e^{-\varrho'|\sigma|} \tag{III.28}$$

with positive constants C''', ϱ'.

Since G is holomorphic in z the maximum principle implies that (III.28) holds for (z, w, σ) in the closure of the set defined by (III.25) (i.e. $z_1 \in \bar{D}_\tau$ instead of $z_1 \in \partial D_\tau$).

Taking again (III.27) into account we find that (III.26) also holds in the closure of (III.25) (possibly with different C'' and ϱ). Finally since f_τ vanishes at the origin we conclude that, for (z, w) in a neighborhood of 0 in $\mathbb{C}^{n+l}$, and σ in a conic neighborhood of σ^0 in $\mathbb{R}^l$, we have

$$|F(z, w, \sigma)| \leqq Ce^{-\varrho|\sigma|},$$

with different positive constant, C and ϱ. The proof of Theorem III.1 is now complete. □

If $\Omega \subset \mathbb{R}^{2n+l}$ is equipped with an abstract CR structure (see Sect. I) with $\dim \mathscr{V}'_\omega = n$, the *characteristic set* of the structure is the subbundle $\sum$ of $T^*\Omega$ defined by

$$(\omega, \theta) \in \sum \Leftrightarrow \langle \theta, L_\omega \rangle = 0, \quad \forall L \in \mathbb{L}.$$

We have $\dim \sum_\omega = l$. A characteristic point $(\omega, \theta) \in \sum$ is of *finite type* if there exist $M_1, M_2, \ldots, M_m \in \mathbb{L} \oplus \bar{\mathbb{L}}$ such that

$$\langle \theta, [M_1, [M_2, \ldots [M_{m-1}, M_m] \ldots]]_\omega \rangle \neq 0. \tag{III.29}$$

The smallest positive integer $m \geqq 2$ for which (III.29) holds is the *type* at (ω, θ), written $m(\omega, \theta)$.

A point $\omega \in \Omega$ is said to be of *finite type* if (ω, θ) is of finite type for every $\theta \in \sum_\omega \backslash \{0\}$ (see Kohn [12], Bloom-Graham [7]).

We assume from now on that, as in Sect. II, we are dealing with a rigid CR structure:

$$\Omega = U \times V, \quad 0 \in U \subset \mathbb{R}^{2n}, \quad 0 \in V \subset \mathbb{R}^l,$$

and the L_j's are given by (II.4) with ϕ satisfying (II.1). The space of characteristic covectors at 0, $\sum_0$, can then be identified with $T_0^* V = \mathbb{R}^l$.

A simple computation shows that if $\sigma \in T_0^* V \backslash \{0\}$ then $(0, \sigma)$ is of finite type if and only if (see (III.1))

$$\sigma \cdot \phi_{(n)} \not\equiv 0.$$

The type at $(0, \sigma)$ is m if there exist $\alpha, \beta \in \mathbb{Z}^n_+$, with $|\alpha| > 0$, $|\beta| > 0$,

$$\sigma \cdot \partial_z^\alpha \partial_{\bar{z}}^\beta \phi(0) \neq 0, \tag{III.30}$$

and $|\alpha| + |\beta| = \mathrm{m}$ is minimum.

In fact if the vector fields L_j are defined by (II.14), then the formal power series $\phi_{(n)}$ can be recovered from all the brackets of L_j and $\bar{L}_j$, $1 \leqq j \leqq n$, as explained below.

As in [5], we introduce the following notation. If $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$ we write

$$\bar{u} \cdot L = \sum_{j=1}^n \bar{u}_j L_j, \quad u \cdot \bar{L} = \sum_{j=1}^n u_j \bar{L}_j.$$

Define

$$q(u,\bar{u})=\frac{1}{2i}(\exp(\mathrm{ad}(\bar{u}\cdot L+u\cdot\bar{L}))[\bar{u}L,u\bar{L}])_0$$
$$=\frac{1}{2i}\left(\sum_{j=0}^{\infty}\frac{1}{j!}\mathrm{ad}^j(\bar{u}L+u\bar{L})[\bar{u}L,u\bar{L}]\right)_0. \quad \text{(III.31)}$$

(If A, B are two vector fields then $(\mathrm{ad}^jA)B=[A,[A\ldots[A,B]\ldots]]$ where A appears j times).

It is easy to see that $q(u,\bar{u})$ is a real formal power series in u and $\bar{u}$ with vector valued coefficients and with no pure terms.

Consider the second order differential operator

$$H=\sum_{\substack{1\leq j\leq n\\1\leq k\leq n}}u_j\bar{u}_k\frac{\partial}{\partial u_j}\frac{\partial}{\partial\bar{u}_k}. \quad \text{(III.32)}$$

Lemma III.2. *Let* $q(u,\bar{u})$ *and* H *be defined by* (III.31) *and* (III.32) *respectively. If* $p(u,\bar{u})$ *is the unique formal power series satisfying*

$$Hp(u,\bar{u})=q(u,\bar{u}), \quad \text{(III.33)}$$

and having no pure terms, then

$$\phi_{(n)}(z,\bar{z})\equiv p(z,\bar{z}).$$

Proof. The proof is a straightforward computation left to the reader. □

Remark III.1. Assume that $(\tilde{L}_1,\ldots,\tilde{L}_n)$ is another canonical basis of $\mathbb{L}$ (Def. I.3). Let $\tilde{q}(u,\bar{u})$ be the formal power series defined by (III.31) with L_j replaced by $\tilde{L}_j$. Let $\tilde{p}(u,\bar{u})$ be the corresponding solution of (III.33) i.e.

$$H\tilde{p}(u,\bar{u})=\tilde{q}(u,\bar{u}),$$

$\tilde{p}$ having no pure terms. Then it follows from Proposition I.2 that there exists a holomorphism $u\mapsto F(u)$ defined in a neighborhood of 0 in $\mathbb{C}^n$, $F(0)=0$, such that

$$\tilde{p}(u,\bar{u})=p(F(u),\overline{F(u)}).$$

Therefore the type at $(0,\sigma^0)$, and the validity of the sector property at σ^0 are independent of the choice of a canonical basis of $\mathbb{L}$. □

The following results are corollaries of Theorem III.1.

Theorem III.2. *Let* $\sigma^0\in T_0^*V\backslash\{0\}$. *Assume that* $(0,\sigma^0)$ *is of finite type then at least one of the following conditions holds*

(i) *Every* CR *distribution* h *in* Ω *is hypo-analytic at* $(0,\sigma^0)$.

(ii) *Every* CR *distribution* h *in* Ω *is hypo-analytic at* $(0,-\sigma^0)$.

In addition, if the type at $(0,\sigma^0)$ *is odd then both* (i) *and* (ii) *hold.*

Proof. Since $(0,\sigma^0)$ is of finite type, the power series $\sigma^0\cdot\phi_{(n)}$ is not identically zero. Thus we have,

$$\sigma^0\cdot\phi_{(n)}(z,\bar{z})=P_m(z,\bar{z})+O(|z|^{m+1}),$$

where P_m is a real valued homogeneous polynomial in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, of degree m not identically zero and without pure terms.

We can write

$$P_m(z, \bar{z}) = \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha| \geqq 1, |\beta| \geqq 1}} c_{\alpha,\beta} z^\alpha \bar{z}^\beta = \sum_{\substack{k+l=m \\ k \geqq 1, l \geqq 1}} Q_{k,l}(z, \bar{z})$$

with

$$Q_{k,l}(z, \bar{z}) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} c_{\alpha,\beta} z^\alpha \bar{z}^\beta .$$

Thus for any $a \in \mathbb{C}^n$, $t \in \mathbb{C}$,

$$P_m(at) = \sum_{\substack{k+l=m \\ k \geqq 1, l \geqq 1}} Q_{k,l}(a, \bar{a}) t^k \bar{t}^l .$$

The polynomials $Q_{k,l}$ are not all identically zero, so we can find $a \in \mathbb{C}^n \backslash 0$ such that for some (k, l), $Q_{k,l}(a, \bar{a}) \neq 0$. Therefore $P_m(at) = |t|^2 Q_{m-2}(t)$, where $Q_{m-2}(t)$ is homogeneous of degree $m-2$. We conclude there exists a sector $\mathcal{S}$ in the t-plane satisfying one of the conditions

$$P_m(at)|_{\mathcal{S}} > 0 \quad \text{or} \quad P_m(at)|_{\mathcal{S}} < 0$$

and

$$\text{angle}\,\mathcal{S} \geqq \frac{\pi}{m-2} > \frac{\pi}{m} .$$

The sector property of Definition III.1 is therefore satisfied for at least one of the vectors σ^0 and $-\sigma^0$. If m is odd then both vectors satisfy the sector property. The desired conclusions follow now immediately from Theorem III.1. □

Corollary III.1. *Assume that the origin is of finite type and that, for every* $\sigma \in \mathbb{R}^l \backslash \{0\}$, *the type at* $(0, \sigma)$ *is odd. Then any* CR *distribution in* Ω *extends holomorphically at the origin.*

Proof. The proof is an immediate consequence of the second part of Theorem III.2. □

For points of finite type on rigid CR manifolds we have the following result.

Theorem III.3. *Let* M *be the submanifold of* $\mathbb{C}^{n+l}$ *defined by* (II.2), *and assume that the origin is of finite type. Then any* CR *distribution defined in a neighborhood of the origin in* M *extends holomorphically to a wedge of the form* (II.14).

Proof. Let h be a CR distribution defined in $\Omega = U \times V$. Making use of Theorem II.2, in order to prove Theorem III.3, it suffices to show that

$$WF_{ha,0} h \subset \Gamma , \tag{III.34}$$

where Γ is a strictly convex closed cone contained in $\mathbb{R}^l$.

Define the following set:

$$S=\{\sigma\in\mathbb{R}^l\setminus\{0\};\ \sigma \text{ does not satisfy the sector property}\}. \tag{III.35}$$

Theorem III.1 states

$$WF_{ha,0}h\subset S. \tag{III.36}$$

We have the following:

Lemma III.3. *If the origin is of finite type, then the set S defined by* (III.35) *is a convex cone of* $\mathbb{R}^l\setminus\{0\}$.

The proof of (III.34) is an easy consequence of (III.36) and Lemma III.3. Indeed let Γ be the convex hull of $WF_{ha,0}h\cup\{0\}$. Since the latter is closed in $\mathbb{R}^l$ and since the convex hull of a closed cone is closed, we conclude that Γ is a closed convex cone of $\mathbb{R}^l$. On the other hand, Lemma III.3 implies that $S\cup\{0\}$ is a convex cone of $\mathbb{R}^l$ which does not contain any line. Therefore using (III.36) we get

$$\Gamma\subset S\cup\{0\},$$

and we conclude that no line is contained in Γ. Thus Γ is strictly convex, which proves (III.34).

It only remains to prove Lemma III.3.

Proof of Lemma III.3. Theorem III.2 and its proof show that if $\sigma\in S$ then $-\sigma\notin S$. Therefore if σ, $\sigma'\in S$ then $\sigma+\sigma'\neq 0$; we must show that $\sigma+\sigma'\in S$.

Let $\gamma:\zeta\to\gamma(\zeta)\in\mathbb{C}^n$ be a holomorphic curve defined in a neighborhood of 0 in $\mathbb{C}$, and $\gamma(0)=0$. Consider the formal real power series

$$f(\zeta,\bar\zeta)=\sigma\cdot\phi_{(n)}(\gamma(\zeta),\overline{\gamma(\zeta)})$$
$$f'(\zeta,\bar\zeta)=\sigma\cdot\phi_{(n)}(\gamma(\zeta),\overline{\gamma(\zeta)}).$$

We must show that either $f+f'\equiv 0$ or

$$f(\zeta,\bar\zeta)+f'(\zeta,\bar\zeta)=Q_{m_0}(\zeta,\bar\zeta)+O(|\zeta|^{m_0+1}) \tag{III.37}$$

with $m_0\in\mathbb{Z}_+$, $m_0\geqq 2$, and Q_{m_0} is a real homogeneous polynomial of degree m_0 which does not satisfy (III.3), (III.4).

Write $(m,m'\in\mathbb{Z}_+\cup\{+\infty\})$

$$f(\zeta,\bar\zeta)=P_m(\zeta,\bar\zeta)+O(|\zeta|^{m+1}),$$
$$f'(\zeta,\bar\zeta)=P'_{m'}(\zeta,\bar\zeta)+O(|\zeta|^{m'+1})$$

(if $m=\infty$ then $f\equiv 0$, same for m'), where P_m and $P_{m'}$ are real homogeneous polynomials of degree m and m' respectively. Since $\sigma\in S$ (resp. $\sigma'\in S$), if m (resp. m') is finite then P_m (resp. $P'_{m'}$) does not satisfy (III.3), (III.4).

It is clear that if $m\neq m'$ then (III.37) holds and Q_{m_0} does not satisfy (III.3), (III.4). If $m=m'=\infty$ then $f+f'\equiv 0$. It only remains to consider the case, $m=m'<\infty$.

Note that if $P_m(\zeta,\bar\zeta)=r^m g(\theta)$ is a real homogeneous polynomial of degree m, ((r,θ) are the usual polar coordinates in the plane), then the reader can check (see proof in [5], Lemma I.2) that P_m does not satisfy (III.3), (III.4) if and only if

$$g(\theta)+g\left(\theta+\frac{\pi}{m}\right)\geqq 0, \quad \forall\theta\in S^1. \tag{III.38}$$

Assume $m=m'$ and write

$$P_m(\zeta,\bar\zeta)=r^m g(\theta), \qquad P'_m(\zeta,\bar\zeta)=r^m g'(\theta).$$

Since both g and g' satisfy (III.38) we conclude that $g+g'$ also satisfies the same inequality, and therefore $P_m+P'_m$ does not satisfy (III.3), (III.4). Since the holomorphic curve γ is arbitrary we have shown that $\sigma+\sigma'$ does not satisfy the sector property i.e. $\sigma+\sigma'\in S$. The proof of the lemma is complete. □

Example III.1. Consider the generic submanifold M of $\mathbb{C}^4$ of codimension 2, defined by ($z\in\mathbb{C}^2$, $w\in\mathbb{C}^2$)

$$\operatorname{Im} w=\phi(z,\bar z),$$

with

$$\phi_1(z,\bar z)=|z_1^2-z_2^3|^2-|z_1|^{10},$$
$$\phi_2(z,\bar z)=|z_1|^2-|z_2|^4.$$

Using Theorem III.1 we can show that any CR distribution h defined on M is hypo-analytic at $(0,\sigma)$ for every $\sigma\in\mathbb{R}^2\backslash\{0\}$, and thus it extends holomorphically at the origin of $\mathbb{C}^4$.

Indeed if $\sigma=(\sigma_1,\sigma_2)\in\mathbb{R}^2\backslash\{0\}$ and $\sigma_2>0$, we can take $\gamma(\zeta)=(0,\zeta)$ in Theorem III.1. If $\sigma_2<0$, take $\gamma(\zeta)=(\zeta,0)$. If $\sigma_2=0$ and $\sigma_1<0$, take $\gamma(\zeta)=(0,\zeta)$. Finally if $\sigma_2=0$ and $\sigma_1>0$, take $\gamma(\zeta)=(\zeta^3,\zeta^2)$.

Note that the latter is a singular curve. The reader can easily check that we cannot prove the hypo-analyticity of h at $(0,\sigma)$, when $\sigma=(1,0)$, by taking only regular holomorphic curves in Theorem III.1, applied to this example. □

Local holomorphic extension of CR functions to one side of a hypersurface in $\mathbb{C}^2$ can yield microlocal hypo-analyticity for rigid CR manifolds with any codimension. We shall give a result in this direction. We start with the following definition.

Definition III.2. Let $P_m(\zeta,\bar\zeta)$ be a real homogeneous polynomial in ζ, $\bar\zeta$ ($\zeta\in\mathbb{C}$) of degree $m\geqq 2$. Let $\sum$ be the hypersurface of $\mathbb{C}^2$ defined by

$$\operatorname{Im}\eta=P_m(\zeta,\bar\zeta), \tag{III.39}$$

where ζ, η denote the coordinates of $\mathbb{C}^2$.

We say that P_m has the *extension property* if for any CR function f defined near the origin on $\sum$, there exists a neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^2$ such that f extends holomorphically to the side of $\sum$ defined by

$$\{(\zeta,\eta)\in\mathcal{O}, \operatorname{Im}\eta<P_m(\zeta,\bar\zeta)\}. \tag{III.40}$$

We can state the following result:

Theorem III.4. *Let $\sigma^0 \in \mathbb{R}^l \setminus \{0\}$ and assume that there exists a holomorphic curve $\zeta \mapsto \gamma(\zeta) \in \mathbb{C}^n$ defined in a neighborhood of* 0 *in* $\mathbb{C}$, *such that*

$$\sigma^0 \cdot \phi_{(m)}(\gamma(\zeta), \overline{\gamma(\zeta)}) = P_m(\zeta, \bar{\zeta}) + O(|\zeta|^{m+1}), \tag{III.41}$$

where P_m is a homogeneous polynomial which has the extension property (Def. III.2). *Then any* CR *distribution defined in Ω is hypo-analytic at* $(0, \sigma^0)$.

If P_m satisfies (III.3) and (III.4) then it has the extension property (Theorem III.1, see also [5]), therefore Theorem III.4 can be considered as a generalization of Theorem III.1. Other sufficient conditions for a polynomial to have the extension property could be found in Bedford [6] and Fornaess-Rea [9].

The following lemma will be needed in the proof of Theorem III.4.

Lemma III.4. *Let $P_m(\zeta, \bar{\zeta})$ be a homogeneous polynomial which has the extension property* (Def. III.2), *and $\sum$ the hypersurface of $\mathbb{C}^2$ defined by* (III.39). *For every $\varrho > 0$ sufficiently small there exists $C > 0$ such that if $H(\zeta, \eta)$ is holomorphic in a neighborhood in $\mathbb{C}^2$ of the set*

$$B = \{(\zeta, \eta) \in \mathbb{C}^2;\ |\zeta| \leqq 1,\ |\eta| \leqq 1\}, \tag{III.42}$$

then

$$\sup_{|\zeta| \leqq \varrho} |H(\zeta, -i\varrho)| \leqq C \sup_{B \cap \Sigma} |H(\zeta, \eta)|. \tag{III.43}$$

Proof. Let E be the Banach space of CR functions defined on $B \cap \sum$ and continuous on its closure $\overline{B \cap \sum}$. Since every function f in E extends to a side of $\sum$ of the form (III.40), using Baire's category theorem as in [3] (proof of Theorem 5.2), we can see that the neighborhood $\mathcal{O}$ in (III.40) can be chosen independent of $f \in E$.

If $\varrho > 0$ is small enough the set

$$\{(\zeta, \eta) \in \mathbb{C}^2, |\zeta| \leqq \varrho, \eta = -i\varrho\}$$

is a compact subset of the open set defined by (III.40). Estimate (III.43) then follows from a standard use of the closed graph theorem. (In fact the constant C in (III.43 can be taken to be one.) □

Proof of Theorem III.4. After repeating Part 1 of the proof of Theorem III.1, we can assume that the curve γ in (III.41) is given by

$$\gamma(\zeta) = (\zeta, 0, \ldots, 0),$$

and that $\sigma^0 = (1, 0, \ldots, 0)$.

We make the change of coordinates (III.19) where the small dilation parameter δ is to be determined later.

By making $\delta > 0$ small, we can assume that the modified FBI transform $F(z, w, \sigma)$ and the holomorphic function $G(z, w, \sigma)$ given by Lemma II.1 are defined for $|z| \leqq 2$, $|w| \leqq 2$.

We shall apply Lemma III.4 to the function

$$H(\zeta, \eta; z', w, \sigma) = G(z, \tilde{w}, \sigma)\Big|_{\substack{z_1 = \zeta \\ \tilde{w}_1 = \eta + i\varrho + w_1,\ \tilde{w}' = w'}} \tag{III.44}$$

($z=(z_1,z')$, $w=(w_1,w')$, $\tilde{w}=(\tilde{w}_1,\tilde{w}')$). We think of $(\zeta,\eta)\in C^2$, $|\zeta|\leqq 1$, $|\eta|\leqq 1$, as variables, and of z', w, σ as parameters.

In fact for all practical purposes we can replace G in (III.44) by F. Indeed in doing so we make an error of exponential decrease in $|\sigma|$, since $F-G$ satisfies (II.21) (with C independent of δ).

After replacing G by F in (III.44) as we have said, the left hand side of (III.43) becomes for each z', w, σ, $|z'|\leqq\frac{1}{2}$, $|w|\leqq\frac{1}{2}$, $\sigma\in\mathbb{R}^l$,

$$\sup_{|z_1|\leqq\varrho}|F(z,w,\sigma)| \tag{III.45}$$

which is the quantity we want to estimate.

Now we estimate the right hand side of (III.43) (after again replacing G by F in (III.44)). We have, for z', w, σ fixed as above,

$$\sup_{(\zeta,\eta)\in B\cap\Sigma}|H(\zeta,\eta;z',w,\sigma)|$$
$$\leqq C\int\sup_{\substack{|\zeta|\leqq 1\\ |\mathcal{R}e\,\eta|\leqq 1}} e^{-|\sigma|E(\zeta,\mathcal{R}e\,\eta,w,\tilde{s},\sigma/|\sigma|)}|\chi(\tilde{s})|d\tilde{s},$$

with C independent of δ, z', w, σ, and

$$E\left(\zeta,\mathcal{R}e\,\eta,w,\tilde{s},\frac{\sigma}{|\sigma|}\right)$$
$$=\frac{\sigma_1}{|\sigma|}[\operatorname{Im}w_1+\varrho+P_m(\zeta,\bar{\zeta})-\phi_1(\zeta,z',\bar{\zeta},\bar{z}')]$$
$$+\frac{\sigma'}{|\sigma|}[\operatorname{Im}w'-\phi'(\zeta,z',\bar{\zeta},\bar{z}')]+(\mathcal{R}e\,w_1+\mathcal{R}e\,\eta-\tilde{s}_1)^2$$
$$-(\operatorname{Im}w_1+\varrho+P_m(\zeta,\bar{\zeta})-\phi_1)^2+(\mathcal{R}e\,w'-\tilde{s}')^2-(\operatorname{Im}w'-\phi')^2.$$

We have used the notation $w=(w_1,w')$, $\tilde{s}=(\tilde{s}_1,\tilde{s}')$, $\phi=(\phi_1,\phi')$ etc....

Making use of (III.20) we get

$$E(\zeta,\mathcal{R}e\,\eta,0,\tilde{s},\sigma^0)=\varrho-\varrho^2+(\mathcal{R}e\,\eta-\tilde{s}_1)^2+\tilde{s}'^2+0(\delta). \tag{III.47}$$

From now on we fix ϱ, $0<\varrho<1$, so that (III.43) holds, and then $\delta>0$ so that the right hand side of (III.47) is >0 for $\tilde{s}\in\operatorname{supp}\chi$.

We conclude from (III.46) and (III.47) that there exists $r>0$ such that, if $\left|\frac{\sigma}{|\sigma|}-\sigma^0\right|<r$, $|w|<r$ and $|z'|\leqq\frac{1}{2}$, then the left hand side of (III.46) is of exponential decay in $|\sigma|$. Thus, so is the quantity (II.45), by using (III.43). The proof of Theorem III.4 is now complete. □

Remark III.1. For tube CR manifolds (which are special cases of rigid manifolds, see Sect. I) defined by

$$\operatorname{Im}w=\phi(y)$$

($z=x+iy\in\mathbb{C}^n$, $w\in\mathbb{C}^l$) with ϕ smooth and satisfying $\phi(0)=0$, $d\phi(0)=0$, sufficient conditions for hypo-analyticity at $(0,\sigma)$, $\sigma\in\mathbb{R}^l\setminus\{0\}$, of CR distributions were given in [4].

When ϕ is real analytic, a necessary and sufficient condition for the hypo-analyticity of any CR distribution at $(0,\sigma)$ is that in any neighborhood of 0 in $\mathbb{R}^n$ the function $y\mapsto\sigma\cdot\phi(y)$ takes strictly negative values. We refer to [4] for the proof of this result. One can also give a different proof by means of the simplified FBI integral used throughout this paper. This is left to the reader.

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Oblatum 9-IV-1985