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A criterion for analytic hypoellipticity of a class of differential operators with polynomial coefficients

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1. Introduction

We shall establish here necessary and sufficient conditions for analytic regularity of solutions of some partial differential equations. Recall that a linear partial differential operator $P = P(x, D)$ with analytic coefficients is *analytic hypoelliptic* if for any distribution with compact support $u \in \mathscr{D}'(\Omega)$,

$$SS_a(Pu) = SS_a(u),$$

where SS_a denotes the analytic singular support. Microlocal analytic hypoellipticity is defined in the same way, by use of the analytic wave front set WF_a. (See e.g. [1] or [27, Definition 6.1] for the definition.)

Analytic hypoellipticity for differential operators with variable coefficients and multiple characteristics has been studied by Trèves [30], Tartakoff [29], Métivier [18] and others, in the case where the characteristic variety is symplectic. A sufficient condition for microlocal analytic hypoellipticity is the injectivity in some Schwartz space $\mathscr{S}(\mathbf{R}^n)$ of a differential operator with polynomial coefficients canonically associated to the operator under consideration at every characteristic point. For example their results apply to operators like

$$P = \Delta_t + |t|^2\Delta_y + \sum a_k D_{y_k} + \sum b_j D_{t_j} + \sum c_{jk} t_j D_{y_k} + \sum d_j t_j + e,$$
$$t \in \mathbf{R}^{n_1}, \quad y \in \mathbf{R}^{n_2},$$

which is in a class considered by Grušin [6]. From those results it follows that P is analytic hypoelliptic if $P_2(t, D_t, \eta) = \Delta_t - |t|^2|\eta|^2 + \Sigma a_k \eta_k$ is injective in $S(\mathbf{R}_t^{n_1})$ for $\eta \in \mathbf{R}^{n_2} \setminus \{0\}$.

In Section 2 we give a criterion of microlocal analytic hypoellipticity for a class of Grušin type operators and prove in particular that the preceding example

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P is analytic hypoelliptic if and only if for some $\varepsilon > 0$ and $C > 0$,

$$P(t, D_t, \eta) = P_2(t, D_t, \eta) + \sum b_j D_{t_j} + \sum c_{jk} t_j \eta_k + \sum d_j t_j + e$$

is injective in $L^2(\mathbf{R}^n)$ for $\eta \in \mathbf{C}^{n_2} \setminus 0$, $|\operatorname{Im} \eta| < \varepsilon |\operatorname{Re} \eta|$, $|\eta| > C$.

In Section 3 we consider differential operators invariant on a two-step Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_2$. For these groups there is a natural family of dilations; hence, an invariant operator L may be decomposed, $L = L_m + L_{m-1} + \cdots + L_0$, with L_j homogeneous of degree j. For $\eta \in \mathfrak{g}_2^*$ let B_η be the bilinear form on $\mathfrak{g}_1$ defined by $B_\eta(X, X') = \eta([X, X'])$. We assume first that B_η is nondegenerate, $\eta \neq 0$, i.e. G an H-group. We assume that L is transversally elliptic, i.e., elliptic in the $\mathfrak{g}_1$ directions. Every $\eta \in \mathfrak{g}^*$ determines an irreducible representation π_η of G. The function $s(\eta) = \dim \ker \pi_\eta(L_m)^* \pi_\eta(L_m)$ is upper semi-continuous with integer values, and it is possible to define locally, in conic sets, the product of the $s(\eta)$ smallest eigenvalues of $\pi_\eta(L)^* \pi_\eta(L)$ as a formal analytic symbol $d(\eta)$. We show that L is analytic hypoelliptic if and only if

(1.1) $\qquad d(\eta)$ is an elliptic symbol.

We also consider groups for which B_η is nondegenerate in an open set and study the corresponding microlocal analyticity.

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2. Grušin type operators

2.1 *Preliminaries.* Here we consider the class of differential operators with polynomial coefficients

$$A = a(t, D_t) = \sum_{|\alpha+\beta| \le m} a_{\alpha\beta} t^\alpha D_t^\beta, \qquad a_{\alpha\beta} \in \mathbf{C}, \tag{2.1}$$

acting on $\mathbf{R}_t^n$, which have the property of ellipticity,

$$\sum_{|\alpha+\beta| = m} a_{\alpha\beta} t^\alpha \tau^\beta \neq 0, \qquad 0 \neq (t, \tau) \in \mathbf{R}_t^n \times \mathbf{R}_\tau^n. \tag{2.2}$$

We collect here, and sketch proofs for, some results, most of which are either already known or can be proved by known methods, such as those of Grušin [7], Sjöstrand [26], Métivier [18] or Melin [15].

Let $H^m(\mathbf{R}^n)$ denote the domain of A considered as an unbounded operator in $L^2(\mathbf{R}^n)$, equipped with the norm

$$\|u\|_{H^m} = \sum_{|\alpha+\beta| \le m} \|t^\alpha D_t^\beta u\|_{L^2}. \tag{2.3}$$

It is straightforward to build a (left) parametrix B for A such that $BA = \mathrm{Id} + R$, where R is an operator whose kernel is a Schwartz function on $\mathbf{R}^{2n}$; hence R is continuous from $\mathscr{S}'(\mathbf{R}^n)$ to $\mathscr{S}(\mathbf{R}^n)$. Here B is a pseudodifferential operator with symbol in $S_{\mathrm{reg}}^{-m}(\mathbf{R}^{2n})$, i.e. $b(t,\tau) \in S_{1,0}^{-m}(\mathbf{R}^{2n})$ and has an asymptotic development in homogeneous functions [4]. This is the main argument to get the following result which was first proved by Grušin [6]. (See also Sjöstrand [26] and Hörmander [12].)

THEOREM 2.1 (Grušin). *If $u \in L^2(\mathbf{R}^n)$ and $Au \in \mathscr{S}(\mathbf{R}^n)$ then $u \in \mathscr{S}(\mathbf{R}^n)$. The operator $A\colon H^m(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ has a finite index which is equal to 0 if $n > 1$ and equal to the winding number of the mapping*

$$(2.4)\qquad (t,\tau) \in \mathbf{R}^2 \setminus \{0\} \to \sum_{|\alpha+\beta|=m} a_{\alpha\beta} t^\alpha \tau^\beta \in \mathbf{C} \setminus \{0\}$$

in the case $n = 1$. (The orientation is fixed so that $D_t - it$ has winding number equal to 1.)

It follows from the first point in Theorem 2.1 that any eigenfunction of A which is in $L^2(\mathbf{R}^n)$ is also in $\mathscr{S}(\mathbf{R}^n)$. More precise estimates are given by the following proposition, which can essentially be found in Melin [15, appendix]. (See also Hörmander [13] or Métivier [18].)

PROPOSITION 2.2. *Let $h(t)$ be an eigenfunction of A. Then h can be extended as an entire function, and there exist $\varepsilon > 0$, $C_1, C_2 > 0$ such that for $t \in \mathbf{C}^n$:*

$$(2.5)\qquad |t^\alpha D_t^\beta h(t)| \le C_1^{|\alpha+\beta|+1} (\alpha!\beta!)^{1/2} \exp\left(-\varepsilon|\operatorname{Re} t|^2 + C_2|\operatorname{Im} t|^2\right).$$

Moreover the Fourier transform $\hat{h}$ can also be extended as an entire function with similar estimates.

This leads to the following:

PROPOSITION 2.3. *Let E be a finite dimensional subspace of $\mathscr{S}(\mathbf{R}^n)$ spanned by eigenfunctions of A, and Π_E the orthogonal projection of $L^2(\mathbf{R}^n)$ on E. Then the kernel and the symbol of Π_E can be extended as entire functions in $\mathbf{C}^{2n}$ satisfying estimates like (2.5).*

It is sufficient to notice that the kernel and the symbol of Π_E are respectively:

$$(2.6)\qquad \Pi_E(t,s) = \sum_{j=1}^{\dim E} h_j(t)\overline{h_j(s)}$$

$$(2.7)\qquad \sigma(\Pi_E)(t,\tau) = \sum_{j=1}^{\dim E} h_j(t)\overline{\hat{h}_j(\tau)}\, e^{-it\tau}$$

where the h_j, $j = 1, \dots, \dim E$, form an orthonormal basis of E. The h_j need not be eigenfunctions of A, but since they are finite linear combinations of eigenfunctions, they still satisfy (2.5).

Now let E_1 be a finite dimensional subspace of $\mathscr{S}(\mathbf{R}^n)$ stable under A^*A. Then $E_2 = (AE_1^\perp)^\perp$ is also a finite dimensional subspace of $\mathscr{S}(\mathbf{R}^n)$ stable under AA^*. Moreover, $AE_1 \subset E_2$, $A^*E_2 \subset E_1$ and $A^*E_2^\perp \subset E_1^\perp$. Now suppose that E_1 is the sum of the eigenspaces of A^*A corresponding to the set Ω of the smallest s eigenvalues, s fixed. Then we have the following.

Lemma 2.4. *There is a classical pseudodifferential operator B with symbol $b(t, \tau)$ such that*

$$BA = I - \Pi_1, \qquad \Pi_1 B = 0,$$
$$AB = I - \Pi_2, \qquad B\Pi_2 = 0,$$

where Π_1 and Π_2 are the orthogonal projections onto E_1 and E_2 respectively. Furthermore, one may choose $B = B_1 A^*$, where

$$b_1(t, \tau) = \frac{1}{2\pi i} \int_\Gamma \zeta^{-1} c(t, \tau, \zeta)\, d\zeta \tag{2.8}$$

with Γ a contour in $\mathbf{C}$ around 0 enclosing Ω, and $c(t, \tau, \zeta)$ the symbol of $(A^*A - \zeta \cdot I)^{-1}$.

Proof. This is a variation of well-known results. (See Métivier [18, Proposition 2.3] for estimates relevant to what follows.) We are using here the fact that A^*A has a complete eigenvalue expansion (see [7]) which guarantees the existence of $(A^*A - \zeta \cdot I)^{-1}$ for ζ on Γ. The formula (2.8) is then the symbolic version of the classical Dunford integral.

2.2 *Parameters and small eigenvalues.* We now assume that we are given a family of operators $A(\eta)$, $\eta \in \Omega \subset \mathbf{R}^{n_2}$, of the form (2.1) with $a_{\alpha\beta} = a_{\alpha\beta}(\eta)$ and so that the mappings $\eta \to A(\eta)$ form an analytic family of unbounded operators in the sense of Kato-Rellich [14]. Later we will extend holomorphically to $\eta \in \mathbf{C}^{n_2}$. We fix $\eta_0 \in \Omega$ and define

$$E_1(\eta_0) = \ker A(\eta_0) = \ker A^*(\eta_0) A(\eta_0),$$
$$E_2(\eta_0) = \ker A^*(\eta_0) = \ker A(\eta_0) A^*(\eta_0).$$

Definition 2.5. We call the *small eigenvalues* of $A^*A(\eta)$, η near η_0, the perturbation of the eigenvalue 0, i.e. the smallest k eigenvalues of $A^*A(\eta)$, where $k = \dim E_1$.

Now let $E_1(\eta)$ be the sum of the eigenspaces corresponding to the small eigenvalues of $A^*A(\eta)$ and $E_2(\eta)$ to those of $AA^*(\eta)$. If $|\eta - \eta_0|$ is sufficiently small, then $A(\eta)|_{E_1(\eta)^\perp}$ is invertible from $E_1(\eta)$ to $E_2(\eta)^\perp$. We define the inverse $B(\eta)$ as in Lemma 2.4.

It is classical that the spaces $E_i(\eta)$, $i = 1,2$, depend analytically on η. Indeed, the orthogonal projections $\Pi_i(\eta)$ on $E_i(\eta)$ are given by

$$\Pi_1(\eta) = \frac{1}{2\pi i}\int_\Gamma (A^*A(\eta) - \zeta)^{-1}\,d\zeta, \tag{2.9i}$$

$$\Pi_2(\eta) = \frac{1}{2\pi i}\int_\Gamma (AA^*(\eta) - \zeta)^{-1}\,d\zeta. \tag{2.9ii}$$

Every function in $E_i(\eta)$ for fixed η satisfies the estimate (2.5), and it is easy to see in the proof that the constants can be chosen independent of η for $|\eta - \eta_0| < \delta$. We remark also that it follows from Kato [14, II Theorem 1.10] that if $\eta \in \Omega \subset \mathbf{R}^1$ then one can choose the basis of the $E_i(\eta)$ to consist of eigenfunctions of $A^*A(\eta)$ or $AA^*(\eta)$.

Now denote by $M(\eta)$ the matrix of $A(\eta)$: $E_1(\eta) \to E_2(\eta)$ expressed in the basis constructed in Proposition 3.3. It is immediate that $A(\eta)$ is injective in $L^2(\mathbf{R}^n)$ if and only if $M(\eta)$ is injective. Note also that $M(\eta_0) = 0$ and that the index of $A(\eta)$ is $\dim E_1(\eta) - \dim E_2(\eta)$. Finally, suppose $A(\eta)$ is self-adjoint for some η fixed near η_0. Then $E_1(\eta) = E_2(\eta)$ and the determinant of $M(\eta)$ is equal to the product of the small eigenvalues of $A(\eta)$.

2.3 *Main results.* We shall study microlocal analytic hypoellipticity for a class of pseudodifferential operators $P(t, D_t, D_y)$ acting on $\mathcal{D}'(\mathbf{R}^{n_1+n_2}_{t,y})$. They are of multiple characteristics on the symplectic manifold

$$t = \tau = 0, \tag{2.10}$$

where (τ, η) denotes the dual variable of $(t, y) \in \mathbf{R}^{n_1}_t \times \mathbf{R}^{n_2}_y$. They are similar to the differential operators studied by Grušin in [6] for C^∞ hypoellipticity, and we use a similar notion of homogeneity, but we allow our operators to be a finite sum of homogeneous terms.

Consider the operator of degree $m + \mu$, $m \in \mathbf{N}$, $\mu \in \mathbf{R}$,

$$P = \sum_{|\alpha+\beta| \le m} a_{\alpha\beta}(D_y)t^\alpha D_t^\beta \tag{2.11}$$

with pseudodifferential coefficients $a_{\alpha\beta}(D_y)$ whose symbols are defined in a conic neighborhood of $\eta_0 \in \mathbf{R}^{n_2} \setminus 0$.

We write

$$P(\eta) = \sum_{|\alpha+\beta| \le m} a_{\alpha\beta}(\eta)t^\alpha D_t^\beta \tag{2.12}$$

and we suppose that P is a finite sum of homogeneous operators à la Grušin:

$$P(\eta) = P_m(\eta) + P_{m-1}(\eta) + ,\dots, + P_0(\eta) \tag{2.13}$$

where

$$P_{m-j}(\eta) = |\eta|^{\mu+\frac{m-j}{2}} \sum_{|\alpha+\beta|<m-j} C_{\alpha\beta}^{m-j}(\eta)(t|\eta|^{1/2})^\alpha(|\eta|^{-1/2}D_t)^\beta \tag{2.14}$$

with $C_{\alpha\beta}^{m-j}$ analytic and homogeneous of degree 0.

We assume that P is transversally elliptic on the characteristic variety; i.e.

$$\sum_{|\alpha+\beta|=m} C_{\alpha\beta}^m(\eta)t'^\alpha\tau'^\beta \neq 0, \qquad (t', \tau') \in \mathbf{R}^{2n}\setminus\{0\}. \tag{2.15}$$

This insures that P is elliptic outside of $t = \tau = 0$, and we shall study microlocal analytic hypoellipticity of P at a point $(t = 0, y_0\ \tau = 0, \eta_0)$.

Before we state the results, let us introduce the notation

$$\omega = \eta/|\eta|, \quad z = |\eta|^{-1/2}, \quad t' = z^{-1}t, \tag{2.16}$$

and write using (2.14)

$$z^{-2\mu-m}P(\eta) = A_{(z,\omega)}(t', D_{t'}) = \sum_{j=0} z^jA_{m-j}(1,\omega)(t', D_{t'}), \tag{2.17}$$

(2.18)

$$A_{(0,\omega)}(t', D_{t'}) = A_m(1,\omega)(t', D_{t'}) = \sum_{|\alpha+\beta|\le m} C_{\alpha\beta}^m(\omega)t'^\alpha D_{t'}^\beta = P_m(\eta/|\eta|).$$

Now $A_{(z,\omega)}(t', D_{t'})$ is an analytic family of differential operators with polynomial coefficients of the type studied in Part 2.1 (see [23]). The ellipticity property (2.2) follows from the transversal ellipticity property (2.15).

We define, for $|z| < \delta$, $|\omega - \omega_0| < \delta$, the spaces $E_1(z,\omega)$ and $E_2(z,\omega)$ which are the perturbations of respectively the kernel of $A(0,\omega_0)$ and $A^*(0,\omega_0)$. We define also their orthonormal bases $h_j^i(t', z, \omega)$, $j = 1,\dots,\dim E_i(z,\omega)$. Finally we write $B(z,\omega) = A(z,\omega)^\perp|_{E_1(z,\omega)^{-1}}$ and $M(z,\omega)$ for the matrix of $A(z,\omega)$: $E_1(z,\omega) \to E_2(z,\omega)$, with

$$M(\eta) = M(|\eta|^{-1/2}, \eta/|\eta|), \qquad |\eta| > \delta^{-2}, \quad \left|\frac{\eta}{|\eta|} - \frac{\eta_0}{|\eta_0|}\right| < \delta, \tag{2.19}$$

for the matrix of the restriction of $|\eta|^{-\mu-\frac{m}{2}}P(\eta)$ expressed in the basis

$$h_j^i\left(t|\eta|^{1/2}, |\eta|^{-1/2}, \frac{\eta}{|\eta|}\right); \quad i = 1,2; \quad j = 1,\dots,\dim E_i\left(|\eta|^{-1/2}, \frac{\eta}{|\eta|}\right). \tag{2.20}$$

THEOREM 2.5. *The pseudodifferential operator $P(t, D_t, D_y)$ is microlocally analytic hypoelliptic (resp. C^∞ hypoelliptic) at $(0, y_0; 0, \eta_0)$ if and only if the matrix $M(D_y)$ is microlocally analytic hypoelliptic (resp. C^∞ hypoelliptic) at (y_0, η_0).*

Before proving this theorem we shall state some other results. It follows from the discussion at the end of Part 2.2 that the index of $P(\eta)$ is equal to $\dim E_1 - \dim E_2$ and can be computed as in Theorem 2.1 using the winding number of the expression (2.4) in the case $n_1 = 1$. The following was proved by Sjöstrand [26] in a more general case and can be viewed as a corollary of Theorem 2.5.

COROLLARY 2.6. *If the index of $P(\eta)$ is positive, then $P(t, D_t, D_y)$ is microlocally neither C^∞ nor analytic hypoelliptic at $(0, y_0; 0, \eta_0)$.*

In order to apply Theorem 2.5 we need the following criterion for the analytic hypoellipticity of a square system of constant coefficient pseudodifferential operators. By a *constant coefficient analytic symbol* we mean a formal sum $a(\xi) \sim \sum_{j=0}^{\infty} a_j(\xi)$ such that the a_j's are positively homogeneous of degree $m - j/2$ and can be simultaneously extended into a conic complex neighborhood $\Gamma_{\mathbf{C}}$ of ξ_0 with the estimates

$$\sup_{\xi \in \Gamma_{\mathbf{C}}} |\xi|^{-m+j/2} |a_j(\xi)| < C^{j+1} (j/2)^{j/2},$$

with C independent of j. We assume $a_0 \not\equiv 0$, and say that any pseudodifferential operator $A(D)$ with symbol $a(\xi)$ is elliptic at ξ_0 if $a_0(\xi_0) \neq 0$.

THEOREM 2.7. *For a square matrix $(a^{ij}(D))$ of constant coefficients analytic pseudodifferential operators the following are equivalent:*

(i) *$a^{ij}(D)$ is analytic hypoelliptic.*

(ii) *$\mathrm{Det}(a^{ij}(\xi))$ is elliptic at ξ_0.*

(iii) *For any realization $a^{ij}(\xi)$ of the formal symbol, there exist $C > 0$ and $\varepsilon > 0$ such that $a^{ij}(\xi)$ is invertible for all $\xi \in \mathbf{C}^n$, $|\xi| > C$, $|\xi/|\xi| - \xi_0/|\xi_0|| < \varepsilon$.*

Proof. The equivalence of analytic hypoellipticity and ellipticity for constant-coefficient differential operators was proved by Petrowsky [19]. The proof that ellipticity implies analytic hypoellipticity is given in Boutet de Monvel-Krée [3] (in the case of classical operators) and the converse may be proved by methods similar to that for the case of differential operators (see [11, proof of Corollary 4.41]).

THEOREM 2.8. *If the index of $P(\eta)$ is zero, then the following are equivalent:*

(i) *$P(t, D_t, D_y)$ is analytic hypoelliptic at $(0, y_0, 0, \eta_0)$.*

(ii) *$P^*P(t, D_t, D_y)$ is analytic hypoelliptic at $(0, y_0, 0, \eta_0)$.*

(iii) *There exists* $\delta > 0$ *such that* $P(\eta)$ *is injective for all* $\eta \in \mathbf{C}^{n_2}$, $|\eta| > \delta^{-2}$, $|\eta/|\eta| - \eta_0/|\eta_0|| < \delta$.

(iv) *The product of the small eigenvalues of* $P^*P(\eta)$ *is elliptic at* η_0.

Proof. The hypothesis implies that the matrix M is square and hence the equivalence of (i) with (iii) or (iv) or that of (ii) with (iii) or (iv) follows from Theorems 2.5 and 2.7.

If the index of P is negative, we do not have that the implication (i) implies (ii) in Theorem 2.8, although the converse is still true. However if $P_m(\eta_0)$ is injective, P is microlocally analytic hypoelliptic at η_0 since in Theorem 2.5 $\dim E_1 = 0$. This was proved before by Trèves [30] (for double characteristics), Tartakoff [29], Métivier [18]. Their results may be applied to a more general class of operators; indeed they allow the coefficients $a_{\alpha\beta}$ to be analytic pseudo-differential operators depending also on y. But for the class (2.11) our results are more precise because they take account of the influence of the lower order term $P_{m-j}(\eta)$, $j = 1,\ldots,m$, and, in many cases, give a necessary and sufficient condition for microlocal analytic hypoellipticity (see also Métivier [16] for other results on necessary conditions for analytic hypoellipticity).

We shall now prove Theorem 2.5.

PROPOSITION 2.9. *For* $i = 1,2$; $j = 1,\ldots,\dim E_i$, *the function*

$$\tilde{h}_j^i(t,\eta) = g(\eta)h_j^i(t|\eta|^{1/2}, |\eta|^{-1/2}, \eta/|\eta|)$$

is an analytic symbol of degree 0 *and type* $(\rho,\delta) = (\frac{1}{2},\frac{1}{2})$. *Moreover it is of degree* $-\infty$ *outside of* $t = 0$.

Proof. This follows from (2.5) of Proposition 2.2 by use of an argument of Métivier [18, Lemma 4.4].

Now let $A(\eta) = |\eta|^{-(m+\mu)}P(\eta)$ and let

$$q(t,\tau,\eta) = g(\eta)b(t|\eta|^{1/2}, \tau|\eta|^{-1/2}, |\eta|^{-1/2}, \eta/|\eta|) \tag{2.21}$$

where $b(t,\tau,(|\eta|^{-1/2}, \eta/|\eta|))$ is defined as in Lemma 2.4 for fixed η. Finally, we put

$$\tilde{P} = |D_y|^{-(m+\mu)}P. \tag{2.22}$$

Clearly $\tilde{P}$ is microlocally analytic hypoelliptic or C^∞ hypoelliptic if and only if P is.

Next, we define the following operators.

$$Q: \mathscr{D}'(\mathbf{R}^{n_1+n_2}_{t,y}) \to \mathscr{D}'(\mathbf{R}^{n_1+n_2}_{t,y}) \text{ with symbol } q(t,\tau,\eta),$$

$$M: (\mathscr{D}'(\mathbf{R}^{n_2}_y))^{\dim E_1} \to (\mathscr{D}'(\mathbf{R}^{n_2}_y))^{\dim E_2} \text{ with symbol } M(\eta),$$

and for $i = 1, 2$, the so-called Hermite operators (see Boutet de Monvel [2]):

$$H_i: \left(\mathscr{D}'(\mathbf{R}^{n_2}_y)\right)^{\dim E_i} \to \mathscr{D}'(\mathbf{R}^{n_1+n_2}_{t,y}) \text{ defined by}$$

$$(2.23) \qquad H_i v(y,t) = \sum_{j=1}^{\dim E_i} (2\pi)^{-n_2} \int e^{iy\cdot\eta} h^i_j(t,\eta) \hat{v}_j(\eta)\, d\eta,$$

$$H_i^*: \mathscr{D}'(\mathbf{R}^{n_1+n_2}_{t,y}) \to \mathscr{D}'(\mathbf{R}^{n_2}_y)^{\dim E_i} \text{ defined by}$$

$$(2.24) \qquad (H_i^* u)_j(y) = (2\pi)^{-n_2} \int e^{iy\eta} \overline{\tilde{h}^i_j(t,\eta)} u\hat{\ }(t,\eta)\, dt\, d\eta.$$

Notice that the L^2 norm of $\tilde{h}^i_j(\cdot,\eta)$ is $|\eta|^{n_2/4}$ when $g(\eta) = 1$.

Proposition 2.10. *Let* $u \in \mathscr{D}'(\mathbf{R}^{n_1+n_2}_{t,y})$ *and* $v \in \mathscr{D}'(\mathbf{R}^{n_2}_y))^{\dim E_i}$. *Then*

(i) $WF_a(Qu) \subset WF_a(u)$,

(ii) $WF_a(Mv) \subset WF_a(v)$,

(iii) $WF_a(H_i u) \subset \{(t = 0, y_0, \tau = 0, \eta_0): (y_0, \eta_0) \in WF_a(u)\}$,

(iv) $WF_a(H_i^* v) \subset \{(y_0, \eta_0): (t = 0, y_0 - 0, \eta_0) \in WF_a(u)\}$.

Also, (i) *to* (iv) *remain valid if* WF_a *is replaced by the* C^∞ *wave front* WF.

Proof. First, (ii) is well-known (see Boutet de Monvel-Krée [3]) since $M(\eta)$ is a semi-classical analytic symbol. To prove (iii) it is sufficient to calculate the wave front of the kernel of H_i,

$$(2.25) \qquad (H_i(t,y,y'))_j = (2\pi)^{-n_2} \int e^{i(y-y')\eta} \tilde{h}^i_j(t,\eta)\, d\eta.$$

By Proposition (3.17) the essential support of $\tilde{h}^i_j(t,\eta)$ is included in $t = 0$, $\eta \in$ support g. Then using Métivier [18, Lemma 3.3] and a functional property of WF_a (see e.g. Bony [1, Appendice]), we get:

$$(2.26) \quad WF_a(H_i(t,y,y')) = \{(t = 0, y, y'; \tau = 0, \eta, -\eta), \eta \in \operatorname{supp} g\}$$

and property (iii) of H_i. The proof of (iv) is essentially the same.

Finally we shall prove (i). From (2.8) and the definition (2.21) of $q(t,\tau,\eta)$ we have

$$(2.27) \quad q(t,\tau,\eta) = g(\eta) \frac{1}{2\pi i} \int_\Gamma \zeta^{-1} c\left(t|\eta|^{1/2}, \tau|\eta|^{-1/2}, (|\eta|^{-1/2}, \eta/|\eta|), \zeta\right) d\zeta.$$

Now observe that $c(t|\eta|^{1/2}, \tau|\eta|^{-1/2}, (|\eta|^{-1/2}, \eta/|\eta|), \zeta)$ is the symbol of a parametrix for $\tilde{P}^*\tilde{P} + \zeta$ for all $\zeta \in \Gamma$. Indeed, this follows immediately from the fact that $c(t,\tau,\eta,\zeta)$ is the symbol of an inverse for $A^*_{(\eta)}A_{(\eta)} - \zeta$ and from the definition (2.22) of $\tilde{P}$. Now it follows from the criterion of Métivier [18, Theorem 5.1] that for each ζ on Γ, $\tilde{P}^*\tilde{P} + \zeta$ is analytic hypoelliptic. (In the C^∞ case the result is given in [4].) Furthermore the proof (see in particular [18, Lemma 2.2]) shows that the parametrix for $\tilde{P}^*\tilde{P} + \zeta$ is an analytic pseudo-differential operator satisfying estimates uniform in ζ. Hence it is not hard to show, since the integration (2.27) is finite, that Q is also an analytic pseudo-differential operator.

Now we establish some important relations between the operators we have built. Let I denote the identity of $\mathscr{D}'(\mathbf{R}^{n_1+n_2}_{t,y})$ and I_{E_i}, $i = 1, 2$, the identity on $(\mathscr{D}'(\mathbf{R}^{n_2}_y))^{\dim E_i}$. As in [26] or [8], we may prove the following.

PROPOSITION 2.11. *The following equivalences are true modulo analytic smoothing operators*:

$$\begin{pmatrix} \tilde{P} & H_2 \\ H_1^* & 0 \end{pmatrix}\begin{pmatrix} Q & H_1 \\ H_2^* & -M \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 0 & I_{E_1} \end{pmatrix},$$

$$\begin{pmatrix} Q & H_1 \\ H_2^* & -M \end{pmatrix}\begin{pmatrix} \tilde{P} & H_2 \\ H_1^* & 0 \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 0 & I_{E_2} \end{pmatrix}.$$

We may now finish the proof of Theorem 2.5. The method is essentially the same as in Sjöstrand [26] for C^∞ hypoellipticity (see also Helffer [8]). We recall some of the details.

Let $u \in \mathscr{D}'(\mathbf{R}^{n_1+n_2}_{y,t})$, $v \in (\mathscr{D}'(\mathbf{R}^{n_2}_y))^{\dim E_1}$, and let $\equiv$ denote the equality between distributions, modulo distributions whose analytic wave front sets avoid (y_0, η_0) or $(t = 0, y_0, \tau = 0, \eta_0)$. By Proposition 2.10 this equivalence is respected by all the operators $\tilde{P}$, Q, H, H^*, M.

Let $Pu \equiv 0$. By using Proposition 2.11 one gets

i) $u \equiv Q\tilde{P}u + H_1H_1^*u \equiv H_1H_1^*u$,

ii) $0 \equiv H_2^*\tilde{P}u - MH_1^*u \equiv -MH_1^*u$.

If M is microlocally analytic hypoelliptic at (y_0, η_0) one gets $H_1^*u \equiv 0$ and $u \equiv 0$, showing that $\tilde{P}$ is microlocally analytic hypoelliptic at $(0, y_0, 0, \eta_0)$. If furthermore M has a left parametrix N, then $Q + H_1NH_2^*$ is a left parametrix for $\tilde{P}$. The converse is similarly proved.

3. Invariant differential operators on nilpotent Lie groups

In this section we study differential operators (left) invariant on some two-step nilpotent Lie groups and give representation-theoretic criteria for analytic hypoellipticity.

3.1 *Statement of main results.* Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a two step Lie algebra

$$[\mathfrak{g}, \mathfrak{g}_1] \subset \mathfrak{g}_2 \quad \text{and} \quad [\mathfrak{g}, \mathfrak{g}_2] = [\mathfrak{g}_2, \mathfrak{g}_2] = (0).$$

We shall identify $\mathfrak{g}$ with its simply connected Lie group G by the exponential map. We denote by (x, y) the coordinates on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, and by (ξ, η) the dual coordinates on $\mathfrak{g}_1^* + \mathfrak{g}_2^*$. Recall that there is a family δ_t, $t > 0$, of dilations of G which are automorphisms, given by $\delta_t(x, y) = (tx, t^2y)$, $t > 0$, and acting on $\mathfrak{g}^*$ by duality.

For $\eta \in \mathfrak{g}_2^* - \{0\}$ let B_η be the skew symmetric bilinear form defined by

$$B_\eta(X, X') = \eta([X, X']), \qquad X, X' \in \mathfrak{g}_1. \tag{3.1}$$

Since B_η depends linearly on η, the subset $U \subset \mathfrak{g}_2^*$ on which the rank of B_η is the largest is a nonempty Zariski open subset. Following Métivier [17], we say that G is an *H-group* if B_η is nondegenerate for all $\eta \in \mathfrak{g}_2^* - \{0\}$. More generally we say that G is *generically an H-group* if B_η is nondegenerate on a nonempty Zariski open subset $U \subset \mathfrak{g}_2^*$. In this case $\mathfrak{g}_1$ is of even dimension, which we denote $2n_1$, and the center of $\mathfrak{g}$ is $\mathfrak{g}_2$. We let $n_2 = \dim \mathfrak{g}_2$.

We introduce a basis $\{X_1,\ldots,X_{2n_1}, Y_1,\ldots,Y_{n_2}\}$ for $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$. Then any invariant differential operator L on G may be written

$$L = L_m + L_{m-1} + \cdots + L_0, \tag{3.2}$$

where each L_j is homogeneous of degree j; i.e., L_j is of the form

$$L_j = \sum a_{\alpha\beta} X^\alpha Y^\beta, \qquad |\alpha| + 2|\beta| = j$$

where $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{2n_1}^{\alpha_{2n_1}}$ and $Y^\beta = Y_1^{\beta_1} Y_2^{\beta_2} \cdots Y_{n_2}^{\beta_{n_2}}$. We shall require that L be transversally elliptic; i.e.

$$\sum_{|\alpha|=m} a_{\alpha,0}\xi^\alpha \neq 0, \quad \text{if } \xi \neq 0. \tag{3.3}$$

Finally, in the special case where $n_1 = 1$ and $\dim \mathfrak{g}_2 > 1$, i.e. $G = H_3 \times \mathbf{R}^{n_2-1}$, where H_3 is the three dimensional Heisenberg algebra, we shall need the following assumption:

If $n_1 = 1$ and $n_2 > 1$, the winding number of the mapping

$$\xi = (\xi_1, \xi_2) \in \mathfrak{g}_1^* - \{0\} \mapsto \sum_{|\alpha|=m} a_{\alpha,0}\xi^\alpha \in \mathbf{C}^* \tag{3.4}$$

is zero.

Remark. As in Section 2, it will follow from Sjöstrand [26] that if $n_1 = 1$ and $n_2 > 1$ then L is not C^∞ hypoelliptic if the winding number is positive.

By using the invariance of L by left translation on G, we see that it is sufficient to study microlocal analytic hypoellipticity of L in the fiber of $T^*(G)$ above the origin, i.e., at a point $(0, 0; \xi, \eta)$. Now if L is transversally elliptic it is microlocally elliptic and hence analytic hypoelliptic at any point $(0, 0; \xi, \eta)$ for $\xi \in \mathbf{R}^{2n_1} - \{0\}$. Hence we may assume $\xi = 0$. Finally, recall (see e.g. [20]) that for $\eta \in U \subset \mathfrak{g}_2^*$ one may associate to the orbit of $(0, \eta)$ a representation $\pi_\eta \in \hat{G}$ by the Kirillov theory. Then $\pi_\eta(L)$ is a differential operator with polynomial coefficients acting on $L^2(\mathbf{R}^{n_1})$. It follows from a result of Grušin [6] (see Theorem 2.1) that $\pi_\eta(L)$ is an operator with index. If, in addition, L is self-adjoint then the spectrum of $\pi_\eta(L)$ is discrete. We shall show that $\eta \to \pi_\eta(L)$ is an analytic family of operators with discrete spectrum, and define the small eigenvalues of $\pi_\eta(L)$ near $|\eta| = \infty$, as in Section 2, Definition 2.5.

Our main result in Section 3 is the following.

THEOREM 3.1. *Suppose G is generically an H-group and L a left invariant differential operator in G satisfying* (3.3) *and* (3.4). *Let $\eta_0 \in U \subset \mathfrak{g}_2^*$. Then the following are equivalent.*

(i) *L is microlocally analytic hypoelliptic at* $(0,0;0,\eta_0)$.

(ii) *L^*L is microlocally analytic hypoelliptic at* $(0,0;0,\eta_0)$.

(iii) *The product of the small eigenvalues of $\pi_\eta(L^*L)$ is an elliptic symbol at η_0.*

Even if (3.4) *fails* (iii) *is equivalent to* (ii) *and implies* (i).

COROLLARY. *If G is an H-group and L is transversally elliptic then L is analytic hypoelliptic at $(0,0;0,\eta_0)$ if and only if the product of the small eigenvalues of $\pi_\eta(L^*L)$ is an elliptic symbol at $\eta = \eta_0$.*

Proof of corollary. Since (3.1) is satisfied with $U = \mathfrak{g}^* - \{0\}$ for an H-group, the conclusion will follow from Theorem 3.1 whenever (3.4) is satisfied, e.g., whenever $n_1 > 1$. However an H-group for which $n_1 = 1$ is necessarily a three dimensional Heisenberg group. Hence the proof of the corollary is completed by the following.

PROPOSITION 3.2. *Suppose that G is a Heisenberg group and L is transversally elliptic. Then the following are equivalent.*

(i) *L is microlocally analytic hypoelliptic at* $(0,0;0,\eta_0)$.

(ii) $\ker \pi_{r\eta_0}(L) = 0$ *for* $r \gg 0$.

(iii) *L is microlocally C^∞ hypoelliptic at* $(0,0;0,\eta_0)$.

(iv) $\ker L \cap L^2(G) = (0)$.

Proof. The implication (ii) implies (i) follows from Theorem 3.1 as does (i) implies (iii). We shall prove (ii) and (iv) are equivalent and then show that (iii) implies (ii). For this, let $U_1, U_2, \ldots, U_{n_1}, V_1, V_2, \ldots, V_{n_1}$, be a basis of $\mathfrak{g}_1$, Y of $\mathfrak{g}_2$ with $[U_i, V_j] = \delta_{ij}Y$, all other brackets vanishing. Now for $\eta \in \mathbf{R}^1 \setminus 0$, let

$$\pi_\eta(U_j) = |\eta|^{1/2}\partial/\partial t_j, \quad \pi_\eta(V_k) = i(\operatorname{sgn}\eta)|\eta|^{1/2}t_k \quad \text{and} \quad \pi_\eta(Y) = i\eta$$

define the irreducible representations of G on $L^2(\mathbf{R}^{n_1})$. Then if $z = |\eta|^{-1/2}$, $z \mapsto |\eta|^{-m/2}\pi_\eta(L) = A(\eta)$ defines an analytic family of operators as in Part 2.3. Hence the eigenvalues of $A^*A(\eta)$ are analytic in z and are discrete. Then any nonvanishing eigenvalue has only isolated zeros, which proves the equivalence of (ii) and (iv).

Finally, if there is an eigenvalue of $A^*A(\eta)$ which vanishes identically, there is an analytic family of vectors $v_{\eta^{-1/2}}(t)$, for $|z|$ small, in $L^2(\mathbf{R}^{n_2})$ with

$$v_{\eta^{-1/2}} = v_0 + \eta^{-1/2}v_1 + \eta^{-2/2}v_2 + \cdots,$$

$A^*(\eta)A(\eta)v_{\eta^{-1/2}} \equiv 0$, $|\eta| > N$. (Here, for simplicity we assume $\eta > 0$.) We may normalize so that $\|v_0\| = 1$. Let f be the function on G defined by

$$f(g) = \int_N^\infty \left(\pi_\eta(g)v_{\eta^{-1/2}}, v_0\right)\eta^{-(m+2)}\,d\eta,$$

where (,) denotes the inner product on $L^2(\mathbf{R}^{n_1})$. Then $f \in \ker L^*L \cap L^2(G) = \ker L \cap L^2(G)$. As in [22, Lemma 4.6] we restrict to $\exp yY$ and write

$$h(y) = f(\exp yY) = \int_N^\infty e^{i\eta y}(v_{\eta^{-1/2}}, v_0)\eta^{-(m+2)}\,d\eta,$$

and show $h \notin C^\infty$. Hence L is not C^∞ hypoelliptic, which shows (iii) implies (ii).

A special case of Proposition 3.2 was obtained by Stein [28].

We now give a version of Theorem 3.1 involving complex representations. The Kirillov theory associates to the orbit of $(0, \eta) \in \mathfrak{g}^*$ an equivalence class of unitary representations $[\pi_\eta]$. By an explicit choice of a realization π_η, one may complexify and define π_ζ for $\zeta \in (\mathfrak{g}_2)^*_{\mathbf{C}}$. However, π_ζ will depend on the choice of the realization and not be canonical (see Helffer [9]).

THEOREM 3.3. *Suppose that G is generically an H-group and L satisfies* (4.3) *and* (4.4). *Let $\eta_0 \in U$. One may define representations π_ζ of $\mathfrak{g}$ on $L^2(\mathbf{R}^{n_1})$ for all $\zeta \in (\mathfrak{g}_2^*)_{\mathbf{C}}$, agreeing with those given by the Kirillov theory if $\zeta \in \mathfrak{g}_2^*$, such that L is analytic hypoelliptic at $(0,0;0,\eta_0)$ if and only if there exist $C > 0$ and $\varepsilon > 0$ such that $\pi_\zeta(L)$ is injective in $L^2(\mathbf{R}^{n_1})$ for all $\zeta \in \mathbf{C}^{n_2}$, $|\zeta| > C$ and* $\left|\frac{\zeta}{|\zeta|} - \frac{\eta_0}{|\eta_0|}\right| < \varepsilon$.

3.2. *Microlocal reduction to Grušin-type operators.* We shall prove Theorems 3.1 and 3.3 by using a microlocal Fourier integral operator, near the point $(0,0;0,\eta_0) \in T^*(G)$, to transform L into a Grušin-type operator of the form studied in Section 3. After a linear change of basis one may assume that $\eta_0 = (1,0,\ldots,0)$. We write X, x, and ∂_x for the column vectors of $\{X_i\}$ $\{x_i\}$, and $\{\partial_{x_i}\}$. Then in the exponential coordinates corresponding to the given bases of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, with $Y_i = \partial_{y_i}$, we have

$$X = \partial_x - \frac{i}{2}B(D_y)x,$$

where $B(\eta) = B_\eta$ is the bilinear form defined by (3.1). Now we may write

$$L = \sum_{|\alpha| < m} a_\alpha(D_y) X^\alpha = L(x, D_x, D_y).$$

Next let J be the $2n_1 \times 2n_1$ skew symmetric matrix $J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$. Then there exists an analytic matrix-valued function $\eta \to \Phi(\eta)$ defined in a conic neighborhood of η_0, homogeneous of degree zero in η such that each $\Phi(\eta)$ is invertible, and $\Phi(\eta)B(\eta)^t\Phi(\eta) = J\eta_1$. Indeed, $B(\eta)$ and $J\eta_1$ are both analytic, nondegenerate, and homogeneous of degree 1 in a conic neighborhood of η_0.

PROPOSITION 3.4. *There is an elliptic Fourier integral operator F, with analytic phase function such that*

$$FLF^{-1} \equiv \sum_{|\alpha| \le m} a_\alpha(D_{y'})\left({}^t\Phi^{-1}(D_{y'})X'\right)^\alpha = \sum_{|\alpha| \le m} C_\alpha(D_{y'})X'^\alpha, \tag{3.5}$$

where $X' = \partial_{x'} - \frac{i}{2}JD_{y_1'}x'$ in a conic neighborhood of $(0,0;0,\eta_0)$; L is analytic hypoelliptic at $(0,0;0,\eta_0)$ if and only if FLF^{-1} is analytic hypoelliptic at $(0;0,\eta_0)$.

Proof. Let F be the invertible Fourier integral operator defined microlocally by
(3.6)

$$Ff(x', y') = (2\pi)^{-n}\int e^{i[(\Phi(\eta)x' - x)\cdot\xi + (y' - y)\cdot\eta]}\Phi^{-1}(\eta)f(x, y)\,dx\,dy\,d\xi\,d\eta.$$

The equality, which is to be understood to be modulo analytic smoothing operators, is easily verified and the last statement follows from the standard properties of transformation of WF_a. (See e.g. [25] or [18, Lemma 3.3].)

Next, we must reduce the number of x variables to obtain an operator of the form (2.11). We introduce the following vector fields on $\mathbf{R}^{n_1+n_2}_{t,y}$.

$$T_j = \partial/\partial t_j, \quad j = 1, 2, \ldots, n_1, \quad \text{and} \quad T_{n_1+j} = t_j\partial/\partial y_1, \quad j = 1, \ldots, n_1. \tag{3.7}$$

Note that $[X_j', X_k'] = [T_j, T_k]$.

PROPOSITION 3.5. *Let*

$$\tilde{P}(x', \partial_{x'}, \partial_y) = \sum b_\alpha(D_y)X'^\alpha$$

and

$$P(t, \partial_t, \partial_y) = \sum b_\alpha(D_y)T^\alpha.$$

Then $\tilde{P}$ is analytic hypoelliptic at $(0,0;0,\eta_0)$ if and only if P is analytic hypoelliptic at $(0,0;0,\eta_0)$.

Proof. $\tilde{P}$ may be transformed to P via an analytic Fourier integral operator, F', such that $F'X'F'^{-1} = T$, which involves a reduction of variables. See Métivier [18, proof of Theorem 5.1] for details.

Proof of Theorems 3.1 *and* 3.3. By Propositions 3.4 and 3.5 we have reduced the study of L to that of a Grušin-type operator, $P(t, D_t, D_y)$. To complete the proof of Theorem 3.1, we must construct representations π_η corresponding to the orbit of $(0, \eta) \in \mathfrak{g}^*$ so that

$$\pi_\eta(L) = P(t, D_t, \eta). \tag{3.8}$$

For this, let π_η be the unitary representation on $L^2(\mathbf{R}_t^{n_1})$ determined by

$$\pi_\eta({}^t\Phi(\eta)\cdot X_j) = \partial/\partial t_j, \quad \text{and} \quad \pi_\eta({}^t\Phi(\eta)\cdot X_{j+n_1}) = i\eta_1 t_j \quad \text{for } j \geq n_1, \tag{3.9}$$

$$\pi_\eta(Y_k) = i\eta_k, \qquad k = 1, 2, \ldots, n_2, \tag{3.10}$$

3.3 *Further results and examples.*

PROPOSITION 3.6. *Suppose that G is generically an H-group and L satisfies* (3.3) *and* (3.4), *and $\eta_0 \in U$. Then one of the following three possibilities occurs.*

(i) *$\pi_{\eta_0}(L_m)$ is injective; then L is analytic hypoelliptic at* $(0, 0; 0, \eta_0)$.

(ii) *$\pi_{\eta_0}(L_m)$ is not injective, but in any conic neighborhood Γ of η_0 there exists η with $\pi_\eta(L_m)$ injective; then L is not analytic hypoelliptic at* $(0, 0; 0, \eta_0)$.

(iii) *$\pi_\eta(L_m)$ is not injective for all η in a conic neighborhood of η_0, and the question of analytic hypoellipticity depends on the terms $L_{m-1} + L_{m-2} + \cdots + L_0$ of lower homogeneity.*

Proof. (i) follows from the results of Métivier [17]. For (ii) we must show that if $\pi_{\eta'}(L_m)$ is injective for some $\eta' \in \Gamma$, then the product d_η of the small eigenvalues of $\pi_\eta(L^*L)$ is not elliptic in Γ. Indeed,

$$d_\eta = |\eta|^k\big(a_0(\omega) + |\eta|^{-1/2}a_1(\omega) + \cdots\big), \quad \text{where } \omega = \eta/|\eta|.$$

Then $a_0(\omega_0) = 0$, but $a_0(\omega') \neq 0$. Hence d_η is not elliptic. The statement (iii) is evident.

Example 3.7. Let $\mathfrak{g}$ be the direct sum of two three-dimensional Heisenberg algebras. Then $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ where X_1, X_2, X_3, X_4 is a basis of $\mathfrak{g}_1$, Y_1, Y_2 a basis of $\mathfrak{g}_2$, with $[X_1, X_2] = Y_1$, $[X_3, X_4] = Y_2$. Introduce coordinates $\eta = (\eta_1, \eta_2)$ on $\mathfrak{g}_2^*$ so that $\eta(Y_i) = \eta_i$, and let

$$L = L_2 + L_1 + L_0 = \big(X_1^2 + X_2^2 + X_3^2 + X_4^2 + iY_1 + 3iY_2\big) + L_1 + L_0. \tag{3.11}$$

Let $U_1 = \{\eta_1 > 0 \text{ and } \eta_2 > 0\}$, $U_2 = \{\eta_1 < 0 \text{ and } \eta_2 < 0 \text{ with } |\eta_2| < \frac{1}{4}|\eta_1|\}$, and $U_3 = \{\eta_1 > 0,\ \eta_2 < 0,\ |\eta_1| = |\eta_2|\}$. Then L is analytic hypoelliptic at

$(0,0;0,\eta)$ if $\eta \in U_1$ but not if $\eta \in U_3$. If $\eta \in U_2$ hypoellipticity depends on L_1 and L_0.

For H-groups we can give a more precise result for second order operators.

PROPOSITION 3.8. *Let G be an H-group, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ with* $\dim \mathfrak{g}_2 > 1$, *$X_1, \ldots, X_{2n_1}$ a basis of $\mathfrak{g}_1$, and $Y_1, \ldots, Y_{n_2}$ a basis of $\mathfrak{g}_2$. Let*

$$L = \sum_{j=1}^{2n_1} X_j^2 + i\sum_{j,k} a_{jk}\left[X_j, X_n\right] + L_1 + L_0 = L_2 + L_1 + L_0.$$

Suppose $\eta_0 \in \mathfrak{g}_2^*$ with $\pi_{\eta_0}(L_2)$ not injective. Then L is not analytic hypoelliptic at $(0,0;0,\eta_0)$. That is, L is analytic hypoelliptic at $(0,0;0,\eta)$ if and only if L_2 is analytic hypoelliptic at $(0,0;0,\eta)$.

Proof. By Proposition 3.6 it suffices to show that there cannot be an open neighborhood V of η_0 in U such that $\pi_\eta(L_2)$ is not injective at every $\eta \in V$. This follows from the fact that $U = \mathfrak{g}_2^* - \{0\}$ is connected; details are given in Rothschild-Tartakoff [23, Proposition 7.1].

It is easy to see that Proposition 3.8 is false for $n_2 = 1$. The following shows it also fails for higher order operators even if $n_2 > 1$.

Example 3.9. Let G be an H-group, $\dim \mathfrak{g}_2 = n_2 > 1$. Then there is a polynomial q_y, with coefficients which are polynomials in Y, such that if $L_m = q_y(\Sigma X_j^2)$, then exactly one eigenvalue of $\pi_\eta(L_m)$ vanishes identically, and no other eigenvalue vanishes. Hence if $L = L_m + L_{m-1} + \cdots + L_0$, one has the following.

(3.12) *If $L = L_m$, L is not analytic hypoelliptic.*

(3.13) *If $L = L_m + C$, $C \neq 0$, then L is analytic hypoelliptic.*

(3.14) *If* $L = L_m + \sum_{j=1}^{n_2-1} Y_j^2 + Y_{n_2}$,

then L is C^∞ hypoelliptic but not analytic hypoelliptic.

To see this, note that $\pi_\eta(C) = C$, which is an elliptic symbol, and $\pi_\eta(\Sigma_{j=1}^{n_2-1} Y_j^2 + Y_{n_2}) = -\Sigma_{j=1}^{n_2-1}\eta_j^2 + i\eta_{n_2}$, which is not an elliptic symbol, but is the symbol of a C^∞ hypoelliptic operator. Hence (3.12), (3.13), and (3.14) will be proved if the polynomial q_Y can be constructed. For this, let q_η be the polynomial with Y replaced by η and note first that the eigenvalues of $\pi_\eta(L_m)$ are $q_\eta(m_\alpha(\eta))$, where $m_\alpha(\eta)$ are the eigenvalues of $\pi_\eta(\Sigma Y_j^2)$; i.e.,

$$m_\alpha(\eta) = \sum_{j=1}^{n_1} (2\alpha_j + 1)\rho_j, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n_1}), \tag{3.15}$$

where $\pm i\rho_j$ are the eigenvalues of the matrix B_η and therefore homogeneous of degree 1. (See e.g. Rothschild-Stein [22] for this calculation.) Assume for simplicity that generically the ρ_j are all distinct. Let q_η be the monic polynomial of degree 2^{n_1} whose roots are the 2^{n_1} linear expressions of the form $\pm\rho_1 \pm \rho_2 \pm \cdots \pm \rho_n$. Since q_η is invariant under any permutation of the roots of det B_η, the coefficient of degree $2^n - j$ of q_η is a homogeneous polynomial in η of degree j. Clearly $q_\eta(m_\alpha(\eta)) \equiv 0$ for $\alpha = (0,0,\ldots,0)$ and $q_\eta(m_\alpha(\eta))$ is nonvanishing for $\alpha \neq (0,0,\ldots,0)$. Finally, if the eigenvalues are not generically distinct, a similar construction can be made with distinct eigenvalues.

Remark. It follows from a result of L. Corwin and the second author [21] that L_m is not locally solvable.

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