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Cauchy-Riemann functions on manifolds of higher codimension in complex space

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§1. Introduction and main results

A smooth manifold M is called CR if there exists a subbundle $\mathscr{V}$ of CTM, the complexified tangent bundle of M, such that

$$\mathscr{V} \cap \bar{\mathscr{V}} = 0 \quad \text{and} \quad [\mathscr{V}, \mathscr{V}] \subset \mathscr{V} . \tag{1.1}$$

A function or distribution on M is called CR if it is annihilated by all the sections of $\mathscr{V}$. If $\dim_{\mathbf{R}} M = 2n + l$ and $\dim_{\mathbf{C}} \mathscr{V} = n$ then we say that M is of *CR dimension* n and *CR codimension* l.

If $(M, \mathscr{V})$ is a CR manifold then N is called a *CR submanifold* of M (of the same CR dimension) if N is a submanifold of M satisfying

$$\mathscr{V}|_N \subset CTN . \tag{1.2}$$

Note that it follows from (1.1) and (1.2) that

$$2\dim_{\mathbf{C}} \mathscr{V} \leqq \dim_{\mathbf{R}} N \leqq \dim_{\mathbf{R}} M , \tag{1.3}$$

and M and N have the same CR dimension. In what follows it will be understood that by a CR submanifold of M we will mean that (1.2) is satisfied.

We introduce the following definition.

Definition. If M is a CR manifold then M is *minimal at* $m_0 \in M$ if there is no CR submanifold N containing m_0 with $\dim_{\mathbf{R}} N < \dim_{\mathbf{R}} M$.

The importance of this notion of minimality was introduced by Tumanov [14], who showed that if M is a generic embedded submanifold of $\mathbf{C}^{n+l}$ and $m_0 \in M$ then if M is minimal at m_0 every CR function on M, defined in a neighborhood of m_0, is the boundary value of a holomorphic function in an open wedge of $\mathbf{C}^{n+l}$ of edge M.

It is easy to see that if M is of finite type in the sense of Bloom-Graham [6] then M is minimal at m_0. Indeed finite type means that the Lie algebra of the sections of

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$\mathscr{V}$ and their conjugates span the complexified tangent space at m_0. If these are also tangent to N then the germs of N and M are the same at m_0. If $(M, \mathscr{V})$ is real analytic then minimal implies finite type; indeed, if M is not of finite type, then the Nagano leaf [9] passing through m_0 would be a proper real analytic CR submanifold, contradicting minimality. If M is only smooth, it could be minimal without being of finite type.

The main result of this paper is to prove that Tumanov's minimality condition is also necessary for holomorphic extendability. Recall that a CR manifold M of CR dimension n and CR codimension l is *locally embeddable* if in a neighborhood of every point there exist $n + l$ smooth CR functions with linearly independent differentials.

Theorem 1. *Let M be a locally embeddable CR manifold which is not minimal at $m_0 \in M$ and N a CR submanifold of M containing m_0, with $\dim_{\mathbf{R}} N < \dim_{\mathbf{R}} M$. Then there exists a CR distribution T defined in a neighborhood U of m_0 with* $\operatorname{supp} T = N \cap U$.

A slightly more general version of Theorem 1 is given in §6 as Theorem 4.

A submanifold M of $\mathbf{C}^{n+l}$ of dimension $2n + l$ is called *generic* if it is locally defined, near $m_0 \in M$, by $\rho_j = 0$, $j = 1, \ldots, l$, where the ρ_j are smooth, real functions such that their complex differentials $\partial\rho_j$ are linearly independent. Such a manifold M equipped with $\mathscr{V}$, the induced tangential Cauchy-Riemann bundle of $\mathbf{C}^{n+l}$, is a CR manifold of CR dimension n and CR codimension l.

Recall that a wedge of edge M is an open set of $\mathbf{C}^{n+l}$ of the form

$$\mathscr{W}(\mathscr{O}, \Gamma) = \{Z \in \mathscr{O}, \rho(Z) \in \Gamma\},$$

where $\mathscr{O}$ is a sufficiently small open neighborhood of m_0 in $\mathbf{C}^{n+l}$, Γ an open cone of $\mathbf{R}^l$, and $\rho = (\rho_1, \ldots, \rho_l)$ where the ρ_j are the defining functions of M near m_0 as above. We can now state our nonextendability result.

Theorem 2. *If M is an embedded generic CR manifold which is not minimal at m_0 then there is a smooth CR function defined in a neighborhood of m_0 which does not extend holomorphically to any wedge with edge U, where U is a neighborhood of m_0 in M.*

Combining Theorem 2 with the sufficient condition of Tumanov we obtain the following.

Corollary 1. *If M is an embedded generic CR manifold and $m_0 \in M$, then every germ of a CR function at m_0 extends holomorphically to a wedge of edge M if and only if M is minimal at m_0.*

The following result shows uniqueness of a CR submanifold of minimal dimension and gives two intrinsic characterizations of such a submanifold.

Theorem 3. *If M is a generic CR manifold and $m_0 \in M$, there is a unique germ N_0 of a CR submanifold contained in M, $m_0 \in N_0$, of minimal dimension. Also N_0 may be described as follows.*

(i) *For every sufficiently small neighborhood U of m_0 in M, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ consists of all points $m \in U'$ which can be reached from m_0 by a finite sequence of integral curves contained in U of sections of* $\operatorname{Re} \mathscr{V}$.

(ii) *For every sufficiently small neighborhood U of m_0 in M, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ is the union of sets of the form $Z(bD)$, where D is the unit disc in $\mathbf{C}$ and Z: $\bar{D} \to \mathbf{C}^{n+l}$ is continuous, holomorphic in D and satisfying $m_0 \in Z(bD) \subset U \cap M$.*

We note that the uniqueness of N_0 and its characterization given by (i) in Theorem 3 hold even in the case of a nonembeddable CR manifold.

The study of extendability of CR functions began with the celebrated paper of Hans Lewy [8] in the '50's in which he showed extendability to one side from a strictly pseudoconvex hypersurface. Other sufficient conditions were subsequently obtained by a number of mathematicians. Recently, a necessary and sufficient condition for every CR function to extend to at least one side of a hypersurface of class C^2 was obtained by Trépreau [12]. The authors obtained some sufficient conditions [2] for extendability in the higher codimension case; these results have been generalized by Tumanov, as explained above. They also obtained in the real analytic case (loc. cit.) necessary conditions, i.e. that M must be of finite type, in order that extendability to a wedge hold. Since Tumanov proved that finite type is also sufficient for extendability, the case of real analytic generic manifolds was completed by Tumanov's work. Our Theorem 2 above now settles the smooth case.

Singular solutions for real analytic vector fields supported on submanifolds were constructed by Zachmanoglou [15] in the study of analytic hypoellipticity and propagation of zeroes. Our solutions in Theorem 1 have the same general form as his, but in our case the vector fields are not real analytic.

We would like to thank Jean-Marie Trépreau for several interesting conversations and, in particular, for suggesting to us the formulation of Theorem 3, and the use of Sussmann's work [11] in its proof.

§2. Local coordinates

We assume here that M is an embedded generic CR manifold which is not minimal and N a CR submanifold of M passing through m_0. We first introduce local coordinates around m_0 which will be used in the proof of Theorem 1. Assume that

$$\dim_{\mathbf{R}} M = 2n + l, \quad \dim_{\mathbf{R}} N = 2n + l_1 \text{ and } \dim_{\mathbf{C}} \mathscr{V} = n\,, \tag{2.1}$$

with $0 \leqq l_1 < l$.

We may choose a local embedding so that M is parametrized in $\mathbf{C}^{n+l}$, where the coordinates are denoted by $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_l)$, by

$$\operatorname{Im} w_j = \phi_j(z, \bar{z}, s), \quad 1 \leqq j \leqq l, \quad s = \operatorname{Re} w, \ \phi(0) = d\phi(0) = 0\,. \tag{2.2}$$

A basis of CR vector fields L_j, $j = 1, \ldots, n$, can then be written in the form

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{1 \leqq k \leqq l} \alpha_{jk}(z, \bar{z}, s) \frac{\partial}{\partial s_k}\,, \tag{2.3}$$

with $\alpha_{jk}(0) = 0$.

Assume that N is given by $\rho_j(z, \bar{z}, s) = 0$, $1 \leqq j \leqq l_2$, where $l_2 = l - l_1$ and the differentials $d\rho_j$ are linearly independent. Since the L_j are tangent to N, we must have $L_j\rho_k = 0$ on N, and $\bar{L}_j\rho_k = 0$ on N, since the ρ_j are real valued. Hence, by (2.3), $\frac{\partial \rho_j}{\partial x_k}(0) = \frac{\partial \rho_j}{\partial y_k}(0) = 0$. We conclude that *rank* $\left(\frac{\partial \rho_j}{\partial s_k}(0)\right)_{\substack{1 \leq j \leq l_2 \\ 1 \leq k \leq l}} = l_2$. After a linear change of variables and using the implicit function theorem, we can assume that N is given as a subset of M by

$$s_{l_1+j} = \psi_j(z, \bar{z}, s_1, \ldots, s_{l_1}), \quad 1 \leqq j \leqq l_2, \quad \psi_j(0) = d\psi_j(0) = 0 . \tag{2.4}$$

Put $t_j = s_{l_1+j} - \psi_j(z, \bar{z}, s)$, $j = 1, \ldots, l_2$, and take for coordinates on M,

$$(x, y, s, t) = (x, y, s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}) .$$

In these coordinates the L_j become

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{l_1} \beta_{jk} \frac{\partial}{\partial s_k} + \sum_{\substack{1 \leq k \leq l_2 \\ 1 \leq r \leq l_2}} \mu_{jkr}\, t_k \frac{\partial}{\partial t_r} , \tag{2.5}$$

where β_{jk} and μ_{jkr} are functions of (x, y, s, t). We introduce the vector fields L_j^0, obtained from L_j by setting $t = 0$:

$$L_j^0 = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{l_1} \beta_{jk}(x, y, s, 0) \frac{\partial}{\partial s_k} . \tag{2.6}$$

The L_j^0 form a basis of the sections of the CR bundle restricted to N. We set

$$\begin{aligned} f_j &= s_j + i\phi_j(z, \bar{z}, s, t + \psi(x, y, s)), \quad 1 \leqq j \leqq l_1 , \\ g_j &= t_j + \psi_j(z, \bar{z}, s) + i\phi_{j+l_1}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)), \quad 1 \leqq j \leqq l_2 , \end{aligned} \tag{2.7}$$

where the ϕ_j are given by (2.2) and ψ_j by (2.4). Note that we have $L_j f_k = L_j g_k = 0$.

For a smooth function $h(x, y, s, t)$ we write

$$h(x, y, s, t) = h^0(x, y, s) + \sum_{k=1}^{l_2} h^k(x, y, s)t_k + O(|t|^2) .$$

Since $L_j f_k = 0$ we conclude that $L_j^0 f_k^0 = 0$, $1 \leqq j \leqq n$, $1 \leqq k \leqq l_1$. As in [3] we define l_1 vector fields R_j of the form $R_j = \sum_{k=1}^{l_1} a_{jk}(x, y, s)\frac{\partial}{\partial s_k}$ satisfying the following relations:

$$[L_k^0, R_j] = 0, \qquad [R_j, R_p] = 0, \qquad R_j f_p^0 = \varepsilon_{jp} , \tag{2.8}$$

where ε_{jp} is the Kronecker symbol. It follows from (2.7) and (2.8) that we have

$$L_j^0 = \frac{\partial}{\partial \bar{z}_j} - \sum_{k=1}^{l_1} f_{k\bar{z}_j}^0 R_k . \tag{2.9}$$

Since the R_k are linear independent we obtain, for $j = 1, \ldots, n$,

$$L_j = L_j^0 + \sum_{\substack{1 \leq k \leq l_2 \\ 1 \leq p \leq l_1}} \lambda_{jkp} t_k R_p + \sum_{\substack{1 \leq k \leq l_2 \\ 1 \leq r \leq l_2}} \mu_{jkr} t_k \frac{\partial}{\partial t_r} . \tag{2.10}$$

§3. Construction of singular solutions. Proof of Theorem 1

We shall use the coordinates (x, y, s, t) introduced in §2 and take a basis for the sections of $\mathscr{V}$ of the form (2.10). We shall construct a distribution solution T of the system of equations $L_j T = 0$ of the form $T = V(z, \bar{z}, s)\ \delta(t)$, where $\delta(t) = \delta(t_1) \otimes \ldots \otimes \delta(t_{l_2})$ is the dirac measure at the origin in $\mathbf{R}^{l_2}$ and V is a smooth function nonvanishing at 0.

Using the relation

$$t_k \frac{\partial}{\partial t_r} \delta(t) = -\varepsilon_{kr} \delta(t)$$

we conclude that V must satisfy the equations

$$L_j^0 V - \sum_{1 \leq r \leq l_2} \mu_{jrr}^0(z, \bar{z}, s) V = 0, \quad 1 \leq j \leq n \,. \tag{3.1}$$

Since V is nonvanishing, equation (3.1) is equivalent to

$$L_j^0(\operatorname{Log} V) - \sum_{1 \leq r \leq l_2} \mu_{jrr}^0(z, \bar{z}, s) = 0 \,. \tag{3.2}$$

Therefore, it suffices to show that $\sum_{1 \leq r \leq l_2} \mu_{jrr}^0(z, \bar{z}, s)$ is in the range of L_j^0. We begin with the calculation of the coefficients μ_{jrr}^0. It follows from (2.10), that if $L_j h = 0$ then

$$L_j h = \left(L_j^0 + \sum_{\substack{1 \leq k \leq l_2 \\ 1 \leq p \leq l_1}} \lambda_{jkp}^0 t_k R_p + \sum_{\substack{1 \leq k \leq l_2 \\ 1 \leq r \leq l_2}} \mu_{jkr}^0 t_k \frac{\partial}{\partial t_r} \right) \left(h^0 + \sum_{1 \leq k \leq l_2} h^k t_k \right) + O(t^2) \,, \tag{3.3}$$

where we have used the notation $h = h^0 + \sum_{1 \leq k \leq l_2} h^k t_k + O(t^2)$ as in §2.

Setting the coefficient of t_k equal to 0 in (3.3) we obtain

$$L_j^0 h^k + \sum_{1 \leq p \leq l_1} \lambda_{jkp}^0 R_p h^0 + \sum_{1 \leq r \leq l_2} \mu_{jkr}^0 h^r = 0 \,. \tag{3.4}$$

Taking for h the functions f_q, $1 \leq q \leq l_1$, defined in (2.7), and using the relations $R_p f_k^0 = \varepsilon_{pk}$ in (2.8), we obtain,

$$\lambda_{jkp}^0 = -L_j^0 f_p^k - \sum_{1 \leq r \leq l_2} \mu_{jkr}^0 f_p^r, \; 1 \leq j \leq n, \quad 1 \leq k \leq l_2, \quad 1 \leq p \leq l_1 \,. \tag{3.5}$$

In (3.4) we now replace h by g_q, $1 \leq q \leq l_2$, defined in (2.7) and λ_{jkp}^0 by its expression given in (3.5). We obtain

$$L_j^0 g_q^k + \sum_{1 \leq p \leq l_1} \left(-L_j^0 f_p^k - \sum_{1 \leq r \leq l_2} \mu_{jkr}^0 f_p^r \right) R_p g_q^0 + \sum_{1 \leq r \leq l_2} \mu_{jkr}^0 g_q^r = 0 \,. \tag{3.6}$$

Rewriting (3.6) by collecting the coefficients of μ_{jkr}^0 we have

$$\sum_{1 \leq r \leq l_2} \mu_{jkr}^0 \left(g_q^r - \sum_{1 \leq p \leq l_1} f_p^r R_p g_q^0 \right) = -L_j^0 g_q^k + \sum_{1 \leq p \leq l_1} (L_j^0 f_p^k) R_p g_q^0 \,. \tag{3.7}$$

Let $\mathscr{A}$ be the matrix $(g_q^r - \sum_{1 \leq p \leq l_1} f_p^r R_p g_q^0)_{1 \leq r, q \leq l_2}$ and $D(z, \bar{z}, s) = \det \mathscr{A}$. By Cramer's rule applied to (3.7) we obtain

$$\mu_{jkr}^0 = \frac{D_{jkr}}{D}, \quad 1 \leqq j \leqq n, \quad 1 \leqq k, r \leqq l_2 , \tag{3.8}$$

where D_{jkr} is the determinant of the matrix obtained by substituting the rth column in $\mathscr{A}$ by the vector $(-L_j^0 g_q^k + \sum_{1 \leq p \leq l_1} (L_j^0 f_p^k) R_p g_q^0)$, $1 \leqq q \leqq l_2$. Summing (3.8) over r we obtain

$$\sum_{1 \leqq r \leqq l_2} \mu_{jrr}^0 = D^{-1} \sum_{1 \leqq r \leqq l_2} D_{jrr} = \frac{-L_j^0 D}{D} = -L_j^0 \log D . \tag{3.9}$$

Here we have used the identity $(L_j^0 f_p^k) R_p g_q^0 = L_j^0 (f_p^k R_p g_q^0)$ which is a consequence of $[L_j^0, R_p] = 0$ (cf (2.8)) and $L_j^0 g_q^0 = 0$.

Note that $D(0) \neq 0$. From (3.2) and (3.9) it follows that a desired solution of (3.1) is given by $V = D^{-1}$. This completes the proof of Theorem 1.

§4. Proof of Theorem 2

We shall first show that the distribution solution T of Theorem 1 is not the boundary value of a holomorphic function in any wedge with edge M. We shall then use T to construct a CR function of class C^k with the same property. We use the closed graph theorem, together with a result from [4] to show that there exists a smooth CR function which does not extend.

We review here some basic properties of boundary values of holomorphic functions in wedges. If M is an embedded generic CR manifold of $\mathbf{C}^{n+l}$ of dimension $2n + l$ given by $\operatorname{Im} w = \phi(z, \bar{z}, s)$ as in (2.2), and Γ an open strictly convex cone of $\mathbf{R}^l$ and $\mathcal{O}$ an open neighborhood of 0 in $\mathbf{C}^{n+l}$, we define the wedge $\mathscr{W}(\mathcal{O}, \Gamma)$ by

$$\mathscr{W}(\mathcal{O}, \Gamma) = \{(z, w) \in \mathcal{O}, \operatorname{Im} w - \phi(z, \bar{z}, s) \in \Gamma\} . \tag{4.1}$$

If h is a holomorphic function in $\mathscr{W}(\mathcal{O}, \Gamma)$ with slow growth at the edge M, i.e. $|h(m)| \leqq C \operatorname{dist}(m, M)^{-p}$, for $m \in \mathscr{W}(\mathcal{O}, \Gamma)$, then h has a boundary value $b_M h$ as a CR distribution on $M \cap \mathcal{O}$. Furthermore, if $b_M h = 0$ in an open set of $M \cap \mathcal{O}$ and $\mathscr{W}(\mathcal{O}, \Gamma)$ is connected then h vanishes identically. (For a flat wedge see [7], and for the general case see [1] and [2].)

We begin with the following result.

Proposition 4.2. *Under the assumptions of Theorem 2, for any $k \geqq 0$ there is a CR function of class C^k defined in a neighborhood of m_0 which does not extend to any wedge with edge M near m_0.*

Proof. The fact that the distribution solution T of Theorem 1 is not the boundary value of a holomorphic function in any wedge with edge M is proved by the following lemma.

Lemma 4.3. *Let M be an embedded generic CR manifold and S a CR distribution on M, nonzero in any neighborhood of m_0, with the following property. For every neighborhood U of m_0 in M there is an open set $U' \subset U$ such that S vanishes in U'. Then in any neighborhood of m_0, S is not the boundary value of a holomorphic function in a wedge with edge M.*

Proof. Indeed, it follows from the remarks preceding Proposition 4.2 that if S were the boundary value of a holomorphic function h in a connected wedge $\mathscr{W}(\mathcal{O}, \Gamma)$, then, since S vanishes in an open subset of any neighborhood of m_0, h would have to vanish identically, which would imply that S is the zero distribution. Since Γ is assumed to be strictly convex, the wedge $\mathscr{W}(\mathcal{O}, \Gamma)$ is connected if $\mathcal{O}$ is connected and sufficiently small. This proves the lemma.

To construct the C^k CR function of the proposition we shall use the following. As in §2 we can find l vector fields D_j satisfying, for $1 \leqq q \leqq n$, $1 \leqq j, p \leqq l$,

$$[L_q, D_j] = 0, \quad [D_j, D_p] = 0, \quad \text{and} \quad D_j(s_p + i\phi_p(z, \bar{z}, s)) = \varepsilon_{jp}\,,$$

where the ϕ_p are as in (2.2).

Lemma 4.4. *Let S be a CR distribution defined in a neighborhood of m_0 in M. Then for any k there is a CR function $f \in C^k$ defined near m_0 such that*

$$S = \Big(\sum_{1 \leqq j \leqq l} D_j^2\Big)^p f,$$

where the D_j are the vector fields defined above.

This lemma is a consequence of the representation of distributions annihilated by a system of complex vector fields given in [5] (see also [13]).

We may now prove Proposition 4.2. By Theorem 1 we can find a CR distribution T in an open subset U with support in N. By Lemma 4.4 we may write $T = (\sum_{1 \leqq j \leqq l} D_j^2)^p f$ with f a CR function of class C^k. We reason by contradiction. Assume that f is the boundary value of a holomorphic function H in a wedge $\mathscr{W}(\mathcal{O}, \Gamma)$. Since D_j is of the form

$$D_j = \frac{\partial}{\partial w_j} + \sum_{1 \leqq k \leqq l} a_{jk} \frac{\partial}{\partial \bar{w}_k},$$

(see [3]), we have $b\left(\dfrac{\partial H}{\partial w_j}\right) = D_j(bH)$. Note that $\dfrac{\partial H}{\partial w_j}$ is of slow growth, by Cauchy's inequalities. We conclude

$$D_j f = b\left(\frac{\partial H}{\partial w_j}\right), \quad T = \Big(\sum_{1 \leqq j \leqq l} D_j^2\Big)^p f = b\left(\Big(\sum_{1 \leqq j \leqq l}\Big(\frac{\partial^2}{\partial w_j^2}\Big)\Big)^p H\right). \tag{4.5}$$

It would then follow from (4.5) that T is the boundary value of a holomorphic function, contradicting the conclusion of Lemma 4.2. The proof of Proposition 4.2 is now complete.

Proof of Theorem 2. We assume by contradiction that every smooth CR function defined near m_0 extends holomorphically in some wedge. We claim that for every open neighborhood U of $m_0 \in M$ there exists a wedge $\mathscr{W}(\mathscr{O}, \Gamma)$ to which every smooth CR function defined on U extends holomorphically. Indeed, this follows from an argument using the Baire category theorem by a slight modification of the proof of Theorem 7 of [4], where the corresponding result is proved for continuous functions. We shall show that if k is sufficiently large, then any CR function of class C^k defined near m_0 extends to a wedge, contradicting Proposition 4.2.

If U is a neighborhood of m_0, let E be the space of smooth CR functions on U and $\mathscr{W}(\mathscr{O}, \Gamma)$ the wedge associated to U by the claim above and denote by $\mathscr{H}$ the space of bounded holomorphic functions in $\mathscr{W}(\mathscr{O}, \Gamma)$. By the closed graph theorem the mapping $E \to \mathscr{H}$ which to each CR function associates its holomorphic extension is continuous. We can therefore find k, U' relatively compact in U and a constant C such that for all $f \in E$,

$$\sup_{z \in \mathscr{W}(\mathscr{O}, \Gamma)} |\tilde{f}(z)| \leq C \sup_{\substack{m \in U' \\ |\alpha| \leq k}} |D^\alpha f(m)| , \tag{4.6}$$

where $\tilde{f}$ is the holomorphic extension of f. Since every CR function of class C^k is locally a limit, in the C^k topology of entire functions [5], it follows from (4.6) that every CR C^k function on U extends holomorphically to $\mathscr{W}(\mathscr{O}, \Gamma)$. Theorem 2 is then proved by contradiction.

§5. Characterization of minimal CR submanifolds; proof of Theorem 3

The proof of the uniqueness in Theorem 3, as well as the characterization in (i), is essentially based on the following localized version of a result of Sussmann [11]: Let $\{X_1, \ldots, X_p\}$ be a set of smooth real vector fields defined near the origin in $\mathbf{R}^q$, and U' a sufficiently small neighborhood of 0. Then there is a unique submanifold $N \subset U'$, $0 \in N$, such that the X_j are all tangent to N at every point of N and for which dim N is minimal with this property. Its uniqueness follows from the fact that it is the union of all points in U' which can be reached by a finite sequence of integral curves, contained in a slightly bigger neighborhood U, of the vector fields X_j.

We now consider the case of a CR manifold M. We take for $X_1, \ldots, X_{2n}$ a basis for the real and imaginary parts of the local CR vector fields L_j, i.e. for the sections of Re$\mathscr{V}$. A submanifold N of M is a CR submanifold of the same CR dimension if and only if all the X_j are tangent to N. Now the uniqueness and the characterization (i) follow from Sussmann's result stated above.

To show that N_0 can be characterized by (ii), we assume that M is embedded and use the following result of Tumanov [14]: There is a CR submanifold of M through m_0 contained in the union of sets of the form $Z(bD)$, where $Z: \bar{D} \to \mathbf{C}^{n-l}$ is continuous, holomorphic in D and satisfying $m_0 \in Z(bD) \subset U \cap M$, and U is a sufficiently small neighborhood of m_0 in M. We shall show that the image of bD under all such holomorphic discs Z lies in the minimal submanifold N_0. Hence the proof of Theorem 3 will be completed by the following.

Lemma 5.1. *Let M be a generic CR submanifold of $\mathbf{C}^{n+l}$ and N a CR submanifold containing m_0. Then there is a neighborhood U of m_0 in M such that for every*

$Z: D \to C^{n+l}$ with Z holomorphic, continuous in $\bar{D}$ and satisfying $m_0 \in Z(bD) \subset U \cap M$, then $Z(bD) \subset N$.

Proof. We use the coordinates $(x, y, s, t) = (x, y, s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2})$ introduced in §2, so that N is the submanifold $\{t = 0\}$. For a disc Z as in the statement of the lemma, we write $Z(\zeta) = (z(\zeta), w(\zeta))$; we may assume without loss of generality that $Z(1) = m_0$. Then

$$\operatorname{Re} w(\zeta) = -T_1(\phi(z(\cdot), \overline{z(\cdot)}, \operatorname{Re} w(\cdot)))(\zeta), \quad \zeta \in bD, \tag{5.2}$$

where ϕ is given by (2.2) and $T_1 f$ is the harmonic conjugate of f, defined on bD vanishing at 1. As in (2.7) we may write

$$\operatorname{Re} w_j(\zeta) = s_j(\zeta), \quad 1 \leqq j \leqq l_1,$$
$$\operatorname{Re} w_j(\zeta) = t_j(\zeta) + \psi_j(z(\zeta), \overline{z(\zeta)}, s(\zeta)), \quad 1 \leqq j \leqq l_2.$$

Equation (5.2) becomes

$$s_j(\zeta) = -T_1(\phi_j(z, \bar{z}, s, t + \psi(x, y, s)))(\zeta), \quad 1 \leqq j \leqq l_1,$$
$$t_j(\zeta) + \psi_j(z(\zeta), \overline{z(\zeta)}, s(\zeta)) = -T_1(\phi_{j+l_1}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)))(\zeta), \quad 1 \leqq j \leqq l_2. \tag{5.3}$$

We claim that (5.3) implies that $t(\zeta) = 0$ for all $\zeta \in bD$, which is the desired conclusion. We shall show first that

$$L_k^0(\psi_j(z, \bar{z}, s) + i\phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s)) = 0, \quad 1 \leqq j \leqq l_2, \quad 1 \leqq k \leqq n, \tag{5.4}$$

where L_k^0 is given by (2.6). Indeed, since

$$L_k g_j = 0, \quad 1 \leqq j \leqq l_2, \quad 1 \leqq k \leqq n, \tag{5.5}$$

where g_j is given in (2.7), we obtain (5.4) by putting $t = 0$ in (5.5). Now the claim follows from a standard approximation argument as follows. Since $g_j^0 = \psi_j(z, \bar{z}, s) + i\phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s))$ is a CR function for the induced CR structure on N, it is the uniform limit of holomorphic polynomials [5]. By the maximum principle, the pullback of g_j^0 to bD by the map $(z(\zeta), s(\zeta))$ extends holomorphically to D. Therefore we have, since $\psi_j(0) = 0$,

$$\psi_j(z(\zeta), \overline{z(\zeta)}, s(\zeta)) = -T_1(\phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s))(\zeta),$$

which proves that $t(\zeta) \equiv 0$ is a solution of the second equation in (5.3). Since the map

$$t \to T_1[\phi_{j+l_1}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)) - \phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s))]$$

is a contraction in $L^2(bD)$ if the neighborhood U of the lemma is sufficiently small, the solution is unique, which proves the claim, and completes the proof of the lemma.

§6. Other results and remarks

Inspection of the proof of Theorem 1 shows that its conclusion holds under assumptions weaker than embeddability of M. We can also weaken the smoothness assumption for M. If N is a CR submanifold of an abstract CR manifold of class C^2 we shall say that M is *locally embeddable of order* 1 *along* N if around every point $m_0 \in N$ there exist $n + l$ functions Z_j of class C^2 such that for every L section of $\mathscr{V}$

$$LZ_j(m) = O(\mathrm{dist}(m, N)^2)\,.$$

Theorem 4. *The conclusion of Theorem* 1 *holds if* M *is of class* C^2 *and locally embeddable of order* 1 *along* N *or if* $\dim N = 2n$. *In addition, without assuming embeddability, if* N *is of codimension* 1 *in* M *then the function which is identically equal to* 1 *on one side of* N *in* M *and equal to* 0 *on the other side is a singular CR function.*

Proof. As mentioned above, the conclusion when M is locally embeddable of order 1 along N follows by inspection of the proof of Theorem 1. To prove the statement when $\dim N = 2n$, we choose coordinates $u = (u_1, \ldots, u_{2n})$, $t = (t_1, \ldots, t_l)$ such that N is given by $t = 0$. We may write the L_j in the form

$$L_j = \sum_{k=1}^{2n} \alpha_{jk}(u, t)\frac{\partial}{\partial u_k} + \sum_{1 \leq k, r \leq l} \mu_{jkr}(u, t)\, t_k \frac{\partial}{\partial t_r}\,. \tag{6.1}$$

By the Newlander-Nirenberg Theorem [10], we may find coordinates $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ such that after a linear transformation on the basis L_j we have

$$L_j^0 = \sum_{k=1}^{2n} \alpha_{jk}(u, 0)\frac{\partial}{\partial u_k} = \frac{\partial}{\partial \bar{z}_j}\,.$$

We then have

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{\substack{1 \leq k \leq n\\ 1 \leq r \leq l}} \lambda_{jkr} t_r \frac{\partial}{\partial \bar{z}_k} + \sum_{\substack{1 \leq k \leq n\\ 1 \leq r \leq l}} \nu_{jkr} t_r \frac{\partial}{\partial z_k} + \sum_{1 \leq k, r \leq l} \mu_{jkr} t_k \frac{\partial}{\partial t_r}\,. \tag{6.2}$$

As in the proof of Theorem 1, we look for a solution of the form $V(x, y)\delta(t)$. For this, it suffices to solve the system

$$L_j^0 V + \sum_{1 \leq r \leq l_2} \mu_{jrr}^0(z, \bar{z}, s) V = 0, \quad 1 \leq j \leq n\,. \tag{6.3}$$

The equations (6.3) can be solved since the compatibility conditions needed for the existence of a solution follow from the commutation relations $[L_j, L_k] = 0$ for all j, k.

Finally, we consider the case where $\mathrm{codim}_M N = 1$. Since the L_j are tangent to N, the Heaviside function H which is equal to 1 on one side of N and 0 on the other side satisfies $L_j H = 0$ in the distribution sense. This completes the proof of Theorem 3.

Remark 6.4. The conclusion of Theorem 1 need not hold without the embeddability condition, as shown by the following example. Let $M = \mathbf{R}^4$ with coordinates (x, y, s, t) and let

$$L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s} + \mu(z, \bar{z}, s)t\frac{\partial}{\partial t}\,. \tag{6.5}$$

with μ a smooth function in $\mathbf{R}^3$ not in the range of the Lewy operator $L^0 = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}$ (see [8]). It is clear that finding a solution of the form $V(x, y, s)\delta(t)$, with $V(0) \neq 0$ is equivalent to solving the equation $L^0 V - \mu V = 0$. Note that in this example the Heaviside function $H(t)$ is a singular solution.

Proposition 6.6. *If M is a real analytic generic manifold which is not minimal at m_0, and N a real analytic CR submanifold of M, then there is a holomorphic submanifold $\mathscr{H}$ in $\mathbf{C}^{n+l}$, such that $N = M \cap \mathscr{H}$ with $\dim_{\mathbf{R}} N = \dim_{\mathbf{C}} \mathscr{H} + n$. In particular, the minimal CR submanifold through m_0 is of this form.*

Proof. We choose coordinates (x, y, s, t) as in §2. Here the ϕ_j and ψ_k as in (2.7) are real analytic functions in a neighborhood of 0. Since $L_j^0 g_k^0 = 0$ and the g_k^0 are real analytic, there exists a holomorphic function $G_k(z, w_1, \ldots, w_{l_1})$ such that g_k^0 is the restriction of G_k to N. In the new variables $w'_j = w_j$, $1 \leq j \leq l_1$, and $w'_{j+l_1} = w_{j+l_1} - G_j(z, w_1, \ldots, w_{l_1})$, $1 \leq j \leq l_2$, we have $N = M \cap \{w'_{j+l_1} = 0, 1 \leq j \leq l_2\}$. This proves the first statement. To prove the second claim, it suffices to observe that when M is real analytic, the minimal CR submanifold through m_0 is the Nagano leaf (see [9]) of the sections of $\mathrm{Re}\mathscr{V}$, which is real analytic.

Remark 6.7. If M is assumed only to be smooth rather than real analytic, the minimal CR submanifold N need not be of the form $N = M \cap \mathscr{H}$ with $\mathscr{H}$ a holomorphic submanifold. Indeed, consider the generic submanifold of $\mathbf{C}^3$ parametrized by (x, y, s, t) and given by $\{(z, w_1, w_2): w_1 = s + i|z|^2, w_2 = t + h(x, y, s)\}$ where h is a smooth non real analytic function satisfying $\frac{\partial h}{\partial \bar{z}} = iz\frac{\partial h}{\partial s}$. Here N is given by $\{t = 0\}$ and $L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}$. If N were the intersection of M with a complex hypersurface, there would exist a holomorphic function $\mathscr{H}$ and holomorphic coordinates (z', w'_1, w'_2) such that h is the restriction of $\mathscr{H}(z', w'_1, w'_2)$ to N, contradicting the assumption that h is not real analytic.

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