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Directions of Analytic Discs Attached to Generic Manifolds of Finite Type

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In this paper, we construct a finite-dimensional family of analytic discs attached to a generic manifold M of finite type in $\mathbf{C}^N$. More precisely, we determine sufficiently many parameters to guarantee that the associated analytic discs sweep out an open wedge in $\mathbf{C}^N$. This construction, which is based on earlier work of Tumanov, gives an explicit algorithm for determining a wedge of extendibility for all CR functions on M near a point. We also define the notion of an *anisotropic wedge*, which is a more natural region of extendibility in higher codimension.

0. INTRODUCTION

In recent results, Tumanov [12] and the present authors [2] (see also [3]) obtained a complete characterization of all generic CR manifolds M in $\mathbf{C}^N$ for which all CR functions are boundary values of holomorphic functions in an open wedge $\mathscr{W}$ with edge M. However, the precise description of $\mathscr{W}$, i.e., the direction of this wedge from M, has been known only for special cases. Note that the class of manifolds for which extendibility holds, called minimal [12], contains, in particular, all finite type generic manifolds in the sense of Bloom and Graham [7]. In this paper, we give a constructive algorithm for determining a direction of wedge extendibility from any smooth generic manifold of finite type by constructing a suitable family of analytic discs attached to M. The same type of construction also gives a family of analytic discs whose boundaries cover the generic manifold M.

For a limited class of generic manifolds of finite type, called semi-rigid (see [4]), directions of wedge extendibility have been determined. This class includes all hypersurfaces of finite type [6] as well as generic

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manifolds with transversal commutative group action, called rigid in [5]. Other results for special cases of generic manifolds of finite type were given in Boggess [8] (see also [9]), Boggess and Polking [11], and Boggess *et al.* [10].

Recall that a real smooth submanifold M of $\mathbf{C}^N$ of codimension ℓ is called *generic* if it is locally defined as $\{\rho_j(\mathscr{Z}, \bar{\mathscr{Z}})=0, j=1, ..., \ell\}$, where the defining functions ρ_j are smooth, real-valued functions with linearly independent complex differentials $\partial\rho_j$. The *CR dimension of M* is then $n=N-\ell$. The CR bundle $\mathscr{V}$ of a generic manifold is defined by letting $\mathscr{V}_p$, $p \in M$, be the space of all antiholomorphic vectors tangent to M. Then M is *minimal* at $p_0 \in M$ if it contains no nontrivial CR submanifold through p_0; i.e., if there is no proper submanifold $M_1 \subset M$ with $p_0 \in M_1$, and such that $\mathscr{V}_p \subset \mathbf{C}T_pM_1$ for all p near p_0.

By a *wedge with edge M* we mean an open set $\mathscr{W}$ of the form

$$\mathscr{W}=\{\mathscr{Z} \in \mathbf{C}^{n+\ell} \cap \mathcal{O} : \rho(\mathscr{Z}, \bar{\mathscr{Z}}) \in \Gamma\}, \tag{0.1}$$

where Γ is a strictly convex open cone in $\mathbf{R}^\ell$, $\mathcal{O}$ is an open neighborhood of p_0 in $\mathbf{C}^{n+\ell}$, and $\rho(\mathscr{Z}, \bar{\mathscr{Z}})=(\rho_1(\mathscr{Z}, \bar{\mathscr{Z}}), ..., \rho_\ell(\mathscr{Z}, \bar{\mathscr{Z}}))$. Tumanov's theorem [12] states that if M is minimal at p_0 there is a wedge $\mathscr{W}$ with edge M such that every CR function on M is the boundary value of a function holomorphic in $\mathscr{W}$.

In [12] the wedge of extendibility is constructed by the method of analytic discs. An *analytic disc* in $\mathbf{C}^N$ is a continuous mapping $A: \bar{\Delta} \to \mathbf{C}^N$ which is holomorphic in Δ, where Δ is the open unit disc in the plane and $\bar{\Delta}=\Delta \cup S^1$, S^1 being the unit circle. We say that A is *attached* to M if $A(S^1) \subset M$. We always assume all analytic discs to be of Hölder class at least $C^{1,\alpha}(\bar{\Delta})$ for some fixed $\alpha \in (0, 1)$. The norm of A is always taken to be in this Hölder class. We say that A passes through a point $p_0 \in M$, which we fix throughout, if $A(1)=p_0$.

In Section 1 we review analytic discs attached to a generic manifold and state a theorem, due to Tumanov [13], which relates disc directions to the wedge filled by discs attached to M and hence determines a wedge of extendibility for all CR functions.

In Section 2 we begin the study of homogeneous generic manifolds, i.e., those defined by polynomials homogeneous with respect to a family of dilations. We define analytic discs through a fixed point p_0 with holomorphic polynomial components. In this section we also state one of our main results, Theorem 2, which gives a bound on the degree of a family of polynomial analytic discs which are sufficient to determine a wedge of extendibility when the manifold M is of finite type. This bound depends only on the Hörmander numbers of M ([4, 7]), counted with their multiplicity. More precisely, we show that any polynomial disc, with degree bounded as above, whose coefficients do not lie in a real proper algebraic subset determines a direction of extendibility. We also show that the boundaries of such discs cover M.

In Section 3 we begin by reviewing some identities of Tumanov [12] and then give expressions to compute the ranks of the mappings used in Theorem 2.

In Section 4 we introduce three subsets of the space of coefficients of the polynomial analytic discs attached to M mentioned above, subsets on which the disc evaluation map at -1 and the disc direction map are not of maximal rank. We show that these are real algebraic subsets of the space of coefficients. Finally, Section 5 and 6 are devoted to proving that these algebraic subsets are actually proper (Theorem 3). These results, which are crucial in our approach, imply Theorem 2. In the course of the proof of Theorem 3, we use an induction on the codimension of M to produce an algorithm to determine explicitly directions of wedge extendibility for M.

In Section 7 we use the homogeneity of M to construct a union of wedges, which we call an *anisotropic wedge*, contained in the union of the analytic discs attached to M. These results are given in Theorem 4 and its corollary. In Section 8, we study the general finite type case as a perturbation of the homogeneous one. Theorem 5 and its corollary generalize the results obtained in Section 7 for the homogeneous case, giving an explicit method of describing the anisotropic wedge of extendibility for the general finite type case.

1. Analytic Discs Attached to Generic Manifolds and Their Directions

Let M be a generic manifold of codimension ℓ in $\mathbf{C}^{n+\ell}$ and $p_0 \in M$. After a holomorphic change of variables in $\mathbf{C}^{n+\ell}$ near p_0, we may assume that M is given by

$$\{(z, w) \in \mathcal{O} : \operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w)\}, \tag{1.1}$$

where $\mathcal{O}$ is an open neighborhood of 0 in $\mathbf{C}^{n+\ell}$, $z \in \mathbf{C}^n$, $w \in \mathbf{C}^\ell$, and ϕ is a smooth function defined in an open neighborhood of 0 in $\mathbf{R}^{2n+\ell}$ and valued in $\mathbf{R}^\ell$, with $p_0 = 0$, $\phi(0) = 0$, and $d\phi(0) = 0$. We write $s = \operatorname{Re} w$.

Using the coordinates above, any analytic disc $A(\zeta)$ attached to a small neighborhood of 0 in M may be written $A(\zeta) = (z(\zeta), w(\zeta))$, where $z(\zeta)$ and $w(\zeta)$ are analytic discs valued in $\mathbf{C}^n$ and $\mathbf{C}^\ell$, respectively, of class $C^{1,\alpha}(\bar{\Delta})$, with $z(1) = 0$, and for $\zeta \in S^1$, $w(\zeta) = s(\zeta) + i\phi(z(\zeta), \overline{z(\zeta)}, s(\zeta))$ with $s(\zeta)$ satisfying the Bishop equation

$$s(e^{i\theta}) = -T_1(\phi(z(\cdot), \overline{z(\cdot)}, s(\cdot)))\,(e^{i\theta}), \tag{1.2}$$

where $T_1(u)$ is the Hilbert transform of a function u defined on S^1 normalized by the condition $T_1(u)(1)=0$. Note that $s(e^{i\theta})\in C^{1,\alpha}(S^1)$.

If $u\in C^{1,\alpha}(S^1)$ satisfies $u(1)=u'(1)=0$, then

$$\frac{d}{dt}T_1 u(e^{it})\bigg|_{t=0} = -1/\pi\int_0^{2\pi}\frac{u(e^{i\theta})}{|e^{i\theta}-1|^2}\,d\theta. \tag{1.3}$$

As in Tumanov [12], it is convenient to introduce the following notation. If u is as above we write

$$\mathscr{T}u = -1/\pi\int_0^{2\pi}\frac{u(e^{i\theta})}{|e^{i\theta}-1|^2}\,d\theta = \frac{1}{i\pi}\int_{S^1}\frac{u(\zeta)}{(\zeta-1)^2}\,d\zeta. \tag{1.4}$$

If $A(\zeta)$ is a disc attached to M with $A(1)=p_0$, we call the vector $iA'(1)=(d/d\theta)\,A(e^{i\theta})|_{\theta=0}\in T_{p0}M$ the *disc direction* of A. With the coordinates chosen above, it follows that $iA'(1)$ is in $T^c_{p_0}M$, the complex tangent space of M at p_0, if and only if $(\partial s/\partial\theta)(1)=0$. The following is due to Tumanov.

THEOREM 1 (Tumanov [12, 13]). *Let M and (z, w) be as above. If $A_1, ..., A_\ell$ are analytic discs attached to M through p_0 such that their disc directions $iA_1'(1), ..., iA_\ell'(1)$ are linearly independent modulo $T^c_{p_0}M$, i.e., the corresponding vectors $(\partial s_1/\partial\theta)(1), ..., (\partial s_\ell/\partial\theta)(1)$ are linearly independent in $\mathbf{R}^\ell$, then wedge extendibility holds at p_0. More precisely, a wedge of extendibility is obtained as follows. Let Γ' be the convex cone spanned by the vectors $(\partial s_1/\partial\theta)(1), ..., (\partial s_\ell/\partial\theta)(1)$ and Γ any convex open cone in $\mathbf{R}^\ell$ with closure in the interior of Γ'. Then there exists $\mathcal{O}$, an open neighborhood of p_0 in $\mathbf{C}^{n+\ell}$, such that all CR functions on M extend to the wedge*

$$\mathscr{W}_\Gamma = \{(z, w)\in\mathcal{O} : \operatorname{Im} w - \phi(z, \bar z, s)\in\Gamma\}. \tag{1.5}$$

Our main results give explicit constructions of disc directions and hence, by Theorem 1, determine a wedge of extendability.

2. Polynomial Analytic Discs for Homogeneous Generic Manifolds of Finite Type

We consider here generic manifolds of finite type given in normal form by homogeneous polynomials. For this, we recall some notation from [4] (see also [7]).

We define coordinates (x, y, s) in $\mathbf{R}^{2n+\ell}$ with $x, y\in\mathbf{R}^n$; $s\in\mathbf{R}^\ell$;

$s=(s_1, \ldots, s_r)$; $s_k \in \mathbf{R}^{\ell_k}$, $1 \leqslant k \leqslant r$; and $\sum_{j=1}^{r} \ell_j = \ell$. If m_j are positive integers satisfying

$$2 \leqslant m_1 < m_2 < \cdots < m_r < \infty,$$

then for $t>0$ we define a family of dilations in $\mathbf{R}^{2n+\ell}$ by

$$\delta_t(x, y, s) = (tx, ty, t^{m_1}s_1, \ldots, t^{m_r}s_r). \tag{2.1}$$

If $q(x, y, s)$ is a polynomial and m a non-negative integer, we say that q is *homogeneous of weight* m if

$$q(\delta_t(x, y, s)) = t^m q(x, y, s), \qquad (x, y, s) \in \mathbf{R}^{2n+\ell},\ t>0. \tag{2.2}$$

Now suppose that $p_{m_k}(x, y, s)$, $k=1, \ldots, r$, are real homogeneous polynomials of weight m_k valued in $\mathbf{R}^{\ell_k}$. Let M be the generic submanifold of $\mathbf{C}^{n+\ell}$ defined by

$$\begin{aligned} z_j &= x_j + iy_j, && j=1, \ldots, n, \\ w_k &= s_k + ip_{m_k}(x, y, s_1, \ldots, s_{k-1}), && k=1, \ldots, r, \end{aligned} \tag{2.3}$$

with $(z, w) \in \mathbf{C}^n \times \mathbf{C}^\ell$ and $w=(w_1, \ldots, w_r)$, $w_k \in \mathbf{C}^{\ell_k}$. We write $w = s + ip(z, \bar{z}, s)$. We say that M is a *homogeneous generic* submanifold of $\mathbf{C}^{n+\ell}$.

We say that a real polynomial $q(x, y, s)$ homogeneous with respect to the weights of (2.1) is *M-pluriharmonic* if there is a holomorphic polynomial $F(z, w)$ in $\mathbf{C}^{n+\ell}$ such that

$$q(x, y, s) = \operatorname{Re} F(z, w)|_M.$$

Recall [4, Theorem 1] that the generic manifold M given by (2.3) is of finite type if and only if $\eta \cdot p_{m_k}$ is not pluriharmonic for any $\eta \in \mathbf{R}^{\ell_k} \setminus \{0\}$, $k=1, \ldots, r$. The positive integers $m_1, \ldots, m_r$ are called the *Hörmander numbers* of M and are biholomorphic invariants attached to M. We also write $n_1, \ldots, n_\ell$ for the Hörmander numbers $m_1, \ldots, m_r$ written with their multiplicities $\ell_1, \ldots, \ell_r$; i.e. $n_j = m_1$ for $j=1, \ldots, \ell_1$, etc.

By a *polynomial disc* we mean an analytic disc $A(\zeta)$, whose components are holomorphic polynomials in ζ. Note that if $A(\zeta) = (z(\zeta), w(\zeta))$ is a disc attached to M through 0 as above, where the components of z are holomorphic polynomials, then $A(\zeta)$ is necessarily a polynomial disc. Indeed, by the Bishop equation (1.2) the components of $w(\zeta)$ must also be polynomials since the Hilbert transform T_1 preserves polynomials.

We define inductively a sequence of positive integers $N_1, \ldots, N_\ell$ by

$$N_j = (n_j - 1)(N_{j-1} + 1), \qquad N_0 = 0, \tag{2.4}$$

where the n_j are the Hörmander numbers defined with multiplicities as above, and put $K = N_\ell$. Note that $K \leqslant \prod_{j=1}^{\ell} n_j$.

For $\mu \in \mathbf{C}^{n(K+1)}$, $\mu = (\mu_0, \mu_1, ..., \mu_K)$, $\mu_j \in \mathbf{C}^n$, we denote by $A(\mu, \zeta)$ the polynomial disc attached to M given by $(z(\mu, \zeta), w(\mu, \zeta))$, where

$$z(\mu, \zeta) = (\zeta - 1) \sum_{j=0}^{K} \mu_j \zeta^j. \tag{2.5}$$

Using these polynomial discs we may state the following result.

THEOREM 2. *Assume that the homogeneous generic manifold M given by* (2.3) *is of finite type. Then for the mapping*

$$\mathbf{C}^{n(K+1)} \ni \mu \mapsto \Psi(\mu) = \mathscr{T}(p[z(\mu, \cdot), \overline{z(\mu, \cdot)}, s(\mu, \cdot)]) \in \mathbf{R}^\ell \tag{2.6}$$

we have rank of $\Psi'(\mu) = \ell$ outside of a proper real algebraic subset $\mathscr{C}$ of $\mathbf{C}^{n(K+1)}$. Hence any $\mu \notin \mathscr{C}$ determines a wedge of CR extendibility at 0.

In addition, if $\Theta : \mathbf{C}^{n(K+1)} \to M$ is defined by $\Theta(\mu) = A(\mu, -1)$, then the rank of $\Theta'(\mu)$ is $2n + \ell$ for $\mu \notin \mathscr{B}$, where $\mathscr{B}$ is a proper real algebraic subset of $\mathbf{C}^{n(K+1)}$.

Note that in the homogeneous case the Bishop equation (1.2) can be solved explicitly using the identities

$$T_1(\zeta^j) = -i(\zeta^j - 1),\ T_1(\zeta^{-j}) = i(\zeta^{-j} - 1), \qquad j \geqslant 0. \tag{2.7}$$

In the course of the proof of Theorem 2 (see Theorem 3 and its proof), we give a minimal subfamily of the $A(\mu, \zeta)$ for which the conclusions of the theorem still hold. The algebraic sets $\mathscr{B}$ and $\mathscr{C}$ are also explicitly determined from the defining polynomials p_j in (2.3).

3. PRELIMINARIES ON ANALYTIC DISCS

Let M be a generic manifold defined by (1.1). We consider families of discs attached to M and depending on parameters. For $z^0, z^1 \in C^{1,\alpha}(S^1)$, $z^0(1) = z^1(1) = 0$, valued in $\mathbf{C}^n$ with small norm, we obtain a family of discs from the Bishop equations (1.2) with $z(\zeta)$ replaced by $z^0(\zeta) + \lambda z^1(\zeta)$, where λ is a small complex number. We put

$$s_\lambda(\zeta) = -T_1(\phi(z^0(\cdot) + \lambda z^1(\cdot), \overline{z^0(\cdot)} + \overline{\lambda z^1(\cdot)}, s_\lambda(\cdot)))(\zeta), \qquad \zeta \in S^1. \tag{3.1}$$

This gives an analytic disc attached to M through 0 by setting $A_\lambda(\zeta) = (z^0(\zeta) + \lambda z^1(\zeta), w_\lambda(\zeta))$, with $w_\lambda(\zeta) = s_\lambda(\zeta) + i\phi(z^0(\zeta) + \lambda z^1(\zeta),$

$\overline{z^0(\zeta)}+\overline{\lambda z^1(\zeta)}, s_\lambda(\zeta))$, $\zeta\in S^1$. We can differentiate Eq. (3.1) with respect to λ. We define $u(\zeta)$ by

$$u(\zeta)=\frac{\partial s_\lambda}{\partial\lambda}(\zeta)\big|_{\lambda=0}, \tag{3.2}$$

so that $u(\zeta)$ satisfies the equation

$$\begin{aligned} u=-T_1(au+b), \qquad a(\zeta)=\phi_s(z^0(\zeta), \overline{z^0(\zeta)}, s^0(\zeta)),\\ b(\zeta)=\phi_z(z^0(\zeta), \overline{z^0(\zeta)}, s^0(\zeta))\, z^1(\zeta), \end{aligned} \tag{3.3}$$

for $\zeta\in S^1$. Let $v(\zeta)$ be the unique $\ell\times\ell$ matrix valued function with coefficients in $C^{1,\alpha}(S^1)$ satisfying

$$v=I+T_1(va), \tag{3.4}$$

where I is the identity matrix.

A slight modification of an argument given in Tumanov [12] gives the following formulas.

$$u=-(I+a^2)^{-1}\,[v^{-1}T_1(vb)+ab], \tag{3.5}$$

and

$$\mathscr{T}(au+b)=\mathscr{T}(vb), \tag{3.6}$$

where $\mathscr{T}$ is given by (1.4).

We also need to differentiate the mapping $\theta\mapsto s_\lambda(e^{i\theta})$. Writing $\tilde{s}(\theta)=s(e^{i\theta})$ and using (1.3) we have $\tilde{s}'_\lambda(0)=-\mathscr{T}(\phi[z(\lambda,\cdot), \overline{z(\lambda,\cdot)}, s(\lambda,\cdot)])$, which after differentiation yields

$$\begin{aligned} \frac{\partial}{\partial\lambda}\tilde{s}'_\lambda(0)\big|_{\lambda=0}=-\mathscr{T}\Big((\phi_z[z^0(\cdot), \overline{z^0(\cdot)}, s^0(\cdot)]\, z^1(\cdot)\\ +\phi_s[z^0(\cdot), \overline{z^0(\cdot)}, s^0(\cdot)]\frac{\partial s_\lambda}{\partial\lambda}(\cdot))\big|_{\lambda=0}\Big). \end{aligned} \tag{3.7}$$

By (3.6) we obtain

$$\frac{\partial}{\partial\lambda}\tilde{s}'_\lambda(0)\big|_{\lambda=0}=-\mathscr{T}(v(\cdot)\,\phi_z[z^0(\cdot), \overline{z^0(\cdot)}, s^0(\cdot)]\, z^1(\cdot)). \tag{3.8}$$

In view of Theorem 1, a sufficient condition to determine a wedge of extendibility is given by the following lemma, which is proved by (3.8) and the implicit function theorem.

Lemma 3.9. *If $z^0, z^1, ..., z^\ell$ are analytic discs valued in $\mathbf{C}^n$ with small norm, $z^j(1)=0$ for all j, and $r \in \mathbf{R}^\ell$ small, let $z(r, \cdot)$ be the family of discs given by $z(r, \cdot) = z^0(\cdot) + \sum_1^\ell r_j z^j(\cdot)$. For r fixed, let $s(r, \cdot)$ be the corresponding function given by the Bishop equation (1.2), with $z(\zeta)$ replaced by $z(r, \zeta)$ and $v(\zeta)$ the function determined by (3.4), using z^0. If the vectors $\operatorname{Re} \mathscr{T}(v(\cdot)\, \phi_z[z^0(\cdot), \overline{z^0(\cdot)}, s^0(\cdot)]\, z^j(\cdot))$, $j=1, ..., l$, are linearly independent then the image of any sufficiently small open neighborhood of 0 in $\mathbf{R}^\ell$ under the mapping*

$$r \mapsto \mathscr{T}(\phi[z(r, \cdot), \overline{z(r, \cdot)}, s(r, \cdot)]) \tag{3.10}$$

is open in $\mathbf{R}^\ell$.

If $A(\mu, \zeta)$ is a family of analytic discs attached to M with

$$z(\mu, \zeta) = (\zeta - 1) \sum_{j=0}^{K} \mu_j \zeta^j, \qquad \mu = (\mu_0, ..., \mu_K),\ \mu_j \in \mathbf{C}^n, \tag{3.11}$$

as in Section 2, we define the evaluation map Θ and the disc direction map Ψ for $\mu \in \mathbf{C}^{n(K+1)}$ by

$$\Theta(\mu) = A(\mu, -1) \in M, \qquad \Psi(\mu) = \mathscr{T}(\phi[z(\mu, \cdot), \overline{z(\mu, \cdot)}, s(\mu, \cdot)]) \in \mathbf{R}^\ell. \tag{3.12}$$

More generally, for $\zeta \in S^1$, we may define the mapping $\Theta(\mu, \zeta) = A(\mu, \zeta) \in M$ so that $\Theta(\mu, -1) = \Theta(\mu)$. By using (3.5), we may write the Jacobian matrix of $\Theta(\mu, \zeta)$ with respect to μ as

$$\Theta'(\mu, \zeta) = B(\zeta) = (B_0(\zeta) \cdots B_K(\zeta)), \tag{3.13}$$

where the $(2n+\ell) \times (2n)$ matrix $B_j(\zeta)$ is given by

$$B_j(\zeta) = \begin{pmatrix} (1-\zeta)\zeta^j I_{n\times n} & 0 \\ 0 & (1-\bar\zeta)\bar\zeta^j I_{n\times n} \\ -(I+a^2)^{-1}[v^{-1}T_1(vb_j) + ab_j](\zeta) & -(I+a^2)^{-1}[v^{-1}T_1(v\bar b_j) + a\bar b_j](\zeta) \end{pmatrix},$$

with $a = \phi_s(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta))$ and $b_j = \phi_z(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta))\,(1-\zeta)\zeta^j$. Therefore, by row and column operations the rank of $B(\zeta)$ is the same as the rank of the matrix $C(\zeta) = (C_0(\zeta) \cdots C_K(\zeta))$, with

$$C_j(\zeta) = \begin{pmatrix} (1-\zeta)\zeta^j I_{n\times n} & 0 \\ 0 & (1-\bar\zeta)\bar\zeta^j I_{n\times n} \\ T_1(vb_j)(\zeta) + ivb_j(\zeta) & T_1(v\bar b_j)(\zeta) - iv\bar b_j(\zeta) \end{pmatrix}. \tag{3.14}$$

Similarly, using again (3.6), the Jacobian matrix of the mapping $\Psi(\mu)$ is given by $\Psi'(\mu) = (D_0 \cdots D_K)$ where D_j is the $\ell \times (2n)$ matrix given by

$$D_j = (\mathscr{T}(vb_j), \mathscr{T}(v\bar{b}_j)). \tag{3.15}$$

Following Tumanov [12], if $z(\zeta)$ is an analytic disc valued in $\mathbf{C}^n$, with $z(1) = 0$, we say that it is of *defect* 0 if for every $v \in \mathbf{R}^\ell \backslash 0$, the mapping

$$S^1 \ni \zeta \mapsto v \cdot v(\zeta)\, \phi_z(z(\zeta), \overline{z(\zeta)}, s(\zeta)) \in \mathbf{C}^n \tag{3.16}$$

does not extend holomorphically to the unit disc.

4. Families of Polynomial Discs

The proof of Theorem 2 is based on an induction on the codimension of M. We assume here that M is a homogeneous generic submanifold of $\mathbf{C}^{n+\ell}$ given by (2.3). It is convenient to denote by M_1 the homogeneous generic submanifold of $\mathbf{C}^{n+\ell-1}$ given by

$$\begin{aligned} z_j &= x_j + iy_j, & j &= 1, ..., n, \\ w_k &= s_k + ip'_{m_k}(x, y, s_1, ..., s_{k-1}), & k &= 1, ..., r', \end{aligned} \tag{4.1}$$

where $z \in \mathbf{C}^n$ and $w_k \in \mathbf{C}^{\ell'_k}$, $\sum_{k=1}^{r'} \ell'_k = \ell - 1$, determined as follows. If $\ell_r = 1$ in (2.3), then $r' = r - 1$, $\ell'_k = \ell_k$, and $p'_{m_k} = p_{m_k}$, for $k = 1, ..., r-1$. If $\ell_r > 1$ in (2.3), then $r' = r$, $\ell'_k = \ell_k$, and $p'_{m_k} = p_{m_k}$, for $k = 1, ..., r-1$, and $\ell'_r = \ell_r - 1$, and the vector p'_{m_r} is obtained by dropping the last component of the vector p_{m_r}. With $K = N_\ell$ defined by (2.4) and $K_1 = N_{\ell-1}$ we have

$$K = (n_\ell - 1)(K_1 + 1). \tag{4.2}$$

For $\mu \in \mathbf{C}^{n(K+1)}$ let $z(\mu) = z(\mu, \zeta)$, the analytic disc defined by (2.5). We consider three subsets of $\mathbf{C}^{n(K+1)}$ as follows.

$$\begin{aligned} \mathscr{A} &= \{\mu \in \mathbf{C}^{n(K+1)}: z(\mu) \text{ is not of defect 0 for } M\}, \\ \mathscr{B} &= \{\mu \in \mathbf{C}^{n(K+1)}: \Theta'(\mu) \text{ is of rank } < 2n + \ell\}, \\ \mathscr{C} &= \{\mu \in \mathbf{C}^{n(K+1)}: \Psi'(\mu) \text{ is of rank } < \ell\}, \end{aligned} \tag{4.3}$$

where Θ and Ψ are defined in (3.12).

Lemma 4.4. *If M is a homogeneous generic submanifold defined by* (2.3) *then $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ as defined by* (4.3) *are real algebraic sets.*

Proof. We show first that $\mathscr{A}$ is algebraic. We write

$$q(\mu, e^{i\theta}) = v(e^{i\theta})\, p_z[z(\mu, e^{i\theta}), \overline{z(\mu, e^{i\theta})}, s(\mu, e^{i\theta})], \tag{4.5}$$

where v is defined by (3.4), in which z^0 is replaced by $z(\mu)$. Note that v can be explicitly computed from (3.4) in the case where M is homogeneous and that its entries are trigonometric polynomials with coefficients which are polynomials in μ and $\bar{\mu}$. Hence it follows from (4.5) that the same holds for $q(\mu, e^{i\theta})$; i.e.,

$$q(\mu, e^{i\theta}) = \sum_{j=-N}^{N} q_j(\mu, \bar{\mu})\, e^{ij\theta} \tag{4.6}$$

where the $q_j(\mu, \bar{\mu})$ are polynomials in μ and $\bar{\mu}$, and N is independent of μ. Hence, by definition, $z(\mu)$ is not of defect 0 if and only if there exists $v \in \mathbf{R}^{\ell} \backslash 0$, such that $vq_j(\mu, \bar{\mu}) = 0$ for $j = -N, ..., -1$, which is equivalent to the condition that the span of the columns of the $\ell \times n$ real matrices $\operatorname{Re} q_j(\mu, \bar{\mu})$, $\operatorname{Im} q_j(\mu, \bar{\mu})$, $j = -N, ..., -1$, is of real dimension less than ℓ. The latter condition is obtained by the vanishing of subdeterminants obtained from the coefficients of the matrices $\operatorname{Re} q_j$, $\operatorname{Im} q_j$ and is hence given by the vanishing of real-valued polynomials in μ and $\bar{\mu}$. This proves that the set $\mathscr{A}$ is real algebraic.

To prove that $\mathscr{B}$ is algebraic, we note that by (3.13) the Jacobian matrix $\Theta'(\mu)$ is given by $(C_0(-1) \cdots C_K(-1))$, where $C_j(\zeta)$ is the $(2n+\ell) \times (2n)$ matrix given by (3.14). Since all the entries of these matrices are polynomials in μ and $\bar{\mu}$, the result follows.

Finally, to show that $\mathscr{C}$ is algebraic we observe that $\Psi'(\mu) = (D_0 \cdots D_K)$ where D_j is given by (3.15). Again the result follows since all the entries of D_j are polynomials in μ and $\bar{\mu}$. This completes the proof of Lemma 4.4. ∎

Using the notation of (2.3) we write

$$p_s = \begin{pmatrix} p_{m_1 s_1} & p_{m_1 s_2} & \cdots & p_{m_1 s_r} \\ p_{m_2 s_1} & p_{m_2 s_2} & \cdots & p_{m_2 s_r} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m_r s_1} & p_{m_r s_2} & \cdots & p_{m_r s_r} \end{pmatrix}, \tag{4.7}$$

where each $p_{m_j s_k}$ is an $\ell_j \times \ell_k$ matrix. Since $p_{m_j s_k} = 0$ for $k \geqslant j$ we have

$$p_s = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ p_{m_2 s_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{m_r s_1} & p_{m_r s_2} & \cdots & 0 \end{pmatrix}. \tag{4.8}$$

Note that if $z(\zeta)$ is a disc valued in $\mathbf{C}^n$, corresponding to (4.8), we may write the matrix $v(\zeta)$ given by (3.4) as $v(\zeta)=(v_{jk}(\zeta))$ where each $v_{jk}(\zeta)$ is an $\ell_j\times\ell_k$ matrix with $v_{jk}(\zeta)=0$ for $k>j$, and $v_{jj}=I_{\ell_j\times\ell_j}$. We use the following results.

LEMMA 4.9. *Let $z(\zeta)$ be a polynomial analytic disc valued in $\mathbf{C}^n$ of degree q with $z(1)=0$ and let $v=(v_{jk}(\zeta))$ be as above. Let $p_z(z(\zeta),\bar{z}(\zeta),s(\zeta))=(p_{m_jz_k})$, where each $p_{m_jz_k}$ is an $\ell_k\times 1$ column vector. Then the entries of $v_{jk}(\zeta)$ are trigonometric polynomials of degree $q(m_j-m_k)$. Also, if $vp_z=(a_{jk})$, $1\leqslant j\leqslant r$, $1\leqslant k\leqslant n$, with a_{jk} an $\ell_j\times 1$ vector, then the components of a_{jk} are trigonometric polynomials of degree $q(m_j-1)$.*

Proof. First, if $s(\zeta)=(s_1(\zeta),...,s_r(\zeta))$, then by the Bishop equation (1.2) and induction, it follows that the components of the $m_j\times 1$ vector $s_j(\zeta)$ are trigonometric polynomials of degree qm_j. By using (3.4) and the forms of p_s and $v(\zeta)$ given above, one can compute $v_{jk}(\zeta)$ inductively for $j>k$:

$$v_{jk}(\zeta)=T_1(p_{m_js_k}+v_{jj-1}p_{m_{j-1}s_k}+\cdots v_{jk+1}p_{m_{k+1}s_k}). \tag{4.10}$$

Since the operator T_1 preserves the degree of the entries and, by the first remark, the degree of $p_{m_js_k}=p_{m_js_k}(z(\zeta),\bar{z}(\zeta),s(\zeta))$ is $q(m_j-m_k)$, we obtain inductively that the degree of $(v_{jk}(\zeta))$ is

$$\max\{q(m_j-m_k),q[(m_j-m_{j-1})+(m_{j-1}-m_k)],\\ ...,q[(m_j-m_{k+1})+(m_{k+1}-m_k)]\},$$

which is $q(m_j-m_k)$ as claimed. Using this, we can show that the degree of $(vp_z)_{jk}$ is $q(m_j-1)$, by multiplying the matrices and observing that the degree of $p_{m_jz_k}$ is $q(m_j-1)$. ∎

LEMMA 4.11. *Let $z(\mu)$ be given by* (3.11). *Then if $\mu_j=0$, $K_1<j\leqslant K$ (with K_1 as in* (4.2)), *then*

$$\operatorname{rk}\Theta'(\mu)=2n+\operatorname{rk}(E_0\cdots E_{K-1}), \tag{4.12}$$

where E_j is the $\ell\times 2n$ matrix given by

$$E_j=(T_1(vb_j)(-1)+ivb_j(-1)\quad T_1(v\bar{b}_j)(-1)-i\overline{vb_j(-1)}), \tag{4.13}$$

with v given by (3.4) *with z^0 replaced by $z(\mu)$, and $b_j=p_z(z(\mu,\zeta),\overline{z(\mu,\zeta)},s(\mu,\zeta))(\zeta-1)\zeta^j$.*

Proof. By Lemma (4.9), if v is as above, all of the components of

$$v(\zeta)\,p_z(z(\mu,\zeta),\overline{z(\mu,\zeta)},s(\mu,\zeta))$$

are trigonometric polynomials of the form $\sum_{j=-N_1}^{N_2} a_j\zeta^j$ with $N_1 \leqslant (K_1+1)(n_\ell-1)=K$. Hence every component of $v(\zeta)\,p_z(z(\mu,\zeta),\overline{z(\mu,\zeta)}, s(\mu,\zeta))\,\zeta^K$ extends holomorphically. In particular, with $C_j(\zeta)$ as given in (3.14) we have

$$C_K(\zeta)=\begin{pmatrix}(1-\zeta)\,\zeta^K I_{n\times n} & 0\\ 0 & (1-\bar{\zeta})\,\bar{\zeta}^K I_{n\times n}\\ 0 & 0\end{pmatrix}. \tag{4.14}$$

Hence

$$\operatorname{rk}[C_0(-1)\cdots C_K(-1)]=\operatorname{rk}(E_0\cdots E_{K-1})+2n, \qquad E_K=0, \tag{4.15}$$

which proves (4.12). ∎

5. Properties of the Algebraic Sets $\mathscr{A}, \mathscr{B}, \mathscr{C}$

Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be the subsets of $\mathbf{C}^{n(K+1)}$ given by (4.3). By Lemma 4.4 they are algebraic. In this section we give some further properties of these sets, which are used in Section 6 to show that these algebraic sets are proper.

We use a result of Tumanov [12] that we recall here. If M is given by (1.11), we let $\mathscr{E}$ be the real Banach space of all analytic discs $z(\zeta)$ valued in $\mathbf{C}^n$ of class $C^{1,\alpha}(S^1)$ with $z(1)=0$. We consider the maps $\mathscr{F}:\mathscr{E}\to M$ and $\mathscr{G}:\mathscr{E}\to\mathbf{R}^\ell$ defined as follows. For $z\in\mathscr{E}$ with small norm, let $A(\zeta)=(z(\zeta), w(\zeta))$ be the corresponding analytic disc attached to M, with $s(\zeta)=\operatorname{Re} w(\zeta)$. Put

$$\mathscr{F}(z)=A(-1), \qquad \mathscr{G}(z)=\mathscr{T}(z(\cdot), \overline{z(\cdot)}, s(\cdot)), \tag{5.1}$$

where $\mathscr{T}$ is given by (1.4). A main result in [12] is that the following are equivalent for $z\in\mathscr{E}$ with small norm:

(a) z is of defect 0,

(b) the rank of the Fréchet derivative $\mathscr{F}'(z):\mathscr{E}\to T_{\mathscr{F}(\zeta)}M$ is $2n+\ell$,

(c) the rank of $\mathscr{G}'(z):\mathscr{E}\to\mathbf{R}^\ell$ is ℓ.

Furthermore, for every $z\in\mathscr{E}$ with small norm the following holds:

$$\mathscr{F}'(z)\,\mathscr{E}\supset T^c_{\mathscr{F}(z)}M. \tag{5.2}$$

Using the above we can prove the following.

LEMMA 5.3. *The following inclusions hold*:

(i) $\mathscr{A} \subset \mathscr{B}$,

(ii) $\mathscr{A} \subset \mathscr{C}$.

Proof. Since $\Theta(\mu) = \mathscr{F}(z(\mu))$ and $\Psi(\mu) = \mathscr{G}(z(\mu))$ it follows that $\operatorname{rk} \Theta'(\mu) \leqslant \operatorname{rk} \mathscr{F}'(z(\mu))$ and $\operatorname{rk} \Psi'(\mu) \leqslant \operatorname{rk} \mathscr{G}'(z(\mu))$. Inclusions (i) and (ii) of the lemma then follow from the definitions of the sets $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ and the equivalence of (a), (b), and (c) above. ∎

As in Section 4, let K_1 be given by (4.2) and put

$$\begin{aligned} \mathscr{A}_0 &= \mathscr{A} \cap (\mathbf{C}^{n(K_1+1)} \times \{0\}), \\ \mathscr{B}_0 &= \mathscr{B} \cap (\mathbf{C}^{n(K_1+1)} \times \{0\}), \\ \mathscr{C}_0 &= \mathscr{C} \cap (\mathbf{C}^{n(K_1+1)} \times \{0\}). \end{aligned} \tag{5.4}$$

The main result of this section is the following.

PROPOSITION 5.5. *If $\mathscr{A}_0$, $\mathscr{B}_0$, $\mathscr{C}_0$ are defined by* (5.4), *then*

$$\mathscr{A}_0 = \mathscr{B}_0 = \mathscr{C}_0.$$

Proof. Since by Lemma 5.3 we have the inclusion $\mathscr{A}_0 \subset \mathscr{B}_0$ and $\mathscr{A}_0 \subset \mathscr{C}_0$ it suffices to show $\mathscr{B}_0 \subset \mathscr{A}_0$ and $\mathscr{C}_0 \subset \mathscr{A}_0$. We first prove $\mathscr{B}_0 \subset \mathscr{A}_0$. For this, let $\mu \in \mathbf{C}^{n(K_1+1)} \times \{0\}$ with $\mu \notin \mathscr{A}$. We show that $\mu \notin \mathscr{B}$. The assumptions of Lemma 4.11 are satisfied, so that it suffices to show that

$$\operatorname{rk}(E_0, ..., E_{K-1}) = \ell, \tag{5.6}$$

where E_j is given by (4.13).

We apply the following algebraic lemma to the $\ell \times n$ matrix $v(\zeta)\, p_z(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta))(\zeta - 1)$. We write $O(1)$ for a trigonometric polynomial with no negative powers and $O(\zeta)$ for a trigonometric polynomial with no negative powers or constant term.

LEMMA (5.7). *Let $Q(\zeta) = (Q_{ij}(\zeta))$, $\zeta \in S^1$, be an $\ell \times n$ matrix of trigonometric polynomials. Then there exist nonnegative integers a_p, b_p, $1 \leqslant p \leqslant n$, with $\sum_1^n a_p + 2b_p \leqslant \ell$, positive integers t_{jk}, $1 \leqslant j \leqslant a_k$, $1 \leqslant k \leqslant n$, and positive integers r_{jk}, $1 \leqslant j \leqslant b_k$, $1 \leqslant k \leqslant n$, such that for every fixed k the r_{jk} and t_{jk} are all distinct, and an invertible $\ell \times \ell$ real constant matrix A such that the following holds. If $\tilde{Q}(\zeta) = AQ(\zeta)$ and $\alpha_k = \sum_1^k a_p + 2b_p$, $\alpha_0 = 0$, then for $k = 1, ..., n$ we have*

$$\tilde{Q}_{\alpha_{k-1}+j,k}(\zeta)=\zeta^{-t_{jk}}(e^{i\theta_{jk}}+O(\zeta)), \qquad 1\leqslant j\leqslant a_k,$$
$$\tilde{Q}_{\alpha_{k-1}+a_k+2j-1,k}(\zeta)=\zeta^{-r_{jk}}(1+O(\zeta)), \qquad 1\leqslant j\leqslant b_k,$$
$$\tilde{Q}_{\alpha_{k-1}+a_k+2j,k}(\zeta)=\zeta^{-r_{jk}}(i+O(\zeta)), \qquad 1\leqslant j\leqslant b_k,$$
$$\tilde{Q}_{\alpha_k+j,k}(\zeta)=O(1), \qquad 1\leqslant j\leqslant \ell-\alpha_k,$$

with θ_{jk} real. Furthermore, A may be chosen so that for each k fixed the sequences $j\mapsto t_{jk}$ and $j\mapsto r_{jk}$ are both strictly decreasing. In addition, if for every $v\in\mathbf{R}^{\ell}\backslash 0$, at least one component of the vector-valued function $\zeta\mapsto vQ(\zeta)$ does not extend holomorphically, then $\sum_1^n a_p+2b_p=\ell$.

Proof. We begin with the column vector $(Q_{11}, Q_{21}, ..., Q_{\ell 1})$. After a real linear change in the entries, which may be achieved by multiplying Q on the left by a real invertible matrix A_1, we have

$$\tilde{Q}_{j1}(\zeta)=\zeta^{-t_{j1}}(e^{i\theta_{j1}}+O(\zeta)), \qquad 1\leqslant j\leqslant a_1,$$
$$\tilde{Q}_{a_1+2j-1,1}(\zeta)=\zeta^{-r_{j1}}(1+O(\zeta)), \qquad 1\leqslant j\leqslant b_1,$$
$$\tilde{Q}_{a_1+2j,1}(\zeta)=\zeta^{-r_{j1}}(i+O(\zeta)), \qquad 1\leqslant j\leqslant b_1,$$
$$\tilde{Q}_{\alpha_1+j,1}(\zeta)=O(1), \qquad 1\leqslant j\leqslant \ell,$$

with $\alpha_1=a_1+2b_1$ and the integers a_1, b_1, t_{j1}, r_{j1} uniquely determined and satisfying the properties of the lemma. Let $Q^1=A_1Q$. We continue with the same process with the column vector $(Q^1_{\alpha_1+1,2}, ..., Q^1_{\ell,2})$. To put it in the desired form involves multiplication of A_1Q on the left by a real matrix A_2. Since this involves only the last $\ell-\alpha_1$ rows of Q^1, it does not change the property of the first column. If $n=2$ we take $A=A_2A_1$; for $n>2$, the rest of the proof follows by an inductive argument on n. ∎

The following elementary lemma, whose proof makes use of (2.7) and is left to the reader, also is used in the proof of Proposition (5.5).

LEMMA 5.8. *Let $u(\zeta)$ be a smooth function defined on S^1.*

(i) *If $u(\zeta)$ extends holomorphically to Δ then $T_1(u)+iu\equiv iu(1)$.*

(ii) *If $u(1)=0$ and $u(\zeta)=\zeta^{-1}c+a(\zeta)$, where $a(\zeta)$ extends holomorphically to Δ and $c\in\mathbf{C}$, then $(T_1(u))(-1)+iu(-1)=-4ic$.*

We return to the proof of the inclusion $\mathscr{B}_0\subset\mathscr{A}_0$ of Proposition 5.5. We apply Lemma 5.7 to

$$Q(\zeta)=v(\zeta)\,p_z(z(\mu,\zeta),\overline{z(\mu,\zeta)},s(\mu,\zeta))\,(\zeta-1),$$

where μ is chosen as above ($\mu\in\mathbf{C}^{n(K_1+1)}\times\{0\}$, $\mu\notin\mathscr{A}$) to obtain $\tilde{Q}(\zeta)$. By the hypothesis $\mu\notin\mathscr{A}$ and the fact that the components of

$v(\zeta)\, p_z(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta))$ are trigonometric polynomials, we have that $vQ(\zeta) = vv(\zeta)\, p_z(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta))\, (\zeta - 1)$ does not extend holomorphically for any $v \in \mathbf{R}^{\ell} \backslash 0$. By Lemma 4.9 the t_{jk} and r_{jk} are necessarily all less than or equal to $(K_1+1)(m_r - 1) = K$.

In order to show (5.6), it suffices to find nonnegative integers $J_1 \leqslant \cdots \leqslant J_\Gamma \leqslant K$, $\Gamma = \sum_{p=1}^{n} a_p + b_p$, such that $\mathrm{rk}(AE_{J_1}, ..., AE_{J_\Gamma}) = \ell$. We claim that the set of integers $t_{jk} - 1$ and $r_{jk} - 1$, where the t_{jk} and r_{jk} are given by Lemma 5.7, gives a choice for $J_1, ..., J_\Gamma$. For this, note first that

$$\begin{aligned} AE_{J_k} &= [T_1(Avp_z(\zeta-1)\,\zeta^{J_k}) + iAvp_z(\zeta-1)\,\zeta^{J_k}]_{|\zeta=-1} \\ &= [T_1(\tilde{Q}(\zeta)\,\zeta^{J_k}) + i\tilde{Q}(\zeta)\,\zeta^{J_k}]_{|\zeta=-1}. \end{aligned} \tag{5.9}$$

By the form of the columns of $\tilde{Q}$ given in Lemma 5.7 and by Lemma 5.8, we have for any nonnegative integer J,

$$AE_J = (Y_1^J \cdots Y_n^J \overline{Y_1^J} \cdots \overline{Y_n^J}), \qquad \text{with} \quad Y_p^J = \begin{pmatrix} X_p^J \\ D_p^J \\ 0 \end{pmatrix}, \tag{5.10}$$

where X_p^J is an $\alpha_{p-1} \times 1$ vector, D_p^J is a $\beta_p \times 1$ vector, $\beta_p = \alpha_p - \alpha_{p-1} = a_p + 2b_p$, and 0 denotes the $\ell - \alpha_p$ vector with all entries equal to 0. To prove the claim, by the form of the right-hand side of (5.10), it suffices to show that for each p, $1 \leqslant p \leqslant n$,

$$\mathrm{rk}(D_p^{R_1} \cdots D_p^{R_{\gamma_p}} \overline{D_p^{R_1}} \cdots \overline{D_p^{R_{\gamma_p}}}) = \beta_p, \tag{5.11}$$

for $\{R_1, ..., R_{\gamma_p}\} = \{t_{1p} - 1, ..., t_{a_p, p} - 1, r_{1,p} - 1, ..., r_{b_p, p} - 1\}$, $\gamma_p = a_p + b_p$. From the form of $\tilde{Q}$, and the use of Lemma 5.8, one can check (5.11) directly. Since $\sum_p \beta_p = \ell$, the claim is proved and hence so is the inclusion $\mathscr{B}_0 \subset \mathscr{A}_0$ of Proposition 5.5.

Proof of the inclusion $\mathscr{C}_0 \subset \mathscr{A}_0$ *of Proposition* 5.5. This proof is very similar to that of the inclusion $\mathscr{B}_0 \subset \mathscr{A}_0$, which we have just completed.

Let $\mu \in \mathbf{C}^{n(K_1+1)} \times \{0\}$ with $\mu \notin \mathscr{A}$. To show that $\mu \notin \mathscr{C}$, by using the observations at the end of Section 3, it suffices to show that

$$\mathrm{rk}(D_0 \cdots D_K) = \ell, \qquad \text{with} \quad D_j = (\mathscr{T}(vb_j)\, \mathscr{T}(v\bar{b}_j)). \tag{5.12}$$

Since $v(\zeta)\, p_z(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta))$ vanishes at $\zeta = 1$, we may write

$$v(\zeta)\, p_z(z(\mu, \zeta), \overline{z(\mu, \zeta)}, s(\mu, \zeta)) = R(\zeta)\, (\bar{\zeta} - 1),$$

where $R(\zeta)$ is an $\ell \times n$ matrix of trigonometric polynomials. By the definition of $\mathscr{A}$, for every $v \in \mathbf{R}^{\ell} \backslash \{0\}$, we have that $v \cdot R(\zeta)\bar{\zeta}$ does not extend holomorphically. We may now apply Lemma 5.7 to the matrix

$Q(\zeta)=R(\zeta)\bar{\zeta}$, so that there exists A, a real invertible matrix with $\tilde{Q}(\zeta)=AQ(\zeta)$ satisfying the conclusion of the lemma. As in the proof of the inclusion $\mathscr{B}_0\subset\mathscr{A}_0$, it suffices to find nonnegative integers $J_1\leqslant\cdots\leqslant J_\Gamma\leqslant K$, $\Gamma=\sum_{p=1}^n a_p+b_p$, such that $\operatorname{rk}(AD_{J_1}\cdots AD_{J_\Gamma})=\ell$. As before, it suffices to take for $J_1, ..., J_\Gamma$ the set of integers $t_{jk}-1$ and $r_{jk}-1$ given by Lemma 5.7. This completes the proof of Proposition 5.5. ∎

6. Induction on the Codimension of M: Proof of Theorem 2

In this section we show that the algebraic sets $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ (see Lemma 4.4) given by (4.3) are all proper. Theorem 2 then follows. We use an induction on ℓ, the codimension of M, to prove the following.

THEOREM 3. *Assume that M is homogeneous and of finite type. Then the sets $\mathscr{A}_0$, $\mathscr{B}_0$, $\mathscr{C}_0$ defined by* (5.4) *coincide and are proper algebraic subsets of $\mathbf{C}^{n(K_1+1)}\times\{0\}$. In particular, the sets $\mathscr{A},\mathscr{B},\mathscr{C}$ given by* (4.3) *are proper algebraic subsets of $\mathbf{C}^{n(K+1)}$.*

For our induction, we denote by M_1 the homogeneous generic submanifold of $\mathbf{C}^{n+\ell-1}$ given by (4.1). Similarly, we denote by $\mathscr{A}_1$, $\mathscr{B}_1$, $\mathscr{C}_1$ the subsets of $\mathbf{C}^{n(K_1+1)}$ which are the analogues of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ for M_1. Also, we denote by Θ_1 and Ψ_1 the analogues for M_1 of the mappings Θ and Ψ. For more clarity, we denote by $\mathscr{F}_M$ the map $\mathscr{F}:\mathscr{E}\to M$ given by (5.1). Similarly, we denote by $\mathscr{F}_{M_1}$ the analogue for M_1. We first prove the more precise following statement.

PROPOSITION 6.1. *Let $\mathscr{E}_1$ be a closed real subspace of the Banach space $\mathscr{E}$, and denote by $\mathscr{H}_M$ (resp., $\mathscr{H}_{M_1}$) the restriction of $\mathscr{F}_M$ (resp. $\mathscr{F}_{M_1}$) to $\mathscr{E}_1$. Assume there exists $z^0\in\mathscr{E}_1$ of sufficiently small norm such that $\operatorname{rk}\mathscr{H}'_{M_1}(z^0)=2n+\ell-1$. Then there exists $z^1\in\mathscr{E}_1$ arbitrarily close to z^0 such that z^1 is of defect* 0 *for M.*

Proof. We argue by contradiction. Suppose that there is no disc of defect 0 for M in an open neighborhood U of z^0 in $\mathscr{E}_1$. Therefore, by the equivalence of (a) and (b) following (5.1), we have for any $z\in U$, $\operatorname{rk}\mathscr{F}'_M(z)<2n+\ell$. On the other hand, if U is sufficiently small, we clearly have for all $z\in U$,

$$\operatorname{rk}\mathscr{F}'_M(z)\geqslant\operatorname{rk}\mathscr{H}'_M(z)\geqslant\operatorname{rk}\mathscr{H}'_{M_1}(z)=2n+\ell-1. \tag{6.2}$$

Hence all the inequalities in (6.2) are equalities. In particular, $\operatorname{rk}\mathscr{F}'_M(z)=\operatorname{rk}\mathscr{H}'_M(z)=2n+\ell-1$ for $z\in U$. By shrinking U is necessary, we may assume that $\sum = \mathscr{H}_M(U)$ is a submanifold of M of dimension $2n + \ell - 1$. Using (5.2) and the remark above, we have for $z \in U$,

$$\mathscr{H}'_M(z)\,\mathscr{E}_1 = \mathscr{F}'_M(z)\,\mathscr{E} \supset T^c_{\mathscr{H}(z)}M,$$

and hence the manifold $\sum$ is a CR submanifold of M of the same CR dimension as M. This would contradict the fact that M is of finite type at every point and in particular at $\mathscr{H}_M(z^0)$. Hence there must be a disc of defect 0 in every neighborhood of z^0 in $\mathscr{E}_1$. This completes the proof of Proposition 6.1. ∎

Proof of Theorem 3. By Lemma 4.4 and Proposition 5.5, we already know that the sets $\mathscr{A}_0$, $\mathscr{B}_0$, and $\mathscr{C}_0$ coincide and form an algebraic subset of $\mathbf{C}^{n(K_1+1)} \times \{0\}$. It remains only to show that this algebraic subset is proper. For this, we use Proposition 6.1 and an induction on ℓ, the codimension of M. The case $\ell = 0$ is straightforward: $M = \mathbf{C}^n$, $K = 0$, and the sets $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are all empty. Assume the theorem holds for $\ell - 1$; we prove that it holds for ℓ. By the induction hypothesis, $\mathscr{B}_1$ is a proper algebraic subset of $\mathbf{C}^{n(K_1+1)}$. From the remarks above, it suffices to show that $\mathscr{A}_0$ is a proper subset of $\mathbf{C}^{n(K_1+1)} \times \{0\}$. We apply Proposition 6.1 with

$$\mathscr{E}_1 = \left\{ z \in \mathscr{E} : z(\zeta) = z(\mu; \zeta) = (\zeta - 1) \sum_{j=0}^{K_1} \mu_j \zeta^j,\ \mu_j \in \mathbf{C}^n \right\}.$$

Note that the map $\mathbf{C}^{n(K_1+1)} \ni \mu \mapsto z(\mu; \cdot) \in \mathscr{E}_1$ is an injection of $\mathbf{C}^{n(K_1+1)}$ into $\mathscr{E}$. We have $\mathscr{H}_{M_1}(z(\mu, \cdot)) = \Theta_1(\mu)$ for all $\mu \in \mathbf{C}^{n(K_1+1)}$. Let $\mu^0 \in \mathbf{C}^{n(K_1+1)} \backslash \mathscr{B}_1$. By Proposition 6.1 there exists $\mu \in \mathbf{C}^{n(K_1+1)}$ arbitrarily close to μ^0 such that $z(\mu; \cdot)$ is of defect 0 for M. Hence, $(\mu, 0) \notin \mathscr{A}_0$, which shows that $\mathscr{A}_0$ is proper, and this completes the proof of Theorem 3.

Remark 6.3. In the proof of Theorem 3 we have actually shown that the open set

$$((\mathbf{C}^{n(K_1+1)} \backslash B_1) \times \{0\}) \cap ((\mathbf{C}^{n(K_1+1)} \times \{0\}) \backslash \mathscr{A}_0)$$

is dense in $(\mathbf{C}^{n(K_1+1)} \backslash B_1) \times \{0\}$.

Remark 6.4. For a general homogeneous manifold M of finite type the sets $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are distinct proper algebraic subsets of $\mathbf{C}^{n(K+1)}$, even in the case $\ell = 1$, $n = 1$. For instance, the following example shows that $\mathscr{A}$ and $\mathscr{B}$ are distinct.

EXAMPLE 6.5. Let M be the hypersurface in $\mathbf{C}^2$ given by $\operatorname{Im} w = p(z, \bar{z}) = z^4 \bar{z} + z \bar{z}^4$. Then $K = m - 1 = 4$. Let $z^0(\zeta) = (\zeta - 1)(\zeta + 1)\zeta^3 = (\zeta - 1)(\zeta^3 + \zeta^4) = z(\mu^0; \zeta)$, where $\mu^0 = (0, 0, 0, 1, 1) \in \mathbf{C}^5$. One can easily check that $\mu^0 \notin \mathscr{A}$. To see that $\mu^0 \in \mathscr{B}$, one may use (3.13) and observe that $p_z(z^0(\zeta), \overline{z^0(\zeta)})(1-\zeta)\zeta^j$, for $j=0, ..., 4$, is the sum of a holomorphic polynomial and an antiholomorphic polynomial, both vanishing at $\zeta = -1$.

7. Wedges and Anisotropic Wedges Attached to Homogeneous Generic Manifolds

For the rest of the paper we denote by M^0 instead of M the homogeneous generic manifold of finite type in $\mathbf{C}^{n+\ell}$ given by (2.3). (In Section 8 we let M denote a generic manifold of finite type with homogeneous part M^0.) It is convenient to write $\mathscr{Z} = (z, w) = (z, w_1, ..., w_r)$ and to introduce the quasinorm

$$N(\mathscr{Z}) = |z| + |w_1|^{1/m_1} + \cdots + |w_r|^{1/m_r}, \tag{7.1}$$

so that if δ_t is the anisotropic dilation given by (2.1), then $N(\delta_t \mathscr{Z}) = tN(\mathscr{Z})$. If Γ is a cone in $\mathbf{R}^\ell$ and $\eta > 0$, we put

$$\Gamma_\eta = \{\delta_\eta \theta, \theta \in \Gamma\}, \tag{7.2}$$

where we have also denoted by δ_η the restriction of the anisotropic dilations in the obvious way to $\mathbf{R}^\ell$. Note that if Γ is an open cone, so is Γ_η. Also, $\Gamma_\eta = \Gamma$ for some $\eta \neq 1$ if and only if $m_1 = \cdots = m_r$. For $a > 0$ we denote by $\mathscr{W}^0(\Gamma, a)$ the wedge of edge M^0 and direction Γ given by

$$\mathscr{W}^0(\Gamma, a) = \{\mathscr{Z} \in \mathbf{C}^{n+\ell} : N(\mathscr{Z}) < a, \operatorname{Im} w - p(z, \bar{z}, s) \in \Gamma\}. \tag{7.3}$$

An *anisotropic wedge with edge M^0 and direction Γ* is a set of the form

$$\mathscr{U}^0(\Gamma, a) = \left\{\mathscr{Z} \in \mathbf{C}^{n+\ell} : N(\mathscr{Z}) < a, \operatorname{Im} w - p(z, \bar{z}, s) \in \bigcup_{a^{-1}N(\mathscr{Z}) < \eta \leqslant 1} \Gamma_\eta\right\}. \tag{7.4}$$

It is easy to check that the wedges and anisotropic wedges defined by (7.3) and (7.4) are compatible with the dilations, i.e., for $t > 0$ we have

$$\delta_t \mathscr{W}^0(\Gamma, a) = \mathscr{W}^0(\Gamma_t, ta). \tag{7.5}$$

We also have for $0 < t \leqslant 1$,

$$\delta_t \mathscr{U}^0(\Gamma, a) = \mathscr{U}^0(\Gamma_t, ta) \subset \mathscr{U}^0(\Gamma, a) \tag{7.6}$$

and

$$\mathscr{U}^0(\Gamma, a) = \bigcup_{0 < \eta \leqslant 1} \mathscr{W}^0(\Gamma_\eta, \eta a). \tag{7.7}$$

Note that the analogue of the inclusion in (7.6) does not hold, in general, for wedges.

For $a > 0$, we denote by $\mathscr{D}_a^0$ the set of all analytic discs A attached to $M^0 \cap \{N(\mathscr{Z}) < a\}$ (without the restriction that $A(1) = 0$). Note that if $A = (z(\zeta), w(\zeta)) \in \mathscr{D}_1^0$, with $A(1) = 0$, and if $\gamma \in \mathbf{C}^n$ and $\sigma \in \mathbf{R}^\ell$ are sufficiently small, then the disc $A_{\gamma,\sigma} = (z_{\gamma,\sigma}(\zeta), w_{\gamma,\sigma}(\zeta))$ determined inductively by the equations

$$\begin{aligned} &z_\gamma(\zeta) = z(\zeta) + \gamma, \\ &s_{j,\gamma,\sigma}(\zeta)) = -T_1(p_{m_j}(z_\gamma, \bar{z}_\gamma, s_{1,\gamma,\sigma}, ..., s_{j-1,\gamma,\sigma})(\zeta) + \sigma_j, \qquad j = 1, ..., r, \end{aligned} \tag{7.8}$$

is in $\mathscr{D}_1^0$ and passes through the point $z = \gamma$, $\mathrm{Re}\, w = \sigma$. Also note that if $A(\cdot)$ is in $\mathscr{D}_1^0$ then $\delta_a A(\cdot)$ is in $\mathscr{D}_a^0$.

As in Section 5 we let $\mathscr{E}$ be the real Banach space of all analytic discs $z(\zeta)$ valued in $\mathbf{C}^n$ of class $C^{1,\alpha}(S^1)$ with $z(1) = 0$ and $\mathscr{G}$ the map defined by (5.1). We have the following theorem.

THEOREM 4. *Let $z^0 \in \mathscr{E}$ be of defect 0, and assume that the associated analytic disc $A^0(\zeta) = (z^0(\zeta), w^0(\zeta))$ through 0 is in $\mathscr{D}_1^0$. If $\mathscr{G}(z^0) \neq 0$ and Γ is a sufficiently small open cone containing $\mathscr{G}(z^0)$, then for $a > 0$ and small we have*

$$\mathscr{U}^0(\Gamma, a) \subset \bigcup_{A \in \mathscr{D}_1^0} A(\Delta). \tag{7.9}$$

If $\mathscr{G}(z^0) = 0$, then there is a positive number a for which the following holds:

$$\{\mathscr{Z} \in \mathbf{C}^{n+\ell} : N(\mathscr{Z}) < a\} \subset \bigcup_{A \in \mathscr{D}_1^0} A(\Delta). \tag{7.10}$$

We let $\mathscr{A}$ be the proper algebraic subset of $\mathbf{C}^{n(K+1)}$ defined by (4.3) and for $\mu \in \mathbf{C}^{n(K+1)}$, we let $A(\mu, \zeta)$ be the corresponding analytic disc defined by (3.11). As a corollary of Theorem 4 we have the following result, which shows that there is an anisotropic wedge contained in the union of the discs attached to M^0.

COROLLARY 7.11. *If $\mu_0 \notin \mathscr{A}$, with $A(\mu_0, \cdot) \in \mathscr{D}_1^0$, then there exists a positive such that (7.9) holds with some open cone Γ containing $\Psi(\mu_0)$ if $\Psi(\mu_0) \neq 0$, and (7.10) holds if $\Psi(\mu_0) = 0$.*

Proof of Theorem 4. We consider here only the case $\mathscr{G}(z^0) \neq 0$, since the case $\mathscr{G}(z^0) = 0$ is simpler and is left to the reader. We first prove the inclusion

$$\mathscr{W}^0(\Gamma, a) \subset \bigcup_{A \in \mathscr{D}_1^0} A(\Delta). \tag{7.12}$$

By the equivalence of (a) and (c) following (5.1), the rank of $\mathscr{G}'(z^0)$ is ℓ. Hence there exist ℓ discs $z^j \in \mathscr{E}$ such that if

$$z(r, \zeta) = z^0(\zeta) + \sum_{j=1}^{\ell} r_j z^j(\zeta) \tag{7.13}$$

and $A(r, \zeta)$ is the corresponding analytic disc attached to M^0 through 0, then the map $r \mapsto \mathscr{G}(z(r, \zeta))$ is a diffeomorphism from an open neighborhood of 0 in $\mathbf{R}^\ell$ into a neighborhood in $\mathbf{R}^\ell$ of $\mathscr{G}(z^0)$. The reader can check, as in [12], that the image of the map $(\gamma, \sigma, r, \xi) \mapsto A_{\gamma,\sigma}(r, \xi)$, when (γ, σ, r, ξ) varies in a neighborhood of $(0, 0, 0, 1)$ in $\mathbf{C}^n \times \mathbf{R}^\ell \times \mathbf{R}^\ell \times [-1, 1]$, contains a wedge of the form $\mathscr{W}^0(\Gamma, a)$. This proves the inclusion (7.12).

Using (7.5), (7.6), and (7.7), and applying the anisotropic dilation δ_η, $0 < \eta \leqslant 1$, to (7.12), we obtain (7.9). ∎

8. Anisotropic Wedges Attached to Generic Manifolds of Finite Type

We assume here that M is a germ at the origin of a smooth generic manifold in $\mathbf{C}^{n+\ell}$ given by (1.1), with $\phi = (\phi_1, ..., \phi_r)$, and

$$\phi_k(z, \bar{z}, s) = p_{m_k}(z, \bar{z}, s) + \chi_k(z, \bar{z}, s), \qquad k = 1, ..., r, \tag{8.1}$$

where the p_{m_k} are the polynomials given in (2.3), and $\chi_k(z, \bar{z}, s) = \mathcal{O}(m_k + 1)$ is a smooth real vector-valued function whose Taylor series at 0 consists of terms homogeneous of weights $\geqslant m_k + 1$ in the sense of Section 2. We denote by M^0 the associated homogeneous generic manifold given by (2.3). We say that M is of *finite type* if M^0 is of finite type, as defined in Section 2. Note that this definition of finite type coincides with the one given by [7] (see also [4]). Here we use the results obtained for M^0 to give an algorithm for finding an anisotropic wedge of extendibility for M itself.

As in Section 7, if a positive is sufficiently small and $N(\mathscr{Z})$ is given by (7.1), we denote by $\mathscr{W}(\Gamma, a)$ the wedge of edge M and direction Γ given by

$$\mathscr{W}(\Gamma, a) = \{\mathscr{Z} \in \mathbf{C}^{n+\ell} : N(\mathscr{Z}) < a, \operatorname{Im} w - \phi(z, \bar{z}, s) \in \Gamma\}. \tag{8.2}$$

Similarly, the anisotropic wedge $\mathscr{U}(\Gamma, a)$ is given by

$$\mathscr{U}(\Gamma, a) = \left\{\mathscr{Z} \in \mathbf{C}^{n+\ell} : N(\mathscr{Z}) < a, \operatorname{Im} w - \phi(z, \bar{z}, s) \in \bigcup_{a^{-1}N(\mathscr{Z}) < \eta \leqslant 1} \Gamma_\eta \right\}. \tag{8.3}$$

If $a > 0$ is sufficiently small, we denote by $\mathscr{D}_a$ the set of all analytic discs A attached to $M \cap \{N(\mathscr{Z}) < a\}$.

As in Section 5 we let $\mathscr{E}$ be the real Banach space of all analytic discs $z(\zeta)$ valued in $\mathbf{C}^n$ of class $C^{1,\alpha}(S^1)$ with $z(1)=0$. We denote by $\mathscr{G}^0$ the map given by (5.1) for M^0. Then we have the following.

THEOREM 5. *Let M be a generic manifold of finite type given by* (1.1), *M^0 be the associated homogeneous manifold given by* (2.3), *and $z_0 \in \mathscr{E}$ of defect* 0 *for M^0, such that the associated disc attached to M^0 through* 0, *$A^0=(z^0, w^0)$, is in $\mathscr{D}_1^0$. Then there exist positive numbers a, t_0 such that if $0<t\leqslant t_0$ then the following holds*:

(i) *If $\mathscr{G}^0(z^0)\neq 0$ and Γ is a sufficiently small open cone in $\mathbf{R}^{\ell}$ containing $\mathscr{G}^0(z^0)$, then*

$$\mathscr{U}(\Gamma_t, ta)\subset \bigcup_{A\in\mathscr{D}_t} A(\Delta). \tag{8.4}$$

(ii) *If $\mathscr{G}^0(z^0)=0$, then*

$$\{\mathscr{Z}\in\mathbf{C}^{n+\ell}: N(\mathscr{Z})<ta\}\subset \bigcup_{A\in\mathscr{D}_t} A(\Delta). \tag{8.5}$$

As before we let $\mathscr{A}$ be the proper algebraic subset of $\mathbf{C}^{n(K+1)}$, corresponding to M^0 defined by (4.3), and for $\mu\in\mathbf{C}^{n(K+1)}$, we let $A(\mu,\zeta)$ be the corresponding analytic disc attached to M^0 and whose z-component is given by (3.11). The following is an immediate corollary of Theorem 5. We continue to denote by $\Psi(\mu)$ the map defined by (2.6) corresponding to M^0.

COROLLARY 8.6. *Let M be a generic manifold of finite type given by* (1.1) *and M^0 the associated homogeneous manifold given by* (2.3). *If $\mu_0\notin\mathscr{A}$, with $A(\mu_0,\cdot)\in\mathscr{D}_1^0$, then there exist a, t_0 positive such that, for $0<t\leqslant t_0$,* (8.4) *holds with some open cone Γ containing $\Psi(\mu_0)$ if $\Psi(\mu_0)\neq 0$, and* (8.5) *holds if $\Psi(\mu_0)=0$.*

Proof of Theorem 5. Since the case $\mathscr{G}^0(z^0)=0$ is simpler, we consider here only the case $\mathscr{G}^0(z^0)\neq 0$. We define a function ψ_k, for $k=1, ..., r$, valued in $\mathbf{R}^{\ell_k}$ by

$$\psi_k(z,\bar z, s, t)=t^{-(m_k+1)}\chi_k(tz, t\bar z, \delta_t s),$$

where $\chi_k(z,\bar z, s)$ is given in (8.1). We note that since $\chi_k=\mathcal{O}(m_k+1)$, the functions ψ_k are smooth in a neighborhood of 0 in $\mathbf{R}^{2n+\ell+1}$. For sufficiently small t we let M^t be the germ of the generic manifold in $\mathbf{C}^{n+\ell}$ given by

$$\operatorname{Im} w_k = p_{m_k}(z,\bar z, s)+t\psi_k(z,\bar z, s, t), \qquad k=1, ..., r. \tag{8.7}$$

With the above notation, when $t=0$, M^0 is the homogeneous generic manifold associated to M, and $M^1=M$. Note also that for $t>0$, the anisotropic dilation δ_t is a bijection from $M^t \cap \{N(\mathscr{Z})<1\}$ onto $M \cap \{N(\mathscr{Z})<t\}$. For a and t positive and sufficiently small, we denote by $\mathscr{D}_a^t$ the set of all analytic discs A attached to $M^t \cap \{N(\mathscr{Z})<a\}$ so that $\mathscr{D}_a^1=\mathscr{D}_a$. Hence we have for t sufficiently small,

$$\delta_t\Big(\bigcup_{A\in\mathscr{D}_1^t} A(\Delta)\Big)=\bigcup_{A\in\mathscr{D}_t} A(\Delta).$$

It is convenient to add a complex variable $v\in\mathbf{C}$ and to introduce the following generic manifold $\tilde{M}$ in $\mathbf{C}^{n+\ell+1}$ given by (8.7) and $\operatorname{Im} v=0$, $\operatorname{Re} v=t$.

Let $A^0=(z^0, w^0)$ be the disc attached to M^0 given in the statement of the theorem. Since z^0 is of defect 0 for M^0, by the equivalence of (a) and (c) following (5.1), the same argument as in the proof of Theorem 4 gives a family of discs $z(r,\zeta)$, as defined in (7.13), such that $r\mapsto \mathscr{G}^0(z(r,\zeta))$ is a diffeomorphism from an open neighborhood of 0 in $\mathbf{R}^\ell$ into a neighborhood in $\mathbf{R}^\ell$ of $\mathscr{G}^0(z^0)$. Let $A^0(r,\zeta)$ be the associated analytic disc valued in $\mathbf{C}^{n+\ell}$ attached to M^0 through 0. Then the analytic disc $A(r,\zeta)=(A^0(r,\zeta),0)$ valued in $\mathbf{C}^{n+\ell+1}$ is attached to $\tilde{M}$ through 0. Using a standard procedure (see [12]), we construct, starting from $A(r,\zeta)$, a family of discs $A_{\gamma,\sigma,\varepsilon}(r,\zeta)$ valued in $\mathbf{C}^{n+\ell+1}$ attached to $\tilde{M}$ through the point $z=\gamma$, $s=\sigma$, $t=\varepsilon$, in a neighborhood of 0 in $\tilde{M}$, with $A_{0,0,0}=A$. This is done by taking $A_{\gamma,\sigma,\varepsilon}(r,\zeta)=(z(r,\zeta)+\gamma, w_{\gamma,\sigma,\varepsilon}(r,\zeta),\varepsilon)$, with $\operatorname{Re} w_{\gamma,\sigma,\varepsilon}(r,\zeta)=s_{\gamma,\sigma,\varepsilon}(r,\zeta)$ satisfying the Bishop equation,

$$\begin{aligned} s_{\gamma,\sigma,\varepsilon}(r,\zeta)= -T_1\big(p(z(r,\cdot)+\gamma, s_{\gamma,\sigma,\varepsilon}(r,\cdot))\\ +\varepsilon\psi(z(r,\cdot)+\gamma, s_{\gamma,\sigma,\varepsilon}(r,\cdot),\varepsilon)\big)(\zeta)+\sigma, \end{aligned}$$

which admits a unique solution $s_{\gamma,\sigma,\varepsilon}$ for $r,\gamma,\sigma,\varepsilon$ small, with $s_{0,0,0}(r,\zeta)=s(r,\zeta)$.

One can check that there is an open cone $\Gamma\subset\mathbf{R}^\ell$ containing $\mathscr{G}^0(z^0)$ such that for every small $\gamma_0\in\mathbf{C}^n$, $\sigma_0\in\mathbf{R}^\ell$, $\theta\in\Gamma$, and $\varepsilon\in\mathbf{R}$, there exist γ,σ,r, and ξ such that

$$(\gamma_0, \sigma_0+i[p(\gamma_0,\bar{\gamma}_0,\sigma_0)+\varepsilon\psi(\gamma_0,\bar{\gamma}_0,\sigma_0,\varepsilon)+\theta],\varepsilon)=A_{\gamma,\sigma,\varepsilon}(r,\xi). \quad (8.8)$$

Note that Γ is independent of ε. If we denote by $\mathscr{W}^t(\Gamma,a)$ the wedge defined as in (8.2) with edge M^t instead of M, then taking the projection of the set defined by (8.8) in $\mathbf{C}^{n+\ell}$ we obtain

$$\mathscr{W}^t(\Gamma,a)\subset\bigcup_{A\in\mathscr{D}_1^t} A(\Delta), \quad (8.9)$$

for $0 < t \leqslant t_0$, for some a and t_0 sufficiently small. Applying the dilation δ_t to (8.9) yields

$$\mathscr{W}(\Gamma_t, ta) \subset \bigcup_{A \in \mathscr{D}_t} A(\Delta). \tag{8.10}$$

Since for $0 < \eta \leqslant 1$ we have $\mathscr{D}_{\eta t} \subset \mathscr{D}_t$, we conclude that

$$\mathscr{W}(\Gamma_{\eta t}, \eta ta) \subset \bigcup_{A \in \mathscr{D}_t} A(\Delta), \qquad 0 < \eta \leqslant 1. \tag{8.11}$$

The inclusion (8.4) follows from (8.11) since we have $\mathscr{U}(\Gamma_t, ta) = \bigcup_{0 < \eta \leqslant 1} \mathscr{W}(\Gamma_{\eta t}, \eta ta)$. This completes the proof of Theorem 5. ∎

REFERENCES

1. M. S. BAOUENDI, C. H. CHANG, AND F. TREVES, Microlocal hypo-analyticity and extension of *CR* functions, *J. Differential Geom.* **18** (1983), 331–391.
2. M. S. BAOUENDI AND L. P. ROTHSCHILD, Cauchy Riemann functions on manifolds of higher codimension in complex space, *Invent. Math.* **101** (1990), 45–56.
3. M. S. BAOUENDI AND L. P. ROTHSCHILD, Minimality and the extension of functions from generic manifolds, *Proc. Sympos. Pure Math.* **52** (1991), 1–13.
4. M. S. BAOUENDI AND L. P. ROTHSCHILD, Normal forms for generic manifolds and holomorphic extension of *CR* functions, *J. Differential Geometry* **25** (1987), 431–467.
5. M. S. BAOUENDI, L. P. ROTHSCHILD, AND F. TREVES, *CR* structures with group action and extendability of *CR* functions, *Invent. Math.* **82** (1985), 359–396.
6. M. S. BAOUENDI AND F. TREVES, About the holomorphic extension of *CR* functions on real hypersurfaces in complex space, *Duke Math. J.* **51** (1984), 77–107.
7. T. BLOOM AND I. GRAHAM, On 'type' conditions for generic submanifolds of C^n, *Invent. Math.* **40** (1977), 217–243.
8. A. BOGGESS, *CR* extendibility near a point where the first Levi form vanishes, *Duke Math. J.* **48** (1981), 665–684.
9. A. BOGGESS, "*CR* manifolds and the tangential Cauchy Riemann complex," CRC Press, Boca Raton, FL, 1991.
10. A. BOGGESS, R. DWILEWICZ, AND A. NAGEL, The hull of holomorphy in a real hypersurface of finite type, *Trans. Amer. Math. Soc.* **323** (1991), 209–232.
11. A. BOGGESS AND J. C. POLKING, Holomorphic extension of CR functions, *Duke Math. J.* **49** (1982), 757–784.
12. A. E. TUMANOV, Extension of *CR* functions into a wedge from a manifold of finite type, *Mat. Sb.* **136** (1988), 128–139 [in Russian]; *Math. USSR-Sb.* **64** (1989), 129–140, [Engl. Transl].
13. A. E. TUMANOV, Extension of *CR* functions into a wedge, *Mat. Sb.* **181** (1990), 951–964, [in Russian].

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