Cauchy-Riemann functions on manifolds of higher codimension in complex space

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§1. Introduction and main results

A smooth manifold M is called CR if there exists a subbundle ℱ of CTM, the complexified tangent bundle of M, such that

\[ ℱ \cap \bar{ℱ} = 0 \quad \text{and} \quad [ℱ, ℱ] \subset ℱ. \]

(1.1)

A function or distribution on M is called CR if it is annihilated by all the sections of ℱ. If dim₂ M = 2n + l and dimℂ ℱ = n then we say that M is of CR dimension n and CR codimension l.

If (M, ℱ) is a CR manifold then N is called a CR submanifold of M (of the same CR dimension) if N is a submanifold of M satisfying

\[ ℱ|_N \subset CTN. \]

(1.2)

Note that it follows from (1.1) and (1.2) that

\[ 2 \text{dim}_ℂ ℱ \leq \text{dim}_R N \leq \text{dim}_R M, \]

(1.3)

and M and N have the same CR dimension. In what follows it will be understood that by a CR submanifold of M we will mean that (1.2) is satisfied.

We introduce the following definition.

Definition. If M is a CR manifold then M is minimal at \( m_0 \in M \) if there is no CR submanifold N containing \( m_0 \) with \( \text{dim}_R N < \text{dim}_R M \).

The importance of this notion of minimality was introduced by Tumanov [14], who showed that if M is a generic embedded submanifold of \( \mathbb{C}^{n+1}_r \) and \( m_0 \in M \) then if M is minimal at \( m_0 \) every CR function on M, defined in a neighborhood of \( m_0 \), is the boundary value of a holomorphic function in an open wedge of \( \mathbb{C}^{n+1}_r \) of edge M.

It is easy to see that if M is of finite type in the sense of Bloom-Graham [6] then M is minimal at \( m_0 \). Indeed finite type means that the Lie algebra of the sections of

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and their conjugates span the complexified tangent space at $m_0$. If these are also tangent to $N$ then the germs of $N$ and $M$ are the same at $m_0$. If $(M, \mathcal{T})$ is real analytic then minimal implies finite type; indeed, if $M$ is not of finite type, then the Nagano leaf [9] passing through $m_0$ would be a proper real analytic CR submanifold, contradicting minimality. If $M$ is only smooth, it could be minimal without being of finite type.

The main result of this paper is to prove that Tumanov's minimality condition is also necessary for holomorphic extendability. Recall that a CR manifold $M$ of CR dimension $n$ and CR codimension $l$ is locally embeddable if in a neighborhood of every point there exist $n + l$ smooth CR functions with linearly independent differentials.

**Theorem 1.** Let $M$ be a locally embeddable CR manifold which is not minimal at $m_0 \in M$ and $N$ a CR submanifold of $M$ containing $m_0$, with $\dim_{\mathbb{R}} N < \dim_{\mathbb{R}} M$. Then there exists a CR distribution $T$ defined in a neighborhood $U$ of $m_0$ with $\text{supp } T = N \cap U$.

A slightly more general version of Theorem 1 is given in §6 as Theorem 4.

A submanifold $M$ of $\mathbb{C}^{n+l}$ of dimension $2n + l$ is called generic if it is locally defined, near $m_0 \in M$, by $\rho_j = 0$, $j = 1, \ldots, l$, where the $\rho_j$ are smooth, real functions such that their complex differentials $\partial \rho_j$ are linearly independent. Such a manifold $M$ equipped with $\mathcal{T}$, the induced tangential Cauchy-Riemann bundle of $\mathbb{C}^{n+l}$, is a CR manifold of CR dimension $n$ and CR codimension $l$.

Recall that a wedge of edge $M$ is an open set of $\mathbb{C}^{n+l}$ of the form

$$W(\mathcal{O}, \Gamma) = \{Z \in \mathcal{O}, \rho(Z) \in \Gamma\},$$

where $\mathcal{O}$ is a sufficiently small open neighborhood of $m_0$ in $\mathbb{C}^{n+l}$, $\Gamma$ an open cone of $\mathbb{R}^l$, and $\rho = (\rho_1, \ldots, \rho_l)$ where the $\rho_j$ are the defining functions of $M$ near $m_0$ as above. We can now state our nonextendability result.

**Theorem 2.** If $M$ is an embedded generic CR manifold which is not minimal at $m_0$ then there is a smooth CR function defined in a neighborhood of $m_0$ which does not extend holomorphically to any wedge with edge $U$, where $U$ is a neighborhood of $m_0$ in $M$.

Combining Theorem 2 with the sufficient condition of Tumanov we obtain the following.

**Corollary 1.** If $M$ is an embedded generic CR manifold and $m_0 \in M$, then every germ of a CR function at $m_0$ extends holomorphically to a wedge of edge $M$ if and only if $M$ is minimal at $m_0$.

The following result shows uniqueness of a CR submanifold of minimal dimension and gives two intrinsic characterizations of such a submanifold.

**Theorem 3.** If $M$ is a generic CR manifold and $m_0 \in M$, there is a unique germ $N_0$ of a CR submanifold contained in $M$, $m_0 \in N_0$, of minimal dimension. Also $N_0$ may be described as follows.

(i) For every sufficiently small neighborhood $U$ of $m_0$ in $M$, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ consists of all points $m \in U'$ which can be reached from $m_0$ by a finite sequence of integral curves contained in $U$ of sections of $\text{Re } \mathcal{T}$.
(ii) For every sufficiently small neighborhood $U$ of $m_0$ in $M$, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ is the union of sets of the form $Z(bD)$, where $D$ is the unit disc in $C$ and $Z: D^* \to C^{n+1}$ is continuous, holomorphic in $D$ and satisfying $m_0 \in Z(bD) \subset U \cap M$.

We note that the uniqueness of $N_0$ and its characterization given by (i) in Theorem 3 hold even in the case of a nonembeddable CR manifold.

The study of extendability of CR functions began with the celebrated paper of Hans Lewy [8] in the '50's in which he showed extendability to one side from a strictly pseudoconvex hypersurface. Other sufficient conditions were subsequently obtained by a number of mathematicians. Recently, a necessary and sufficient condition for every CR function to extend to at least one side of a hypersurface of class $C^2$ was obtained by Trépreau [12]. The authors obtained some sufficient conditions [2] for extendability in the higher codimension case; these results have been generalized by Tumanov, as explained above. They also obtained in the real analytic case (loc. cit.) necessary conditions, i.e. that $M$ must be of finite type, in order that extendability to a wedge hold. Since Tumanov proved that finite type is also sufficient for extendability, the case of real analytic generic manifolds was completed by Tumanov's work. Our Theorem 2 above now settles the smooth case.

Singular solutions for real analytic vector fields supported on submanifolds were constructed by Zachmanoglou [15] in the study of analytic hypoellipticity and propagation of zeroes. Our solutions in Theorem 1 have the same general form as his, but in our case the vector fields are not real analytic.

We would like to thank Jean-Marie Trépreau for several interesting conversations and, in particular, for suggesting to us the formulation of Theorem 3, and the use of Sussmann's work [11] in its proof.

§2. Local coordinates

We assume here that $M$ is an embedded generic CR manifold which is not minimal and $N$ a CR submanifold of $M$ passing through $m_0$. We first introduce local coordinates around $m_0$ which will be used in the proof of Theorem 1. Assume that

$$\dim_RM = 2n + l, \quad \dim_R N = 2n + l_1 \text{ and } \dim_C V^- = n,$$

with $0 \leq l_1 < l$.

We may choose a local embedding so that $M$ is parametrized in $C^{n+1}$, where the coordinates are denoted by $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_l)$, by

$$\text{Im } w_j = \phi_j(z, \bar{z}, s), \quad 1 \leq j \leq l, \quad s = \text{Re } w, \quad \phi(0) = d\phi(0) = 0.$$  

A basis of CR vector fields $L_j, j = 1, \ldots, n$, can then be written in the form

$$L_j = \frac{\partial}{\partial z_j} + \sum_{1 \leq k \leq l} x_{jk}(z, \bar{z}, s) \frac{\partial}{\partial s_k},$$

with $x_{jk}(0) = 0$.

Assume that $N$ is given by $\rho_j(z, \bar{z}, s) = 0, 1 \leq j \leq l_2$, where $l_2 = l - l_1$ and the differentials $d\rho_j$ are linearly independent. Since the $L_j$ are tangent to $N$, we must
have $L_j \rho_k = 0$ on $N$, and $\tilde{L}_j \rho_k = 0$ on $N$, since the $\rho_j$ are real valued. Hence, by (2.3),
\[ \frac{\partial \rho_j}{\partial x_k}(0) = \frac{\partial \rho_j}{\partial y_k}(0) = 0. \]
We conclude that $\text{rank} \left( \frac{\partial \rho_j}{\partial \mathcal{S}_k} \right)_{1 \leq j \leq l_2} = l_2$. After a linear change of variables and using the implicit function theorem, we can assume that $N$ is given as a subset of $M$ by
\[ s_{1_j + j} = \psi_j(z, \bar{z}, s_1, \ldots, s_{l_1}), \quad 1 \leq j \leq l_2, \quad \psi_j(0) = d\psi_j(0) = 0. \tag{2.4} \]
Put $t_j = s_{1_j + j} - \psi_j(z, \bar{z}, s)$, $j = 1, \ldots, l_2$, and take for coordinates on $M$,
\[(x, y, s, t) = (x, y, s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}). \]
In these coordinates the $L_j$ become
\[ L_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^{l_1} \beta_{jk} \frac{\partial}{\partial s_k} + \sum_{1 \leq k \leq l_2} \mu_{jk} t_k \frac{\partial}{\partial t_r}, \tag{2.5} \]
where $\beta_{jk}$ and $\mu_{jk}$ are functions of $(x, y, s, t)$. We introduce the vector fields $L_j^0$, obtained from $L_j$ by setting $t = 0$:
\[ L_j^0 = \frac{\partial}{\partial z_j} + \sum_{k=1}^{l_1} \beta_{jk}(x, y, s, 0) \frac{\partial}{\partial s_k}. \tag{2.6} \]
The $L_j^0$ form a basis of the sections of the CR bundle restricted to $N$. We write
\[ f_j = s_j + i\phi_j(z, \bar{z}, s, t + \psi(x, y, s)), \quad 1 \leq j \leq l_1, \tag{2.7} \]
\[ g_j = t_j + \psi_j(z, \bar{z}, s) + i\phi_j(z, \bar{z}, s, t + \psi(z, \bar{z}, s)), \quad 1 \leq j \leq l_2, \tag{2.8} \]
where the $\phi_j$ are given by (2.2) and $\psi_j$ by (2.4). Note that we have $L_j f_k = L_j g_k = 0$.
For a smooth function $h(x, y, s, t)$ we write
\[ h(x, y, s, t) = h^0(x, y, s) + \sum_{k=1}^{l_1} h^k(x, y, s) t_k + O(|t|^2). \]
Since $L_j f_k = 0$ we conclude that $L_j^0 f_k^0 = 0$, $1 \leq j \leq n$, $1 \leq k \leq l_1$. As in [3] we define $l_1$ vector fields $R_j$ of the form $R_j = \sum_{k=1}^{l_1} a_{jk}(x, y, s) \frac{\partial}{\partial s_k}$ satisfying the following relations:
\[ [L_k^0, R_j] = 0, \quad [R_j, R_p] = 0, \quad R_j f_p^0 = \epsilon_{jp}, \tag{2.8} \]
where $\epsilon_{jp}$ is the Kronecker symbol. It follows from (2.7) and (2.8) that we have
\[ L_j^0 = \frac{\partial}{\partial z_j} - \sum_{k=1}^{l_1} f_k^0 \frac{\partial}{\partial s_k}. \tag{2.9} \]
Since the $R_k$ are linear independent we obtain, for $j = 1, \ldots, n$,
\[ L_j = L_j^0 + \sum_{1 \leq k \leq l_1} \beta_{jk} t_k R_k + \sum_{1 \leq k \leq l_2} \mu_{jk} t_k \frac{\partial}{\partial t_r}. \tag{2.10} \]
§3. Construction of singular solutions. Proof of Theorem 1

We shall use the coordinates \((x, y, s, t)\) introduced in §2 and take a basis for the sections of \(V\) of the form \((2.10)\). We shall construct a distribution solution \(T\) of the system of equations \(L_T = 0\) of the form \(T = V(z, \tilde{z}, s) \delta(t)\), where \(\delta(t) = \delta(t_1) \otimes \ldots \otimes \delta(t_n)\) is the dirac measure at the origin in \(\mathbb{R}^k\) and \(V\) is a smooth function nonvanishing at 0.

Using the relation
\[
t_k \frac{\partial}{\partial t_r} \delta(t) = -\varepsilon_{kr} \delta(t)
\]
we conclude that \(V\) must satisfy the equations
\[
L_j^0 V - \sum_{1 \leq r \leq l_1} \mu^0_{jr}(z, \tilde{z}, s) V = 0, \quad 1 \leq j \leq n . \tag{3.1}
\]

Since \(V\) is nonvanishing, equation \((3.1)\) is equivalent to
\[
L_j^0 (\log V) - \sum_{1 \leq r \leq l_1} \mu^0_{jr}(z, \tilde{z}, s) = 0 . \tag{3.2}
\]

Therefore, it suffices to show that \(\sum_{1 \leq r \leq l_1} \mu^0_{jr}(z, \tilde{z}, s)\) is in the range of \(L_j^0\). We begin with the calculation of the coefficients \(\mu^0_{jr}\). It follows from \((2.10)\), that if \(L_j h = 0\) then
\[
L_j h = \left( L_j^0 + \sum_{1 \leq k \leq l_2} \lambda^0_{jk} t_k R_p + \sum_{1 \leq k \leq l_2} \mu^0_{jk} t_k \frac{\partial}{\partial t_r} \right) \left( h^0 + \sum_{1 \leq k \leq l_1} h^k t_k \right) + O(t^2) , \tag{3.3}
\]
where we have used the notation \(h = h^0 + \sum_{1 \leq k \leq l_1} h^k t_k + O(t^2)\) as in §2.

Setting the coefficient of \(t_k\) equal to 0 in \((3.3)\) we obtain
\[
L_j^0 h^k + \sum_{1 \leq r \leq l_1} \lambda^0_{jk} R_p h^0 + \sum_{1 \leq r \leq l_1} \lambda^0_{jr} h^r = 0 . \tag{3.4}
\]

Taking for \(h\) the functions \(f_q, 1 \leq q \leq l_1\), defined in \((2.7)\), and using the relations \(R_p f^0_k = \varepsilon_{pk}\) in \((2.8)\), we obtain,
\[
\lambda^0_{jk} = -L_j^0 f^k_p - \sum_{1 \leq r \leq l_1} \mu^0_{jr} f^r_p , \quad 1 \leq j \leq n , \quad 1 \leq k \leq l_2, \quad 1 \leq p \leq l_1 . \tag{3.5}
\]

In \((3.4)\) we now replace \(h\) by \(g_q, 1 \leq q \leq l_2\), defined in \((2.7)\) and \(\lambda^0_{jk}\) by its expression given in \((3.5)\). We obtain
\[
L_j^0 g_q + \sum_{1 \leq p \leq l_1} \left( -L_j^0 f^k_p - \sum_{1 \leq r \leq l_1} \mu^0_{jr} f^r_p \right) R_p g^0_q + \sum_{1 \leq r \leq l_1} \mu^0_{jr} g^r_q = 0 . \tag{3.6}
\]

Rewriting \((3.6)\) by collecting the coefficients of \(\mu^0_{jr}\) we have
\[
\sum_{1 \leq r \leq l_1} \mu^0_{jr} \left( g_q - \sum_{1 \leq p \leq l_1} f^r_p R_p g^0_q \right) = -L_j^0 g_q^k + \sum_{1 \leq p \leq l_1} \left( L_j^0 f^r_p \right) R_p g^0_q . \tag{3.7}
\]
Let $\mathcal{A}$ be the matrix $(g_q^r = - \sum_{1 \leq r \leq l_1} T_p R_p g_q^0)_1 \leq r, q \leq l_2$ and $D(z, \bar{z}, s) = \det \mathcal{A}$. By Cramer's rule applied to (3.7) we obtain

$$
\mu_{jr}^0 = \frac{D_{kr}}{D}, \quad 1 \leq j \leq n, \quad 1 \leq k, r \leq l_2,
$$

(3.8)

where $D_{kr}$ is the determinant of the matrix obtained by substituting the $r$th column in $\mathcal{A}$ by the vector $(-L^0_1 g_q^r + \sum_{1 \leq r \leq l_1} (L^0_1 f_k^r) R_p g_q^0)$, $1 \leq q \leq l_2$. Summing (3.8) over $r$ we obtain

$$
\sum_{1 \leq r \leq l_2} \mu_{jr}^0 D_{jr} = D^{-1} \sum_{1 \leq r \leq l_2} D_{jr} = -\frac{L^0_1 D}{D} = -L^0_1 \log D.
$$

(3.9)

Here we have used the identity $(L^0_1 f_k^r) R_p g_q^0 = L^0_1 (f_k^r R_p g_q^0)$ which is a consequence of $[L^0_1, R_p] = 0$ (cf (2.8)) and $L^0_1 g_q^0 = 0$.

Note that $D(0) \neq 0$. From (3.2) and (3.9) it follows that a desired solution of (3.1) is given by $V = D^{-1}$. This completes the proof of Theorem 1.

§4. Proof of Theorem 2

We shall first show that the distribution solution $T$ of Theorem 1 is not the boundary value of a holomorphic function in any wedge with edge $M$. We shall then use $T$ to construct a CR function of class $C^k$ with the same property. We use the closed graph theorem, together with a result from [4] to show that there exists a smooth CR function which does not extend.

We review here some basic properties of boundary values of holomorphic functions in wedges. If $M$ is an embedded generic CR manifold of $C^{n-1}$ of dimension $2n + l$ given by $\text{Im } w = \phi(z, \bar{z}, s)$ as in (2.2), and $\Gamma$ an open strictly convex cone of $R^l$ and $\mathcal{C}$ an open neighborhood of $0$ in $C^{n-1}$, we define the wedge $\mathcal{W}(\mathcal{C}, \Gamma)$ by

$$
\mathcal{W}(\mathcal{C}, \Gamma) = \{(z, w) \in \mathcal{C}, \text{Im } w - \phi(z, \bar{z}, s) \in \Gamma\}.
$$

(4.1)

If $h$ is a holomorphic function in $\mathcal{W}(\mathcal{C}, \Gamma)$ with slow growth at the edge $M$, i.e. $|h(m)| \leq C \text{dist}(m, M)^{-p}$, for $m \in \mathcal{W}(\mathcal{C}, \Gamma)$, then $h$ has a boundary value $b_M h$ as a CR distribution on $M \cap \mathcal{C}$. Furthermore, if $b_M h = 0$ in an open set of $M \cap \mathcal{C}$ and $\mathcal{W}(\mathcal{C}, \Gamma)$ is connected then $h$ vanishes identically. (For a flat wedge see [7], and for the general case see [1] and [2].)

We begin with the following result.

**Proposition 4.2.** Under the assumptions of Theorem 2, for any $k \geq 0$ there is a CR function of class $C^k$ defined in a neighborhood of $m_0$ which does not extend to any wedge with edge $M$ near $m_0$.

**Proof.** The fact that the distribution solution $T$ of Theorem 1 is not the boundary value of a holomorphic function in any wedge with edge $M$ is proved by the following lemma.
Lemma 4.3. Let $M$ be an embedded generic CR manifold and $S$ a CR distribution on $M$, nonzero in any neighborhood of $m_0$, with the following property. For every neighborhood $U$ of $m_0$ in $M$ there is an open set $U' \subset U$ such that $S$ vanishes in $U'$. Then in any neighborhood of $m_0$, $S$ is not the boundary value of a holomorphic function in a wedge with edge $M$.

Proof. Indeed, it follows from the remarks preceding Proposition 4.2 that if $S$ were the boundary value of a holomorphic function $h$ in a connected wedge $\mathcal{W}(\mathcal{C}, \Gamma)$, then, since $S$ vanishes in an open subset of any neighborhood of $m_0$, $h$ would have to vanish identically, which would imply that $S$ is the zero distribution. Since $\Gamma$ is assumed to be strictly convex, the wedge $\mathcal{W}(\mathcal{C}, \Gamma)$ is connected if $\mathcal{C}$ is connected and sufficiently small. This proves the lemma.

To construct the $C^k$ CR function of the proposition we shall use the following. As in §2 we can find $l$ vector fields $D_j$ satisfying, for $1 \leq q \leq n$, $1 \leq j, p \leq l$,

$$[L_q, D_j] = 0, \quad [D_j, D_p] = 0, \quad \text{and} \quad D_j(s_p + i\phi_p(z, \bar{z}, s)) = \epsilon_{jp},$$

where the $\phi_p$ are as in (2.2).

Lemma 4.4. Let $S$ be a CR distribution defined in a neighborhood of $m_0$ in $M$. Then for any $k$ there is a CR function $f \in C^k$ defined near $m_0$ such that

$$S = \left( \sum_{1 \leq j \leq l} D_j^* \right)^k f,$$

where the $D_j$ are the vector fields defined above.

This lemma is a consequence of the representation of distributions annihilated by a system of complex vector fields given in [5] (see also [13]).

We may now prove Proposition 4.2. By Theorem 1 we can find a CR distribution $T$ in an open subset $U$ with support in $N$. By Lemma 4.4 we may write $T = \left( \sum_{1 \leq j \leq l} D_j^* \right)^k f$ with $f$ a CR function of class $C^k$. We reason by contradiction. Assume that $f$ is the boundary value of a holomorphic function $H$ in a wedge $\mathcal{W}(\mathcal{C}, \Gamma)$. Since $D_j$ is of the form

$$D_j = \frac{\partial}{\partial w_j} + \sum_{1 \leq k \leq l} a_{jk} \frac{\partial}{\partial w_k},$$

(see [3]), we have $b \left( \frac{\partial H}{\partial w_j} \right) = D_j(bH)$. Note that $\frac{\partial H}{\partial w_j}$ is of slow growth, by Cauchy's inequalities. We conclude

$$D_j f = b \left( \frac{\partial H}{\partial w_j} \right), \quad T = \left( \sum_{1 \leq j \leq l} D_j^* \right)^k f = b \left( \left( \sum_{1 \leq j \leq l} \left( \frac{\partial^2}{\partial w_j^2} \right) \right)^k H \right).$$

(4.5)

It would then follow from (4.5) that $T$ is the boundary value of a holomorphic function, contradicting the conclusion of Lemma 4.2. The proof of Proposition 4.2 is now complete.

Proof of Theorem 2. We assume by contradiction that every smooth CR function defined near $m_0$ extends holomorphically in some wedge. We claim that for every
open neighborhood \( U \) of \( m_0 \in M \) there exists a wedge \( \mathcal{W}(\mathcal{C}, \Gamma) \) to which every smooth CR function defined on \( U \) extends holomorphically. Indeed, this follows from an argument using the Baire category theorem by a slight modification of the proof of Theorem 7 of [4], where the corresponding result is proved for continuous functions. We shall show that if \( k \) is sufficiently large, then any CR function of class \( C^k \) defined near \( m_0 \) extends to a wedge, contradicting Proposition 4.2.

If \( U \) is a neighborhood of \( m_0 \), let \( E \) be the space of smooth CR functions on \( U \) and \( \mathcal{W}(\mathcal{C}, \Gamma) \) the wedge associated to \( U \) by the claim above and denote by \( \mathcal{H} \) the space of bounded holomorphic functions in \( \mathcal{W}(\mathcal{C}, \Gamma) \). By the closed graph theorem the mapping \( E \to \mathcal{H} \) which to each CR function associates its holomorphic extension is continuous. We can therefore find \( k, U' \) relatively compact in \( U \) and a constant \( C \) such that for all \( f \in E \),

\[
\sup_{z \in \mathcal{W}(\mathcal{C}, \Gamma)} |\tilde{f}(z)| \leq C \sup_{m \in U'} |D^m f(m)|, \tag{4.6}
\]

where \( \tilde{f} \) is the holomorphic extension of \( f \). Since every CR function of class \( C^k \) is locally a limit, in the \( C^k \) topology of entire functions [5], it follows from (4.6) that every CR \( C^k \) function on \( U \) extends holomorphically to \( \mathcal{W}(\mathcal{C}, \Gamma) \). Theorem 2 is then proved by contradiction.

§5. Characterization of minimal CR submanifolds; proof of Theorem 3

The proof of the uniqueness in Theorem 3, as well as the characterization in (i), is essentially based on the following localized version of a result of Sussmann [11]: Let \( \{X_1, \ldots, X_p\} \) be a set of smooth real vector fields defined near the origin in \( \mathbb{R}^q \), and \( U' \) a sufficiently small neighborhood of 0. Then there is a unique submanifold \( N \subset U', 0 \in N \), such that the \( X_j \) are all tangent to \( N \) at every point of \( N \) and for which \( \dim N \) is minimal with this property. Its uniqueness follows from the fact that it is the union of all points in \( U' \) which can be reached by a finite sequence of integral curves, contained in a slightly bigger neighborhood \( U \), of the vector fields \( X_j \).

We now consider the case of a CR manifold \( M \). We take for \( X_1, \ldots, X_{2n} \) a basis for the real and imaginary parts of the local CR vector fields \( L_j \), i.e. for the sections of \( \text{Re}\gamma' \). A submanifold \( N \) of \( M \) is a CR submanifold of the same CR dimension if and only if all the \( X_j \) are tangent to \( N \). Now the uniqueness and the characterization (i) follow from Sussmann's result stated above.

To show that \( N_0 \) can be characterized by (ii), we assume that \( M \) is embedded and use the following result of Tumanov [14]: There is a CR submanifold of \( M \) through \( m_0 \) contained in the union of sets of the form \( Z(bD) \), where \( Z: D \to \mathbb{C}^{n+1} \) is continuous, holomorphic in \( D \) and satisfying \( m_0 \in Z(bD) \subset U \cap M \), and \( U \) is a sufficiently small neighborhood of \( m_0 \) in \( M \). We shall show that the image of \( bD \) under all such holomorphic discs \( Z \) lies in the minimal submanifold \( N_0 \). Hence the proof of Theorem 3 will be completed by the following.

**Lemma 5.1.** Let \( M \) be a generic CR submanifold of \( C^{n+1} \) and \( N \) a CR submanifold containing \( m_0 \). Then there is a neighborhood \( U \) of \( m_0 \) in \( M \) such that for every
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$Z: D \rightarrow C^{*} \rightarrow 1$ with $Z$ holomorphic, continuous in $\overline{D}$ and satisfying

$m_{0} \in Z(bD) \subset U \cap M$, then $Z(bD) \subset N$.

Proof. We use the coordinates $(x, y, s, t) = (x, y, s, s, t, t, \ldots, t, t)$ introduced in §2, so that $N$ is the submanifold $\{t = 0\}$. For a disc $Z$ as in the statement of the lemma, we write $Z(\zeta) = (z(\zeta), w(\zeta))$; we may assume without loss of generality that $Z(1) = m_{0}$. Then

\[ \text{Re} w(\zeta) = - T_{1}(\phi(z(\cdot), \bar{z}(-\cdot), \text{Re} w(\cdot)))\zeta, \quad \zeta \in bD, \tag{5.2} \]

where $\phi$ is given by (2.2) and $T_{1}$ is the harmonic conjugate of $f$, defined on $bD$ vanishing at 1. As in (2.7) we may write

\[ \text{Re} w_{j}(\zeta) = s_{j}(\zeta), \quad 1 \leq j \leq l_{1}, \]

\[ \text{Re} w_{j}(\zeta) = t_{j}(\zeta) + \psi_{j}(z(\zeta), \bar{z}(-\zeta), s(\zeta)), \quad 1 \leq j \leq l_{2}. \]

Equation (5.2) becomes

\[ s_{j}(\zeta) = - T_{1}(\phi_{j}(z, \bar{z}, s, t + \psi(x, y, s)))\zeta, \quad 1 \leq j \leq l_{1}, \]

\[ t_{j}(\zeta) + \psi_{j}(z(\zeta), \bar{z}(-\zeta), s(\zeta)) = - T_{1}(\phi_{j+s_{1}}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)))\zeta, \quad 1 \leq j \leq l_{2}. \tag{5.3} \]

We claim that (5.3) implies that $t(\zeta) = 0$ for all $\zeta \in bD$, which is the desired conclusion. We shall show first that

\[ L_{k}^{0}(\psi_{j}(z, \bar{z}, s) + i\phi_{j+s_{1}}(z, \bar{z}, s, \psi(z, \bar{z}, s)) = 0, \quad 1 \leq j \leq l_{2}, \quad 1 \leq k \leq n, \tag{5.4} \]

where $L_{k}^{0}$ is given by (2.6). Indeed, since

\[ L_{k}g_{j} = 0, \quad 1 \leq j \leq l_{2}, \quad 1 \leq k \leq n, \tag{5.5} \]

where $g_{j}$ is given in (2.7), we obtain (5.4) by putting $t = 0$ in (5.5). Now the claim follows from a standard approximation argument as follows. Since

$g_{j}^{0} = \psi_{j}(z, \bar{z}, s) + i\phi_{j+s_{1}}(z, \bar{z}, s, \psi(z, \bar{z}, s))$ is a CR function for the induced CR structure on $N$, it is the uniform limit of holomorphic polynomials [5]. By the maximum principle, the pullback of $g_{j}^{0}$ to $bD$ by the map $(z(\zeta), s(\zeta))$ extends holomorphically to $D$. Therefore we have, since $\psi_{j}(0) = 0$,

\[ \psi_{j}(z(\zeta), \bar{z}(-\zeta), s(\zeta)) = - T_{1}(\phi_{j+s_{1}}(z, \bar{z}, s, \psi(z, \bar{z}, s)))\zeta, \]

which proves that $t(\zeta) = 0$ is a solution of the second equation in (5.3). Since the map

\[ t \rightarrow T_{1}[\phi_{j+s_{1}}(z, \bar{z}, s, t + \psi(z, \bar{z}, s) - \phi_{j+s_{1}}(z, \bar{z}, s, \psi(z, \bar{z}, s))] \]

is a contraction in $L^{2}(bD)$ if the neighborhood $U$ of the lemma is sufficiently small, the solution is unique, which proves the claim, and completes the proof of the lemma.

§6. Other results and remarks

Inspection of the proof of Theorem 1 shows that its conclusion holds under assumptions weaker than embeddability of $M$. We can also weaken the smoothness
assumption for \( M \). If \( N \) is a CR submanifold of an abstract CR manifold of class \( C^2 \)
we shall say that \( M \) is \textit{locally embeddable of order 1} along \( N \) if around every point
\( m_0 \in N \) there exist \( n + l \) functions \( Z_j \) of class \( C^2 \) such that for every \( L \) section of \( \mathcal{V} \)
\[ LZ_j(m) = O(\text{dist}(m, N)^2) \, . \]

\textbf{Theorem 4.} The conclusion of Theorem 1 holds if \( M \) is of class \( C^2 \) and locally
embeddable of order 1 along \( N \) or if \( \dim N = 2n \). In addition, without assuming
embeddability, if \( N \) is of codimension 1 in \( M \) then the function which is identically
equal to 1 on one side of \( N \) in \( M \) and equal to 0 on the other side is a singular CR
function.

\textbf{Proof.} As mentioned above, the conclusion when \( M \) is locally embeddable of order 1 along \( N \) follows by inspection of the proof of Theorem 1. To prove the statement
when \( \dim N = 2n \), we choose coordinates \( u = (u_1, \ldots, u_{2n}), t = (t_1, \ldots, t_l) \) such
that \( N \) is given by \( t = 0 \). We may write the \( L_j \) in the form
\[
L_j = \sum_{k=1}^{2n} x_{jk}(u, t) \frac{\partial}{\partial u_k} + \sum_{1 \leq k, r \leq l} \mu_{jkr}(u, t) t_k \frac{\partial}{\partial t_r} \, . \tag{6.1}
\]

By the Newlander-Nirenberg Theorem [10], we may find coordinates
\( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \) such that after a linear transformation on the
basis \( L_j \) we have
\[
L_j^0 = \sum_{k=1}^{2n} x_{jk}(u, 0) \frac{\partial}{\partial u_k} = \frac{\partial}{\partial z_j} \, .
\]

We then have
\[
L_j = \frac{\partial}{\partial z_j} + \sum_{1 \leq k \leq n} \lambda_{jkr} t_r \frac{\partial}{\partial z_k} + \sum_{1 \leq k \leq n} \nu_{jkr} t_r \frac{\partial}{\partial s_k} + \sum_{1 \leq k, r \leq l} \mu_{jkr} t_k \frac{\partial}{\partial t_r} \, . \tag{6.2}
\]

As in the proof of Theorem 1, we look for a solution of the form \( V(x, y)\delta(t) \). For
this, it suffices to solve the system
\[
L_j^0 V + \sum_{1 \leq r \leq l} \mu_{jrs}(z, \bar{z}, s) V = 0, \quad 1 \leq j \leq n \, . \tag{6.3}
\]
The equations (6.3) can be solved since the compatibility conditions needed for
the existence of a solution follow from the commutation relations \( [L_j, L_k] = 0 \) for
all \( j, k \).

Finally, we consider the case where \( \text{codim}_M N = 1 \). Since the \( L_j \) are tangent to
\( N \), the Heaviside function \( H \) which is equal to 1 on one side of \( N \) and 0 on the other
side satisfies \( L_j H = 0 \) in the distribution sense. This completes the proof of
Theorem 3.

\textbf{Remark 6.4.} The conclusion of Theorem 1 need not hold without the embeddability condition, as shown by the following example. Let \( M = \mathbb{R}^4 \) with coordinates
\( (x, y, s, t) \) and let
\[
L = \frac{\partial}{\partial z} - i z \frac{\partial}{\partial s} + \mu(z, \bar{z}, s) t \frac{\partial}{\partial t} \, . \tag{6.5}
\]
with $\mu$ a smooth function in $\mathbb{R}^3$ not in the range of the Lewy operator

$$L^0 = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}$$ (see [8]). It is clear that finding a solution of the form

$$V(x, y, s) \delta(t),$$ with $V(0) = 0$ is equivalent to solving the equation $L^0 V - \mu V = 0$.

Note that in this example the Heaviside function $H(t)$ is a singular solution.

**Proposition 6.6.** If $M$ is a real analytic generic manifold which is not minimal at $m_0$, and $N$ a real analytic $CR$ submanifold of $M$, then there is a holomorphic submanifold $\mathcal{H}$ in $\mathbb{C}^{n + l}$, such that $N = M \cap \mathcal{H}$ with $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} \mathcal{H} + n$. In particular, the minimal $CR$ submanifold through $m_0$ is of this form.

**Proof.** We choose coordinates $(x, y, s, t)$ as in $\S 2$. Here the $\phi_j$ and $\psi_k$ as in (2.7) are real analytic functions in a neighborhood of $0$. Since $L^0 \phi_j = 0$ and the $g^k_0$ are real analytic, there exists a holomorphic function $G_k(z, w_1, \ldots, w_{l_1})$ such that $g^k_0$ is the restriction of $G_k$ to $N$. In the new variables $w_j = w_j, 1 \leq j \leq l_1$, and $w'_j + t_1 = w_j + t_1 - G_j(z, w_1, \ldots, w_{l_1}), 1 \leq j \leq l_2$, we have $N = M \cap \{w'_j + t_1 = 0, 1 \leq j \leq l_2\}$. This proves the first statement. To prove the second claim, it suffices to observe that when $M$ is real analytic, the minimal $CR$ submanifold through $m_0$ is the Nagano leaf (see [9]) of the sections of $Re\psi$, which is real analytic.

**Remark 6.7.** If $M$ is assumed only to be smooth rather than real analytic, the minimal $CR$ submanifold $N$ need not be of the form $N = M \cap \mathcal{H}$ with $\mathcal{H}$ a holomorphic submanifold. Indeed, consider the generic submanifold of $\mathbb{C}^3$ parametrized by $(x, y, s, t)$ and given by $\{(z, w_1, w_2); w_1 = s + |z|^2, w_2 = t + h(x, y, s)\}$ where $h$ is a smooth non real analytic function satisfying $\frac{\partial h}{\partial \bar{z}} = iz \frac{\partial h}{\partial s}$. Here $N$ is given by $\{t = 0\}$ and $L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}$. If $N$ were the intersection of $M$ with a complex hypersurface, there would exist a holomorphic function $\mathcal{H}$ and holomorphic coordinates $(z', w'_1, w'_2)$ such that $h$ is the restriction of $\mathcal{H}(z', w'_1, w'_2)$ to $N$, contradicting the assumption that $h$ is not real analytic.

**References**


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