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Central Idempotent Measures on Connected Locally Compact Groups*

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A measure μ of finite total variation on a locally compact group G is idempotent if $\mu * \mu = \mu$, and is central if invariant under all inner automorphisms of G. Recent results of D. Rider and D. Ragozin concerning compact groups are combined with results of the authors for noncompact groups to determine all central idempotent measures on a connected G in terms of the structural features of G.

In this note we shall characterize all central idempotent measures on a connected locally compact group G. We consider $\mu \in M(G)$, the convolution algebra of all complex measures on G with finite total variation. A measure μ is *central* if $\mu(\alpha_x(E)) = \mu(E)$ (where $\alpha_x(g) = xgx^{-1}$) for every Borel set E and $x \in G$. These measures form the center of the algebra $M(G)$. We shall describe the *idempotent* central measures, those such that $\mu * \mu = \mu$, in terms of the structural features of G.

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Let $B(G)$ be the set of elements in G whose conjugacy class has compact closure; according to Tits [16], $B(G)$ is a closed characteristic subgroup in G. Let $K(G)$ be the largest compact normal subgroup in G. The existence of $K(G)$ is discussed in Iwasawa [10]. Finally, let K_0 be the identity component of $K(G)$. For any compact group N we write $N^\wedge$ for the set of equivalence classes of irreducible unitary representations.

DEFINITION. Let $J(G)$ be the set of all central idempotent measures on G. Then $J(G)$ is a Boolean algebra under the operations

$$\mu \wedge \nu = \mu * \nu \qquad \mu \vee \nu = \mu + \nu - \mu * \nu.$$

The unit in $J(G)$ is δ_e, the point mass at the unit. Every $\mu \in J(G)$ has a complement $\mu^\sim = \delta_e - \mu$.

Let H be a compact normal subgroup of G and m_H the normalized Haar measure on H. Then $m_H \in J(G)$. We get other central idempotents supported on H by taking any irreducible representation $\rho \in K(G)^\wedge$, restricting its character $\chi_\rho(g) = \mathrm{Trace}(\rho(g))$ to H, and making a suitable normalization. We write

$$\phi_{\rho,H}(h) = \frac{d(\rho)}{\int_H |\chi_\rho(h)|^2\, dh} \chi_\rho(h) \qquad \text{for all} \quad h \in H, \tag{1}$$

where $d(\rho)$ is the degree of ρ. Then $\nu_{\rho,H} = \phi_{\rho,H} \cdot m_H$ is a central idempotent. We show that these primitive central idempotents actually generate $J(G)$ as a Boolean algebra.

THEOREM. *Let G be any connected locally compact group. Then $J(G)$ consists of finite Boolean combinations of the primitive idempotents $\nu_{\rho,H}$ where $\rho \in K(G)^\wedge$ and H is a compact normal subgroup of G.*

This theorem was proved for $G = T$ (circle group) by Helson [8] and for $G = T^n$ by Rudin [15]. P. J. Cohen in [2] gave a powerful generalization, proving the result for all locally compact abelian G. The proof of Cohen's result was substantially simplified in a note by Ito and Amemiya [9]. Recently, attention has shifted to noncommutative groups. In [13] Rider proved the result for unitary groups, by reducing the problem to the case of compact abelian groups. Ragozin [12] proved the result valid for any compact simple Lie group, without recourse to the abelian case. Finally, Rider [14] verified the result for arbitrary compact *connected* groups. Until recently there have been only scattered results on arbitrary locally compact groups.

The structure of *all* idempotent measures with $\|\mu\| \leqslant 1$ was determined in Greenleaf [4]. On a group admitting a compact neighborhood of the unit invariant under inner automorphisms (an IN group), each central idempotent has compact support group, as shown in Mosak–Moskowitz [11]. Then in [6] the authors proved the following results for connected locally compact G.

$$\text{Every central measure } \mu \in M(G) \text{ has } \operatorname{supp}(\mu) \subseteq B(G). \tag{2}$$

From this we will show that every central idempotent $\mu \in J(G)$ has $\operatorname{supp}(\mu) \subseteq K(G)$. By combining Rider's results [14] on compact groups with those of the authors on connected noncompact G, we prove the present Theorem. Here Iwasawa's classic theorem [10, Theor. 1′] on automorphisms of a compact group plays an important role.

We will need some technical observations relating $H^\wedge$ and $K^\wedge$ when H is a closed normal subgroup in a compact group K. If $\rho \in K^\wedge$, we form the *normalized character*

$$\psi_\rho(k) = d(\rho)^{-1}\chi_\rho(k) = d(\rho)^{-1} \operatorname{Tr} \rho(k).$$

LEMMA 1. *Let K be a compact group and H a closed normal subgroup. Assume that K is generated by its identity component together with its center Z. Then*

(i) *If $\rho \in K^\wedge$, then there is a $\sigma \in H^\wedge$ such that $\psi_\rho \mid H = \psi_\sigma$.*

(ii) *If $\sigma \in H^\wedge$, then there is a $\rho \in K^\wedge$ such that $\psi_\rho \mid H = \psi_\sigma$.*

(iii) *If $\sigma \in H^\wedge$, then the normalized character ψ_σ is K-invariant:*

$$\psi_\sigma(khk^{-1}) = \psi_\sigma(h) \qquad \textit{for all} \quad h \in H, \quad k \in K.$$

Proof. Let $K \times H^\wedge \to H^\wedge$ be the usual action $k \cdot \sigma(h) = \sigma(khk^{-1})$. By Clifford's Theorem [1; Theor. 1], if $\rho \in K^\wedge$, its restriction decomposes $\rho \mid H = \sum_{i=1}^{p} m\sigma_i$ where $\{\sigma_1, ..., \sigma_p\}$ form a single K-orbit in $H^\wedge$ and $m \geqslant 1$. By hypothesis, K acts as a connected group on $H^\wedge$, which is discrete (Z acts trivially). Thus K-orbits in $H^\wedge$ are single points; i.e., $\rho \mid H = m\sigma$ for some $\sigma \in H^\wedge$. Now $d(\rho) = m \cdot d(\sigma)$, so that

$$\psi_\rho \mid H = d(\rho)^{-1} \chi_\rho \mid H = d(\rho)^{-1} m \, \chi_\sigma = d(\sigma)^{-1} \chi_\sigma = \psi_\sigma$$

on H. For (ii), if K were separable Mackey's version of the subgroup theorem for induced representations would imply that every $\sigma \in H^\wedge$ appears in the decomposition of $\rho \mid H$ for some $\rho \in K^\wedge$. For general K

this is proved by Grosser and Moskowitz [7, Theor. 5.1]. In (iii) invariance of the character follows from the fact that K leaves points $\sigma \in H^\wedge$ fixed. Q.E.D.

If K is as in Lemma 1 and H normal in K, we get the same set of central idempotents in $M(K)$ by two different constructions.

(i) Taking $\rho \in K^\wedge$, forming $\psi_\rho \cdot m_H$, and normalizing to get $\nu_{\rho,H}$ as in (1).

(ii) Taking $\sigma \in H^\wedge$ and scaling $\psi_\sigma \cdot m_H$ by $d(\sigma)$.

Obviously construction (ii) gives an idempotent; it is *central in* $M(K)$ by Lemma 1(iii).

Next we must show that the Theorem is true for certain quotients if it is true for the original group. Such a result is used implicitly, and without proof, in [14]. However, there seem to be some subtle points to the proof, so we shall give the details here.

LEMMA 2. *Let K be a compact group generated by its identity component together with its center. Let R be a closed central subgroup in K. Then the Theorem is true for K/R if it is true for K.*

Proof. Let $\pi: K \to K/R$ be the quotient homomorphism, m_R normalized Haar measure on R, and let $j(\mu) = m_R * \mu = m_R * \mu * m_R$ for $\mu \in M(K)$. This is a norm decreasing homomorphism of $M(K)$ *onto* the subalgebra $M(K; R) = m_R * M(K)$ of measures "constant on cosets of R." There is a canonical isometric isomorphism $\pi^*: M(K; R) \to M(K/R)$, see [5, Appendix, A.8]. Lemma 2 follows from the following assertion.

ASSERTION. *The homomorphism $\phi = \pi^* \circ j: M(K) \to M(K/R)$ maps $J(K)$ onto $J(K/R)$, preserves Boolean operations, and maps the set of primitive central idempotents $\mathscr{P} = \{\nu_{\rho,H} : \rho \in K^\wedge, H$ normal in $K\}$ on K onto the set of primitive central idempotents $\mathscr{P}' = \{\nu_{\rho',H'} : \rho' \in (K/R)^\wedge, H'$ normal in $K/R\}$ on K/R.*

Proof of Assertion. Clearly $\phi(\delta_k) = \delta_{\pi(k)}$ for point masses. Since ϕ is a convolution homomorphism and $\phi(\delta_e) = \delta_{eR}$, ϕ preserves Boolean operations. Since $\phi(\delta_g * \mu) = \delta_{\pi(g)} * \phi(\mu)$ and measures are central if and only if $\delta_g * \mu = \mu * \delta_g$, it follows that ϕ maps central measures in $M(K)$ to central measures in $M(K/R)$. But ϕ also maps idempotents to idempotents, so it maps $J(K)$ into $J(K/R)$. This map is onto, for if $\mu' \in J(K/R)$ corresponds to $\mu \in M(K; R)$ under π^*, then $\mu = m_R * \mu * m_R$. Now μ is idempotent; it is central on K because

$$\delta_{\pi(k)} * \mu' = \mu' * \delta_{\pi(k)} \Rightarrow \delta_k * \mu = \delta_k * m_R * \mu = \mu * \delta_k * m_R$$
$$= \mu * m_R * \delta_k = \mu * \delta_k$$

for $k \in K$. Finally, $\phi(\mu) = \pi^*(j(\mu)) = \pi^*(\mu) = \mu'$.

Next we show $\phi(\mathscr{P}) = \mathscr{P}'$. *The main problem is to show* $\phi(\mathscr{P}) \subseteq \mathscr{P}'$. If $\nu_{\rho',H'} \in \mathscr{P}'$, let $\rho = \rho' \circ \pi \in K^\wedge$ and $H = \pi^{-1}(H')$. Clearly $\nu_{\rho,H} \in \mathscr{P} \cap M(K; R)$, since ψ_ρ is constant on R cosets and $m_R * m_H = m_{RH} = m_H$. Furthermore, $\phi(\nu_{\rho,H}) = \nu_{\rho',H'}$ because routine arguments show that

$$d(\rho) = d(\rho') \qquad \text{and} \qquad \int |\chi_\rho(h)|^2 \, dm_H = \int |\chi_{\rho'}(h')|^2 \, dm_{H'}.$$

Thus $\phi(\mathscr{P}) \supseteq \mathscr{P}'$.

Conversely, consider $\nu_{\rho,H}$ where $\rho \in K^\wedge$ and H normal in K. It is easy to see that $m_R * m_H = m_{HR} \in M(K; R)$ and that m_{HR} goes to $m_{H'}$ $(H' = HR/R)$ under the isomorphism π^*. Let

$$c = d(\rho)^2 \cdot \left(\int_H |\chi_\rho(h)|^2 \, dm_H\right)^{-1},$$

so that $\nu_{\rho,H} = c \cdot (\psi_\rho \cdot m_H)$. We shall ignore c and show that $j(\psi_\rho \cdot m_H) = (\psi_\rho \cdot m_H) * m_R$ is given by $\psi_\sigma \cdot m_{HR} = \psi_\sigma \cdot (m_H * m_R)$ for some $\sigma \in HR^\wedge$ with ψ_σ equal to ψ_ρ on H and constant on cosets of R. First recall that $\psi_\rho(kr) = \psi_\rho(k)\,\psi_\rho(r)$ for $k \in K$, $r \in R$ since ψ_ρ is a normalized character on K and R is central. For any $k \in H \cap R$ and $f \in C(K)$, a continuous function on K, we get

$$\langle (\psi_\rho m_H) * m_R, f\rangle = \int_H \int_R f(sr)\,\psi_\rho(s)\, dm_H(s)\, dm_R(r),$$
$$= \int_H \int_R f(skr)\,\psi_\rho(sk)\, dm_H(s)\, dm_R(r),$$
$$= \psi_\rho(k) \int_H \int_R f(sr)\,\psi_\rho(s)\, dm_H(s)\, dm_R(r).$$

Since this is true for all $k \in H \cap R$ and $f \in C(K)$ we must have

$$\text{Either} \quad \psi_\rho \mid H \cap R \equiv 1 \qquad \text{or} \qquad \pi^*(\psi_\rho m_H) = 0. \tag{3}$$

In the latter case, $\phi(\nu_{\rho,H}) = 0 \in \mathscr{P}'$. In the former case we get a well defined continuous function $F\colon HR \to \mathbf{C}$ by extending $\psi_\rho \mid H$ constant along R cosets: $F(sr) = \psi_\rho(s)$ for all $s \in H$, $r \in R$. This is well defined due to (3). Now

$$\langle \pi^*(\psi_\rho m_H), f\rangle = \int_H\int_R f(sr)F(sr)\,dm_H(s)\,dm_R(r) = \langle F \cdot m_H * m_R\,, f\rangle,$$

so that $\pi^*(\psi_\rho m_H) = F m_{HR}$. Now on HR we have $F = \psi_\sigma$ for some $\sigma \in HR^\wedge$ because F satisfies the functional identity characteristic of normalized irreducible characters on a compact group (see Dixmier [3, Sect. 15.3.10] for details):

$$F(x)F(y) = \int_{HR} F(txt^{-1}y)\,dm_{HR}(t) \qquad \text{for all} \quad x, y \in HR. \tag{4}$$

To show (4), write $x = x'r'$ and $y = y'r''$ where $x', y' \in H$. Then $F(x) = F(x'r') = F(x') = \psi_\rho(x')$ and similarly, $F(y) = \psi_\rho(y')$. The integral above is

$$\begin{aligned}\int_{HR} F(srxr^{-1}s^{-1}y)\,dm_H(s)\,dm_R(r) &= \int_H\int_R F(sx'r's^{-1}y'r'')\,dm_H(s)\,dm_R(r),\\ &= \int_H F(sx's^{-1}y' \cdot r'r'')\,dm_H(s),\\ &= \int_H \psi_\rho(sx's^{-1}y')\,dm_H(s) = \psi_\rho(x')\,\psi_\rho(y'),\end{aligned}$$

as required.

Again by Lemma 1(ii), there is a $\tau \in K^\wedge$ such that $\psi_\tau \mid HR = F = \psi_\sigma$. Now ψ_τ (on K) is constant on R cosets; in particular, $\psi_\tau \equiv 1$ on R. This implies that $\tau \equiv I$ on R. Indeed, $\tau \mid R$ is a multiplicative character times I, $\tau(r) = \alpha(r) \cdot I$ because R is central and τ irreducible. Obviously there is a $\rho' \in (K/R)^\wedge$ such that $\tau = \rho' \circ \pi$, so that $\psi_\tau = \psi_{\rho'} \circ \pi$. Now

$$\phi(\psi_\rho m_H) = j((\psi_\rho m_H) * m_R) = j(F \cdot m_{HR}) = j(\psi_\tau m_{HR}) = \psi_{\rho'} m_{H'}\,,$$

where $H' = HR/R$.

The normalizing constants

$$c = d(\rho)^2 \left(\int |\chi_\rho(h)|^2\,dm_H\right)^{-1} \qquad c' = d(\rho')^2 \left(\int |\chi_{\rho'}(h')|^2\,dm_{H'}\right)^{-1}$$

must agree since $\phi(\nu_{\rho,H}) = c \cdot \phi(\psi_\rho \cdot m_H)$ is idempotent and a scalar multiple of $\nu_{\rho',H'} = c'(\psi_{\rho'} \cdot m_{H'})$, which is also idempotent. Thus $\phi(\nu_{\rho,H}) = \nu_{\rho',H'} \in \mathscr{P}'$ and $\phi(\mathscr{P}) \subseteq \mathscr{P}'$. Q.E.D.

Before proving the Theorem, we state Iwasawa's result.

THEOREM (Iwasawa, [10, Theor. 1′]). *Let K be a compact group and* Aut(K) *the group of all bicontinuous automorphisms. Let* $\mathrm{Aut}_0(K)$ *be its identity component and* $\mathrm{Int}(K) \subseteq \mathrm{Aut}(K)$ *the inner automorphisms. Then* $\mathrm{Aut}_0(K)$ *is the set of inner automorphisms* $\{\alpha_k : k \in K_0$, *the identity component of* $K\}$. *In particular, any connected subgroup* $\mathfrak{A}$ *in* Aut(K) *lies within* $\mathrm{Aut}_0(K) \subseteq \mathrm{Int}(K)$.

Proof of Theorem. We must first show that

$$\text{Every central idempotent measure } \mu \in J(G) \text{ has } \mathrm{supp}(\mu) \subseteq K(G). \tag{5}$$

If $\mu \in J(G)$ then $\mathrm{supp}(\mu) \subseteq B(G)$ by (2). Now $G^* = G/K(G)$ is a Lie group (Hilbert's fifth problem; $K(G)$ contains all small normal subgroups); G^* has no proper compact normal subgroups and, from the definition of $B(G)$, $B(G) = \pi^{-1}(B(G^*))$ where $\pi: G \to G^*$ is the quotient map. For any Lie group G^* in which $K(G^*)$ is trivial, the nilradical is simply connected [6, Lemma 3.1]; from Tits [16] it follows that $B(G^*)$ has the simple structure $B(G^*) = Z(G^*) \cdot V$ where $Z(G^*)$ = center of G^*, and V is a closed vector subgroup (contained in the center of the nilradical). Thus, in our situation, $B(G^*)$ is a closed abelian subgroup of G^*, and is isomorphic to $\mathbf{Z}^p \times \mathbf{R}^q$. Thus $B(G)$ is an extension

$$\{e\} \longrightarrow K(G) \longrightarrow B(G) \xrightarrow{\pi} B(G^*) \cong \mathbf{Z}^p \times \mathbf{R}^q \longrightarrow \{e\}.$$

Consequently $B(G)$ has a compact invariant neighborhood of the unit. It follows from [11] that the "support group" of μ = closed subgroup generated by supp(μ), is compact. This support group is obviously normal, hence contained in $K(G)$, as required for (5).

If H is compact and normal in the locally compact connected group G, it is easily seen that the measures $\nu_{\rho,H}$ in (1) have been normalized so they are idempotents in $M(G)$. Clearly $H \subseteq K(G)$ and $\nu_{\rho,H}$ are invariant under the inner automorphisms of $K(G)$ acting on itself. The inner automorphisms of G acting on itself leave $K(G)$ invariant and induce a connected group of automorphisms $\mathfrak{A} \subseteq \mathrm{Aut}(K(G))$. Therefore $\mathfrak{A} \subseteq \mathrm{Int}(K(G))$, so that $\nu_{\rho,H}$ is a central measure in $M(G)$.

Our result 2(ii) shows that the measures $\mu \in J(G)$ are precisely the $\mathfrak{A}$-invariant idempotent measures in $M(K(G))$. By the remarks above, these are just the central idempotent measures on $K(G)$, $J(G) = J(K(G))$.

Rider's result [14, Theor. 9.2] does not quite identify $J(K(G))$ since $K(G)$ might not be connected. However, using Iwasawa's result again we find the relationship between $K(G)$ and its connected component K_0.

$$K(G) = K_0 \cdot Z \qquad \text{where} \quad Z \text{ is the center of } K(G)$$

In fact, $\mathfrak{A} = \operatorname{Int}(G) \mid K(G)$ is equal to $\{\alpha_{k_0} : k_0 \in K_0\}$. Thus if $g \in G$ there is a $k_0 \in K_0$ such that

$$gkg^{-1} = k_0 k k_0^{-1}; \quad \text{i.e.,} \quad (k_0^{-1}g)k = k(k_0^{-1}g) \quad \text{all} \quad k \in K(G).$$

Thus $k_0^{-1}g \in C$ = centralizer of $K(G)$ in G, so that $G = K_0 \cdot C$. In particular $K(G) \cap C = Z$ and we get

$$K(G) = K_0 \cdot (K(G) \cap C) = K_0 \cdot Z.$$

Therefore $K(G)$ is isomorphic to a quotient, $K(G) \cong (K_0 \times Z)/R$, where $R = \{(k_0, z) : k_0 = z^{-1}\}$. Clearly R is central in $K_0 \times Z$ and $p(k_0, z) = k_0 z$ is an onto homomorphism $p: K_0 \times Z \to K(G)$ with kernel R. But K_0 may be written [17, Sect. 32] as $K_0 = (S \times A)/D$ where $S = \prod_{i \in I} S_i$ is a direct product of connected compact simple Lie groups, A is compact abelian, and D central in $S \times A$. Let $q: S \times A \to K_0$ be the quotient homomorphism. We have a system of *onto* homomorphisms

$$S \times A \times Z \xrightarrow{q \times id} K_0 \times Z \xrightarrow{\quad p \quad} K(G).$$

Now $\operatorname{Ker}(q) = D$ is central in $S \times A$, so $\operatorname{Ker}(q \times id)$ is central in $S \times A \times Z$; furthermore, $R = \operatorname{Ker}(p)$ is central in $K_0 \times Z$. Rider's results [14, Theor. 2.2] actually apply to direct products whose factors are compact connected simple Lie groups and (not necessarily connected) compact abelian groups. Therefore our Theorem is valid for $S \times A \times Z$. Two successive applications of Lemma 2 show the Theorem true for $K(G)$. Since $J(G) = J(K(G))$, the Theorem is proved in full. Q.E.D.

EXAMPLES. The affine group of the real line ($ax + b$ group) is modeled on the plane $\mathbf{R}^2$, equipped with the multiplication $(a, s) \cdot (b, t) = (a + b, s + e^a t)$. It is easily seen that $B(G) = K(G) = \{e\}$, so there are only the trivial central idempotents $\mu = 0$ and $\mu = \delta_e$. More generally, if G is a simply connected solvable Lie group it is well known that $K(G)$ must be trivial [20, Theor. 1.2, p. 135]. Thus the central idempotent measures are trivial on such groups.

The Heisenberg group N is $\mathbf{R}^3$ equipped with the operation $(a, b, c) \cdot (u, v, w) = (a + u, b + v, c + w + av)$; N is nilpotent with center $Z(N) = \{(0, 0, c) : c \in \mathbf{R}\}$. Let $\Gamma = \{(0, 0, n) : n \in \mathbf{Z}\}$ and $G = N/\Gamma$. Now $Z(G)$ is a toroid $T = Z(N)/\Gamma$; also, $K(G) = T$. As always, $B(G) = \pi^{-1}(B(G^*))$ where $\pi: G \to G^* = G/K(G)$ is the quotient map. Here $G/K(G) \cong \mathbf{R}^2$ is abelian, so $B(G^*) = G^*$ and $B(G) = G$. All central idempotents on G live on the toroid T. They are precisely the central idempotent measures for the group T itself (without reference to the action of G), which were analyzed in [15]. The group N has no proper compact subgroups, so its central idempotents are trivial.

Appendix

In certain cases it is possible to prove (2), which is our tool for handling noncompact groups, without resorting to the complicated structural analysis of unbounded conjugacy classes presented in [6]. We shall briefly outline this method, based on the Krein–Milman Theorem. It is useful in specific examples, but unfortunately applies only when the space of conjugacy classes $G/\mathrm{Int}(G)$ is known to be "sufficiently regular."

This orbit space has a natural Borel structure determined by the $\mathrm{Int}(G)$-invariant Borel sets in G. This structure is *countably separated* if there are countably many Borel sets $E_n \subseteq G/\mathrm{Int}(G)$ that separate points in this space; thus every orbit ζ is the intersection of those E_n which contain it. If G is a connected Lie group and if $G/\mathrm{Int}(G)$ is countably separated, then $G/\mathrm{Int}(G)$ is an analytic Borel space (cf. [18; Sect. 1] for definitions and basic results). It is difficult, at present, to state general conditions guaranteeing regularity of this orbit space, though it is usually not difficult to check regularity for a specific group. Not all groups have regular orbit spaces; but nilpotent and solvable groups of exponential type do.

Suppose that G is a connected Lie group such that $G/\mathrm{Int}(G)$ has a regular structure (it is enough that the set of *unbounded* orbits be countably separated). If μ is a nonnegative finite central measure, so is its restriction $\nu = \mu \mid G \sim B(G)$ to the union of the unbounded conjugacy classes. Can we have $\nu \neq 0$? Let $M(X)$ be all finite measures on the locally compact space $X = G \sim B(G)$, and let $\Sigma = \{\nu \in M(X)$: $\nu \geqslant 0$, $\|\nu\| \leqslant 1$, and ν is $\mathrm{Int}(G)$ invariant$\}$. Then $M(X) = C_0(X)^*$ and Σ is a weak-$*$ closed, compact convex set. By Krein–Milman, Σ is the weak-$*$ closed convex hull of its extreme points $\mathrm{Ext}(\Sigma)$. If

$\lambda \in \text{Ext}(\Sigma)$ then $\lambda = 0$ or $\| \lambda \| = 1$. If $\lambda \neq 0$ in $\text{Ext}(\Sigma)$ and if E is an $\text{Int}(G)$-invariant Borel set in X, then the measures $\lambda_1 = \lambda \mid E$ and $\lambda_2 = \lambda \mid (X \sim E)$ cannot both be nonzero; they are nonnegative, $\text{Int}(G)$-invariant, and after suitable scaling would yield a decomposition of λ as a convex sum of measures in Σ, which is impossible. Thus λ is *ergodic* under the action of $\text{int}(G)$. Since we assume that the Borel structure on $G/\text{int}(G)$ is countably separated, ergodic measures must live on single orbits [18, p. 17]. That is, there is a single conjugacy class $C_y \subseteq X$ such that $\lambda(C_y) = \| \lambda \| = 1$.

In many groups it can be shown directly that any conjugacy class which supports a finite $\text{Int}(G)$-invariant measure must be compact. (In general, this was proved for solvable connected Lie groups by Mostow, Homogeneous spaces with finite invariant measure, *Ann. of Math.*, **75** (1962) 17–37, and for semisimple groups without compact factors by Borel, Density properties for certain subgroups of semisimple groups without compact components, *Ann. of Math.*, **72** (1960) 179–188.) In such groups we conclude that y and C_y lie in $B(G)$; they cannot lie in X. Thus, for such groups, we conclude that $\text{Ext}(\Sigma) = \{0\}$, $\Sigma = \{0\}$, and that all central measures on G are supported within $B(G)$, if we also know that the space of conjugacy classes is regular.

Actually, the authors have proved [19] that in any connected Lie group, every conjugacy class which supports a finite $\text{Int}(G)$-invariant measure must be compact. However, their proof required the full machinery worked out in their study of unbounded conjugacy classes [6], and so cannot be invoked in obtaining an alternative proof of (2).

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