COMPACTNESS OF CERTAIN HOMOGENEOUS SPACES OF FINITE VOLUME.

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1. Introduction. We consider connected Lie groups G and closed subgroups H such that G/H has a finite G-invariant measure (finite volume). The question of whether this implies G/H is compact has been approached in various ways. One is to restrict G, allowing arbitrary subgroups H; Mostow [9] has shown that if G is solvable then finite volume implies compactness for any G/H. However, this result is not valid for other classes of groups, such as semisimple ones. We shall allow G to be arbitrary and require H to be the centralizer of some point $x \in G$.

Theorem 1. *Let G be a connected Lie group and H the centralizer of some point $x \in G$. If G/H has finite volume, then G/H is compact.*

This result overlaps another theorem of Mostow [9; Theorem 7.1], which says that if G is a connected Lie group, and H any closed subgroup with *finitely many connected components*, then finite volume implies compactness for G/H. It is not obvious how this result could be utilized to yield Theorem 1 when H is an arbitrary centralizer. Of course if G is an algebraic group then the centralizer of a point is an algebraic subgroup and so has a finite number of components.

Various applications are given in the last two sections of the paper. One of these generalizes that form of the Borel Density Theorem [1] which originally motivated it. Our result applies to a broader class of groups than semisimple groups without compact factors. We let $\mathcal{C}(x) = \mathcal{C}_G(x)$ denote the conjugacy class of x where $x \in G$ (or more generally if we are considering some action of G on a space X we let $\mathcal{C}_G(x)$ denote the G-orbit of x) and consider the characteristic subgroup $B(G) = \{x: x \in G,\ \mathcal{C}(x) \text{ has compact closure}\}$. The class of groups we are interested in are those for which $B(G)$ is trivial, i.e., coincides

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with the center $Z(G)$. These groups include among others semisimple groups without compact factors, simply connected complex analytic groups, complex analytic linear groups and connected Lie groups whose radical is simply connected of type E. (See Theorem 9.4 of [2]). In the present paper we also prove the following (Theorem 3, corollaries 2 & 3):

Let G be a connected Lie group, H a closed subgroup such that G/H has finite volume. Then the centralizer $Z_G(H) \subseteq B(G)$. In particular if $B(G)=Z(G)$ then $Z_G(H)=Z(G)$. Moreover if G has no compact connected normal subgroup then $B(G)$ and therefore also $Z_G(H)$ is abelian. If G is an arbitrary connected Lie group then $B(G)$ and therefore also $Z_G(H)$ have compact derived group.

Another application concerns restriction of lattices Γ to the center.

COROLLARY 4. *Let G be a connected Lie group, $G^\sim$ its universal covering group and suppose $B(G^\sim)=Z(G^\sim)$. If Γ is a lattice in G then $\Gamma \cap Z(G)$ is a uniform lattice in $Z(G)$.*

2. Proof of Theorem 1. If $x \in G$, $\mathcal{O}(x)$ is a σ-compact Borel set and the canonical bijection θ: $G/C_x \to \mathcal{O}(x)$ is continuous. Since G/C_x and $\mathcal{O}(x)$ are standard Borel spaces, θ is a Borel isomorphism, cf. [7], even though it is not always bicontinuous. Furthermore, θ is equivariant with respect to the actions of G:

$$\lambda_g: xH \to gxH \qquad \alpha_g: x \to gxg^{-1}, \quad g, x \in G;$$

thus, (finite) λ_G-invariant Borel measures on G/C_x may be transferred to (finite) $\mathcal{I}(G)$-invariant measures on $\mathcal{O}(x)$, and vice-versa. The latter measures μ may be regarded as *central* (that is, $\mathcal{I}(G)$-invariant) measures on G which are supported on the Borel set $\mathcal{O}(x)$ in the sense that $\mu(G \sim \mathcal{O}(x))=0$. Theorem 1 can therefore be recast as follows, by applying Lemma 1 below.

THEOREM 1′. *Let G be a connected Lie group and $\mathcal{O}$ any conjugacy class in G. If $\mathcal{O}$ supports a nonzero finite $\mathcal{I}(G)$-invariant measure, then $\mathcal{O}$ is a compact set.*

LEMMA 1. *Let G be a connected locally compact group. Fix $x \in G$ and let C_x be its centralizer. The following conditions are equivalent.*

(i) *G/C_x is compact*
(ii) *The conjugacy class $\mathcal{O}(x)=\{gxg^{-1}: g \in G\}$ is compact.*

Proof. The equivariant bijection $\theta: G/C_x \to \mathcal{O}(x)$ is continuous, so (i) implies (ii) is trivial. Conversely, if $\mathcal{O}(x)$ is compact, then standard Baire category arguments for homogeneous locally compact spaces, such as one finds in [4; pp. 7–8], show that θ is bicontinuous and G/C_x compact.

We note that G may have bounded conjugacy classes which are not closed. Theorem 1′ will show that those which are not closed cannot support a finite volume.

In the proof of Theorem 1 we utilize Tits' description [11] of $B(G)$ in terms of the structural features of G together with the authors' recent study (Theorem 2, below) of unbounded conjugacy classes in connected Lie groups. In G, let $Z(G)$ be the center, $N=N(G)$ the nilradical, $R=R(G)$ the radical, and $K=K(G)$ the maximal compact normal subgroup. The connected component $K_0=K(G)_0$ is the largest compact normal connected subgroup in G. $K(G)$ was first defined in [5].

THEOREM 2. ((1.5) *of* [2]). *If G is a locally compact connected group and μ any finite $\mathcal{I}(G)$-invariant measure on G, then the closed support $supp(\mu)$ lies in $B(G)$.*

Let $\mathcal{O}(x), x\in G$ be a conjugacy class with finite invariant volume μ. From Theorem 2 it follows that $x\in \mathcal{O}_x \subseteq \text{supp}(\mu)\subseteq B(G)$. We shall prove Theorem 1 first in the case when K_0 is trivial. Then Tits shows ([11] Théorèm 1) $B(G)=VZ(G)$ for some closed vector group V and $V=B(G)_0$. We write $B(G)=\cup_i z_i V$ (coset decomposition). If $z\in Z(G)$ the diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha_g} & V \\ {\scriptstyle \lambda_z}\downarrow & & \downarrow{\scriptstyle \lambda_z} \\ zV & \xrightarrow{\alpha_g} & zV \end{array}$$

commutes for each $g\in G$. The following easy lemma which will be used several times in the sequel enables us here to restrict attention to V.

LEMMA 2. *Let X and Y be locally compact Hausdorff G-spaces and π: $X\to Y$ a surjective continuous open G-equivariant map. If $\mathcal{O}_G(x)$, $x\in X$, supports a finite G-invariant measure then $\pi(\mathcal{O}_G(x))=\mathcal{O}_G(\pi(x))$ also supports a finite G-invariant measure.*

Proof. Let E be a Borel set in $\mathcal{O}(\pi(x))$. Then $\pi^{-1}(E)$ is a Borel set in $\mathcal{O}(x)$ and we define $\nu(E)=\mu(\pi^{-1}(E))$. Clearly ν is a finite regular Borel measure and

$$\nu(gE)=\mu(\pi^{-1}(g\cdot E))=\mu(g\cdot\pi^{-1}(E))=\mu(\pi^{-1}E)=\nu(E)$$

since $\pi^{-1}(g\cdot E)=g\cdot\pi^{-1}(E)$ for each E and $g\in G$.

Let $G_V = \mathcal{G}(G)|V$. Then G_V is a subgroup of $Gl(V)$. Since $V \subseteq B(G)$ each point $v \in V$ has bounded orbit. It follows easily that G_V has compact closure in $Gl(V)$. If G_V is given its natural Lie topology then the identity maps G_V faithfully and continuously into a compact group. By the Freudenthal-Weil theorem [4, p. 145] we can write $G_V = K_1 \times V_1$ the direct product of a compact group and a vector group. Let V/K_1 be the locally compact Hausdorf space of K_1-orbits in V equipped with the quotient topology. Let $\pi\colon V \to V/K_1$ be the canonical map, which is continuous and *open*. In these spaces we shall consider the topological Borel structures; both are standard, and π is a Borel map. The jointly continuous action $\varphi\colon G_V \times V \to V$ preserves K_1-orbits since K_1 is normal, and induces an action $\varphi'(g,\pi(v)) = g\cdot\pi(v) = \pi(g\cdot v)$ which makes the diagram below commute.

$$\begin{array}{ccccc} G_V & \times & V & \xrightarrow{\varphi} & V \\ \downarrow & id \times \pi & \downarrow & & \downarrow \pi \\ G_V & \times & V/K_1 & \xrightarrow{\varphi'} & V/K_1 \end{array}$$

Since $\pi(\mathcal{O}(x)) = \mathcal{O}(\pi(x))$ and G_V acts transitively on $\mathcal{O}(x)$, it acts transitively on $\mathcal{O}(\pi(x))$. But because this comes about by identifying points under the action of K_1, V_1 alone acts transitively on $\mathcal{O}(\pi(x))$. By lemma 2 the finite G_V-invariant volume on $\mathcal{O}(x)$ induces a G_V-invariant and therefore V_1 invariant volume on $\mathcal{O}(\pi(x))$. By lemma 2 we get a V_1 translation invariant volume on $V_1/\mathrm{Stab}(\pi(x))$. Since V_1 is abelian the stability group is normal and therefore $V_1/\mathrm{Stab}(\pi(x))$ is compact. The compactness of K_1 implies that $\pi^{-1}(\mathcal{O}(\pi(x))$ is compact. Now $\mathcal{O}(x)$ is K_1 saturated, i.e. $\mathcal{O}(x) = \pi^{-1}(\mathcal{O}(\pi(x))$. Thus $\mathcal{O}(x)$ is compact.

If K_0 is nontrivial we write $K_0 = Z(K_0)_0[K_0, K_0]$ the decomposition theorem for a compact connected Lie group (see e.g. p. 144 of [4]). Here $Z(K_0)_0$ is the identity component of the center of K_0 and $[K_0, K_0] = S_0$ the derived group. As is easily seen $Z(K_0)_0$ is a torus, S_0 is semisimple and both $Z(K_0)_0$ and S_0 are characteristic in G. It follows that $Z(K_0)_0$ is central in G. Let $\pi\colon G \to G^* = G/R$ be the canonical map and decompose G^* into the product of simple factors which we group as follows:

$$G^* = H_1 H_2 H_3$$

where H_3 is the noncompact part of G^*, H_2 the product of those compact simple factors in G^* which do not admit normal local cross sections with respect to π and H_1 the product of those compact simple factors that do.

Clearly $H_1 = \pi(S_0)$. The H_i and their products are normal and therefore closed (see [8]) pairwise commuting connected Lie subgroups of G^*. The H_i's intersect pairwise in finite central subgroups. Let $H = \pi^{-1}(H_2 H_3)$. Then H is a closed normal connected Lie subgroup of G containing R. Moreover one has

(i) $G = S_0 H$

(ii) $[S_0, H] = (e)$

(iii) $S_0 \cap H$ is a (finite) central subgroup of G.

Proof of (i), (ii), and (iii). Let $\mathfrak{g}$, $\mathfrak{r}$, $\mathfrak{s}_0$ and $\mathfrak{h}_i$ denote the Lie algebras of G, R, S_0 and H_i respectively. Then $\mathfrak{g}/\mathfrak{r} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$. If $x \in \mathfrak{s}_0 \cap (d\pi)^{-1}(\mathfrak{h}_2 \oplus \mathfrak{h}_3)$, then $d\pi(x) \in \mathfrak{h}_1 \cap (\mathfrak{h}_2 \oplus \mathfrak{h}_3) = (0)$ so that $x \in \mathfrak{r}$, that is $x \in \mathfrak{s}_0 \cap \mathfrak{r} = (0)$. This means $S_0 \cap H$ is discrete.

Since $S_0 \cap H$ is normal and G is connected, $S_0 \cap H$ is central. Since S_0 is normal $[S_0, H] \subseteq S_0$. Moreover, for $s_0 \in S_0$ and $h \in H$, $\pi[s_0, h] = [\pi(s_0), \pi(h)] \in [H_1, H_2 H_3] = (e)$. It follows that $[S_0, H]$ is a connected subgroup of $S_0 \cap R \subseteq S_0 \cap H$. Since the latter is discrete $[S_0, H] = (e)$. (i) is clear.

We prove Theorem 1 when S_0 is trivial (G without *normal* compact simple subgroups). For any $x \in G$ define the closed subgroups:

$$\begin{aligned} H_x &= \{ g \in G : \alpha_g(x) = gxg^{-1} \in xK_0 \} \\ &= \{ g \in G : \alpha_g(x) \equiv x (\operatorname{mod} K_0) \} \supseteq C_x = \{ g \in G : \alpha_g(x) = x \}; \end{aligned}$$

then consider an x such that G/C_x has finite volume. It follows that G/H_x and H_x/C_x have finite invariant volumes (under actions of G, H_x respectively); cf. [9, Lemma 2.5]. We note that this lemma, valid for general locally compact groups, has a simple self-contained proof independent of all other considerations in [9]. Thus our analysis does not depend in any important way on [9]. If $G' = G/K_0$, let $\pi\colon G \to G'$ be the canonical map. Then the diagram below commutes.

$$\begin{array}{ccc} G & \xrightarrow{\alpha_g} & G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G' & \xrightarrow[\alpha_{\pi(g)}]{} & G' \end{array} \qquad \text{all } g \in G.$$

By lemma 2 $\mathcal{O}(\pi(x))$ has a finite volume. Since $K(G')_0$ is trivial Theorem 1 is valid for G' and hence $\mathcal{O}(\pi(x))$ is compact. G acts transitively on $\mathcal{O}(\pi(x))$ via G' with isotropy group H_x. Hence G/H_x is compact by lemma 1.

For $g \in H_x$ let $\varphi(g) = [g, x]$. Then $\varphi\colon H_x \to K_0$ is a continuous map and an easy calculation shows that φ satisfies the cocycle identity

$$\varphi(g_1 g_2) = \alpha_{g_1}(\varphi(g_2))\varphi(g_1), \qquad g_1, g_2 \in H_x$$

Since $S_0 = (e)$ and K_0 is central in G, φ is an anti-homomorphism and therefore a homomorphism whose kernel is C_x. Since C_x is normal in H_x, H_x/C_x is a compact group. Therefore G/C_x is compact.

Next we assume $Z(S_0) = (e)$. Then $S_0 \cap H$ is a finite central subgroup of G by (iii) above and therefore $S_0 \cap H \subseteq Z(S_0) = (e)$. This means that $G = S_0 \times H$ (direct product) and if $x = s_0 h_0$, $\mathcal{O}_G(x) = \mathcal{O}_{S_0}(s_0)\mathcal{O}_H(h_0)$. Clearly $\mathcal{O}_{S_0}(s_0)$ is compact. Using the projection P_H: $S_0 \times H \to H$ and applying lemma 2 we get a finite H-invariant measure on $\mathcal{O}_H(h_0)$. Since H is a group without compact normal simple factors $\mathcal{O}_H(h_0)$ is compact by the above. Hence $\mathcal{O}_G(x)$ is compact.

Finally we consider the general case. Let $x \in G$ have the property that G/C_x has finite volume. Form π: $G \to G/Z(S_0) = G'$. Then $G' = \pi(S_0)\pi(H) = S_0'H'$. This is precisely the decomposition (i), (ii), (iii) above with G' replacing G. Using lemma 2 we get a finite G'-invariant measure on $\mathcal{O}(\pi(x))$. Since S_0 is semisimple $Z(S_0')$ is trivial and therefore $\mathcal{O}(\pi(x))$ is compact by the above. We consider the following diagram

$$\begin{array}{ccc} G & \xrightarrow{j_1} & G/C_x \\ {\scriptstyle j_2}\downarrow & \swarrow_{j_3} & \downarrow{\scriptstyle \theta} \\ \mathcal{O}(\pi(x)) & \xleftarrow[\pi]{} & \mathcal{O}(x) \end{array}$$

where j_1 is the canonical map, $j_3 = \pi \cdot \theta$ and $j_2 = j_3 j_1$. $\mathcal{O}(\pi(x))$ is a transitive G-space where G acts through G' by conjugation. Also j_2 coincides with the map of G onto the orbit induced by the action

$$j_2(g) = \pi\theta j_1(g) = \pi\alpha_g(x) = \alpha_{\pi(g)}(\pi(x))$$

The argument used in Lemma 1 (i.e. [4] p. 7–8) shows that j_2 is continuous and open. Since the same is true of j_1 and $j_3 j_1 = j_2$ we see that j_3 is a continuous and open surjection of G/C_x onto the compact space $\mathcal{O}(\pi(x))$. Since $Z(S_0)$ is finite the inverse image of each point under j_3 is finite. It follows that G/C_x is compact. Q.E.D.

3. Density Properties and Centralizers of Subgroups. Borel [1], extending work of Selberg, discussed certain "density theorems" concerning the centralizer of a closed subgroup H such that G/H has finite volume. When G is a connected semisimple Lie group without compact factors the centralizer $Z_G(H)$ coincides with the center $Z(G)$. We prove that similar results are valid for a broad class of groups.

THEOREM 3. *Let G be a connected locally compact group and H a closed subgroup such that G/H has finite volume. Then the centralizer* $Z_G(H) \subseteq B(G)$. *In particular if* $B(G)=Z(G)$ *then* $Z_G(H)=Z(G)$.

Proof. If $x \in Z_G(H)$ then $H \subseteq C_x$ so that we have the continuous open map π: $G/H \rightarrow G/C_x$. Since G acts equivalently and transitively by lemma 2 we get a finite G-invariant measure on G/C_x and therefore on $\mathcal{O}(x)$. This means by Theorem 2, $x \in B(G)$.

We remark that since Theorem 3 depends only on Theorem 2 and since for semisimple groups without compact factors a very short proof of Theorem 2 is given in [10], Theorem 3 constitutes a rapid proof of the Selberg form of the Borel density theorem.

COROLLARY 1. *Let G be a connected Lie group with* $B(G)=Z(G)$, *and* $\mathfrak{g}$ *its Lie algebra. If G/H has finite volume, then each* $X \in \mathfrak{g}$ *fixed under* Ad(H) *is in the center* $\mathfrak{z}(\mathfrak{g})$.

Proof. Consider the one parameter group $\{\exp(tX): t \in \mathbf{R}\}$. Then if $h \in H$,

$$\alpha_h(\exp(tX)) = \exp(\mathrm{Ad}(h)(tX)) = \exp(t\,\mathrm{Ad}(h)X)$$

$$= \exp(tX) \qquad \text{all } t \in \mathbf{R},$$

since $\mathrm{Ad}(H)X = X$. Thus, by Theorem 3, $\exp(tX) \in Z_G(H) = Z(G)$ for all $t \in \mathbf{R}$, and so $X \in \mathfrak{z}(\mathfrak{g})$.

In more general groups, we may expect finite volume of G/H to insure that H is sufficiently pervasive in G that $Z_G(H)$ is abelian, if not equal to $Z(G)$. As before N denotes the nilradical and R the radical of G.

COROLLARY 2. *Let G be a connected Lie group, H a closed subgroup such that G/H has finite volume. If* $K(G)_0=(e)$ *then* $Z_G(H)$ *is abelian; in fact* $B(G) \subseteq Z(N)\cdot Z(G)$.

In particular:

COROLLARY 2′. *Let G be a connected Lie group, H a closed subgroup such that G/H has finite volume. If (i) G/R has no compact factors and (ii) N is simply connected then* $Z_G(H)$ *is abelian and in fact* $B(G) \subseteq Z(N)Z(G)$.

Corollary 2′ was proved in the special case where G is simply connected solvable and H is discrete by R. Tolimieri [12] and also by S. P. Wang [13]. In its present form it has also been proven by Claire Sit using different methods. Mrs. Sit has also extended the results of Corollary 3 (below) to certain nonconnected and non-Lie groups. These results will appear in her doctoral dissertation (CUNY Graduate Center).

Proof of Corollary 2′. Since G/R has no compact factors $K(G)_0 \subseteq R$. As a compact connected solvable group it follows from the Peter-Weyl theorem together with the global form of Lie's theorem $K(G)_0$ is abelian i.e. a torus. Since the automorphism group of a torus is discrete and G is connected $K(G)_0$ is central. It follows that $K(G)_0 \subseteq N$ and therefore is trivial. This means that lemma 6 of [11] applies. If $x \in B(G)$, then α_x is an automorphism of bounded displacement, and hence by Theorem 3 of [11], $\alpha_x = \alpha_y$ for some $y \in Z(N)$. Thus, $x = yz$ where $y \in Z(N)$, $z \in Z(G)$. Since we have just shown that $B(G) \subseteq Z(N) \cdot Z(G)$, Corollary 2′ follows from Theorem 3 above. Corollary 2 was proven in the course of proving Corollary 2′.

The hypothesis in Theorem 3 is stronger than that of Corollary 2, as is the conclusion. The following example shows that stronger hypotheses than those of Corollary 2 are needed to conclude that $Z_G(H) = Z(G)$, as in Theorem 3.

Example. Let G be the semidirect product group $R \times_\eta R^2$ of Euclidean motions of the plane, and let $H = \{e\} \times Z^2$. The hypothesis of Corollary 2 holds, but not that of Theorem 3, since $B(G) = \{e\} \times_\eta R^2 = N(g)$ and $Z(G) = \{e\} \times \{e\}$. Moreover, $Z_G(H) = B(G)$.

We now recall certain definitions from [3]. If G is a locally compact group and $x \in G$, we say that x is an $[FC]^-$-element if the closure $\mathcal{O}(x)^-$ of the conjugacy class is compact; i.e. if and only if $x \in B(G)$. We say that G is an $[FC]^-$ *group* if $G = B(G)$. More generally, if $\mathfrak{L} \subseteq \mathfrak{A}(G)$, the group of bicontinuous automorphisms of G, we say that G is an $[FC]^-_{\mathfrak{L}}$ *group* if each orbit $\mathfrak{L}(x)$ has compact closure. If the derived group $[G,G]$ of G has compact closure, we say that G is an $[FD]^-$ *group*. It is easy to see that an $[FD]^-$ group is also $[FC]^-$.

PROPOSITION 1. *In any connected Lie group, or for that matter any locally compact connected group G, the subgroup $B(G)$ is a compactly generated $[FD]^-$ group.*

Proof. Let K be the maximal compact normal subgroup of G. Then G/K is an analytic group without compact normal subgroups and as such, $B(G/K) = B(G/K)_0 \cdot Z(G/K)$ (see [11]). Also since K is compact $B(G) = \pi^{-1}(B(G/K))$ where π: $G \to G/K$ is the canonical endomorphism. Therefore to see that $B(G)$

is compactly generated, since K is compact, it suffices to show $B(G/K)$ is compactly generated. Thus we may assume for the purpose of proving $B(G)$ is compactly generated that G is an analytic group and contains no compact normal subgroups. Now $B(G)=B(G)_0\cdot Z(G)$ and since $B(G)$ is closed in G and G is a connected Lie group and therefore second countable, the second isomorphism theorem holds.

$$B(G)/B(G)_0 \cong Z(G)/B(G)_0 \cap Z(G).$$

Here $B(G)_0$ is a vector space contained in the center of the nilradical N, (see [11]) and $B(G)_0 \cap Z(G)$ is the subspace of points fixed under all inner automorphisms of G, acting on $B(G)_0$, and is therefore connected so that $B(G)_0 \cap Z(G) \subseteq Z(G)_0$. The converse inclusion is obvious, so $Z/Z_0 \cong B(G)/B(G)_0$. We will show that Z/Z_0 is finitely generated; the same is then true of $B(G)/B(G)_0$. If W is any symmetric neighborhood of e in $B(G)_0$, and $g_1,\dots,g_k$ are representatives in $B(G)$ of the generators together with their inverses in $B(G)/B(G)_0$, then $\cup_{i,j=1}^{k} g_i W \cap W g_j$ is a compact symmetric neighborhood of e in $B(G)$ which generates $B(G)$.

Now $Z(G)/Z(G)_0$ is a discrete normal subgroup of the Lie group G/Z_0. If Γ is any discrete normal subgroup of an analytic group H and if $(H^{\sim},\pi)$ is the simply connected covering of H then $\pi^{-1}(\Gamma)$ is a discrete normal subgroup of $H^{\sim}$ and hence since $H^{\sim}$ is simply connected, $\pi^{-1}(\Gamma)$ is the fundamental group of $H^{\sim}/\pi^{-1}(\Gamma)$. As the fundamental group of a Lie group $\pi^{-1}(\Gamma)$ is finitely generated [4]. Hence so is Γ.

This proves $B(G)$ is a compactly generated group. If G is a Lie group, then as remarked earlier [11] shows that $B(G)$ is closed, but clearly this closedness is also valid for any connected locally compact connected group G. This implies that $B(G)\in[FC]_{\mathfrak{L}}^{-}$ where $\mathfrak{L}=\mathfrak{I}(G)$ restricted to $B(G)$, and in particular is an $[FC]^{-}$ group. As a compactly generated $[FC]^{-}$ group, $B(G)$ is an $[FD]^{-}$ group by Theorem 3.20 of [3].

We now give a new proof of the following result of D. H. Lee [6] which in turn depended on the Borel Density Theorem as well as certain strengthenings of results of H. C. Wang. The present proof depends instead on [3] and [11].

Corollary 3. *Let G be a Lie group and H a closed subgroup such that G/H has finite volume. Then $Z_G(H)$ is an $[FD]^{-}$ group.*

Proof. By Theorem 3, $Z_G(H)\subseteq B(G)$ and by Proposition 1, $B(G)$ is an $[FD]^{-}$ group. Evidently the closed subgroup $Z_G(H)$ is also an $[FD]^{-}$ group.

Finally we turn to the question of restriction of lattices to the center of G. We first recall that if Γ is a lattice in G then $\Gamma \cap Z(G)$ is a lattice (and hence a uniform lattice) in $Z(G)$ if and only if $Z(G)\Gamma$ is a closed set in G. In fact if $Z\Gamma$ is closed and G/Γ has finite volume so does $Z\Gamma/\Gamma \approx Z/Z \cap \Gamma$ by the lemma of Mostow mentioned earlier. Conversely since Z is abelian finite volume of $Z/Z \cap \Gamma$ implies compactness of this space. Since the map $Z/Z\cap\Gamma \to G/\Gamma$ is continuous the image $Z\Gamma/\Gamma$ is compact and therefore its inverse image $Z\Gamma$ under π: $G \to G/\Gamma$ is closed.

COROLLARY 4. *Let G be a connected Lie group, $G^{\sim}$ its universal covering group and suppose $B(G^{\sim}) = Z(G^{\sim})$. If Γ is a lattice in G then $\Gamma \cap Z(G)$ is a uniform lattice in $Z(G)$.*

We note that the assumption $B(G^{\sim}) = Z(G^{\sim})$ is somewhat weaker than $B(G) = Z(G)$. (This means for example that Corollary 4 applies to any complex analytic group or to any connected Lie group whose radical is of type E.) For suppose $B(G) = Z(G)$ and $x^{\sim} \in B(G^{\sim})$. Then $x = \pi(x^{\sim})$ is clearly in $B(G)$, where π: $G^{\sim} \to G$ is the covering map; hence $x \in Z(G)$, and $[x^{\sim}, G^{\sim}] \subseteq \operatorname{Ker}\pi$, which is discrete. But $[x^{\sim}, G^{\sim}]$ is connected and hence $=(e)$, so that $B(G^{\sim}) = Z(G^{\sim})$. Simple examples show the converse is false. We remark that the case of semisimple groups without compact factors has been dealt with in [14] for uniform lattices and in [15] for nonuniform lattices.

Proof of Corollary 4. First suppose $B(G) = Z(G)$. Let $N_G(\Gamma)$ be the normalizer of Γ in G. Then $N_G(\Gamma)$ is a closed subgroup of G and is therefore a Lie group. Its identity component N_0 is therefore a connected Lie group which normalizes and therefore centralizes the discrete group Γ. By Theorem 3, $N_0 \subseteq Z(G)$. Thus $N_G(\Gamma) \supseteq \Gamma Z(G) \supseteq N_0$ so that $\Gamma Z(G)$ is an open and therefore closed subgroup of $N_G(\Gamma)$. Since $N_G(\Gamma)$ is closed in G so is $\Gamma Z(G)$. Now let π: $G^{\sim} \to G$ be the covering map. As is easily seen $\Gamma^{\sim} = \pi^{-1}(\Gamma)$ is a lattice in $G^{\sim}$. Since π is a covering map $\pi(Z(G^{\sim})) = Z(G)$ and hence $\pi(Z(G^{\sim})\Gamma^{\sim}) = Z(G)\Gamma$. But $Z(G^{\sim})$ and therefore also $Z(G^{\sim})\Gamma^{\sim}$ contains $\operatorname{Ker}\pi$ and therefore its image $Z(G)\Gamma$ in G is closed.

Example. Let G be the simply connected covering group of $Sl(2, \mathbf{R})$. Then $Z(G) \cong \mathbf{Z}$, and $B(G) = Z(G)$ by [10]. If Γ is any discrete subgroup such that G/Γ has finite volume, then $\Gamma \cap Z(G)$ is uniform in $Z(G)$, or equivalently, Γ meets $Z(G)$ nontrivially.

COROLLARY 5. *Let Γ be a lattice in a connected Lie group G and suppose that $B(G\sim) = Z(G\sim)$. Then $Ad(\Gamma)$ is a discrete subgroup of $Gl(\mathfrak{g})$.*

Proof. Consider $\mathrm{Ad}: G \to Gl(\mathfrak{g})$. Clearly $\mathrm{Ad}^{-1}(\mathrm{Ad}(\Gamma)) = \Gamma$. $Z(G)$ is closed (Corollary 4). By [16], $\mathrm{Ad}(G)$ is closed in $Gl(\mathfrak{g})$. Since G is second countable,

$\mathrm{Ad}: G \to \mathrm{Ad}(G)$ is open, and so $\mathrm{Ad}(\Gamma)$ is closed in $Gl(\mathfrak{g})$. Since $\Gamma Z(G)/Z(G) \cong \Gamma/\Gamma \cap Z(G)$ is discrete, so is $\mathrm{Ad}(\Gamma)$.

Note added in proof: Here is a variant of Theorem 3 which enables us to weaken the hypothesis on G to one on $G\sim$.

THEOREM 3′. *Let G be a connected Lie group such that $B(G\sim) = Z(G\sim)$, and H a closed subgroup such that G/H has finite volume. Each analytic subgroup L which centralizes H is central in G.*

Proof. If G is simply connected, this is Theorem 3. If $(G\sim, \pi)$ is the covering group, $\pi^{-1}(H)$ is closed and $G\sim/\pi^{-1}(H)$ and G/H are $G\sim$-equivariantly homeomorphic, so $G\sim/\pi^{-1}(H)$ has finite $G\sim$-invariant volume. If $L\sim$ is the analytic subgroup of $G\sim$ whose Lie algebra of L, then for $l\sim \in L\sim$ and $x\sim \in \pi^{-1}(H)$ we have $\pi(x\sim) = h \in H$, $\pi(l\sim) = l \in L$, and $\pi([l\sim, x\sim]) = [l,h] = e$ since L centralizes H. Thus $[L\sim, x\sim] \subseteq \mathrm{Ker}\,\pi$. Since $L\sim$, and hence also $[L\sim, x\sim]$ are connected, and $\mathrm{Ker}\,\pi$ is discrete, $[L\sim, x\sim]$ reduces to the identity so that $L\sim$ centralizes $x\sim$. Therefore $L\sim$ centralizes $\pi^{-1}(H)$. By the first part of the proof, $L\sim \subseteq Z(G\sim)$, so that $L \subseteq Z(G)$.

Theorem 1 has recently been extended by the authors to cover all second countable (not necessarily connected) Lie groups, see "Automorphisms, orbits, and homogeneous spaces of non-connected Lie groups", *Math. Annalen*, 212 (1974) pp. 145–155. This article streamlines some of the above proofs and applies to centralizers of families of automorphisms rather than the centralizer of a single inner automorphism.

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