A Distribution –
Theoretic Proof of Kirillov’s Character Formula
for Nilpotent Lie Groups

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Let $G$ be a connected, simply connected nilpotent Lie group, and $\pi$ an irreducible unitary representation of $G$. Dixmier [2] showed that the operator $\pi(f) = \int_G \pi(x) f(x) \, dx$ is of trace class for every smooth function $f$ of compact support on $G$; there is a well-defined distribution on $G$ given by $f \mapsto \text{Tr} \, \pi(f)$. Kirillov [4] has given an explicit formula (1) for $\text{Tr} \, \pi(f)$. In this note we give a new proof of Kirillov’s formula, using the Bochner-Schwartz theorem for the Fourier transform of a positive definite distribution. Our proof is inspired by Schwartz’s treatment [7] of representations of the Lorentz group. (See also [3].)

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ the linear dual of $\mathfrak{g}$. $G$ acts on $\mathfrak{g}$ by the adjoint representation Ad and on $\mathfrak{g}^*$ by the co-adjoint representation Ad* given by

$$\langle X, \text{Ad}^* x \cdot Y \rangle = \langle \text{Ad} x^{-1} \cdot X, Y \rangle \quad Y \in \mathfrak{g}^*, \; X \in \mathfrak{g}.$$

Since $\exp : \mathfrak{g} \to G$ is a diffeomorphism, the map $f \mapsto f \circ \exp$ is an isomorphism between $\mathcal{D}(G)$ and $\mathcal{D}(\mathfrak{g})$, the spaces of smooth functions of compact support on $G$ and $\mathfrak{g}$ respectively. Hence the spaces of distributions $\mathcal{D}'(\mathfrak{g})$ and $\mathcal{D}'(G)$ are isomorphic by the map $T \mapsto T \circ \text{exp} = t$, where $t(f) = T(f \circ \exp)$ for $T \in \mathcal{D}'(\mathfrak{g})$, $f \in \mathcal{D}(\mathfrak{g})$.

Now let $dX$ be a Lebesgue measure on $\mathfrak{g}$ and $dx$ the corresponding Haar measure on $G$. $T \in \mathcal{D}'(\mathfrak{g})$ (resp. $T \in \mathcal{D}'(G)$) is called central if it is invariant under the adjoint representation of $G$ (resp. inner automorphisms of $G$). $T$ is called positive definite if for any $F \in \mathcal{D}(\mathfrak{g})$, $(T(F \ast \tilde{F})) \geq 0$, where $\tilde{F}$ is the function $X \mapsto \tilde{F}(-X)$ and convolution is defined by

$$F \ast \tilde{F} = \int_{\mathfrak{g}} F(X) \tilde{F}(Y-X) \, dX.$$

$t$ is called positive definite if for any $f \in \mathcal{D}(G)$, $(f \ast \check{f}) \geq 0$, where $\check{f}$ is the function $x \mapsto \check{f}(x^{-1})$ and the convolution is defined by

$$f \ast \check{f}(x) = \int_{\mathfrak{g}} f(x) \check{f}(y^{-1}) \, dy.$$

$T \in \mathcal{D}'(\mathfrak{g})$ is central if and only if the corresponding $t \in \mathcal{D}'(G)$ is central. For positive definite distributions one has

Schiffman’s Theorem [6]. Let $T$ be a central distribution of $\mathfrak{g}$. Then $T$ is positive definite as a distribution of $\mathfrak{g}$ if and only if $T \circ \exp = t$ is a positive definite distribution on $G$. 
If $\pi$ is an irreducible unitary representation, the distribution $f \mapsto \text{Tr} \pi(f)$ is called the \textit{character} of $\pi$. Kirillov's character formula is given as follows.

\textbf{Theorem \cite[Theorem 7.4]{4}.} Let $G$ be a connected, simply connected nilpotent Lie group and $\pi$ an irreducible unitary representation of $G$ with character $\text{Tr} \pi$. Then there is an $\text{Ad}^* G$-orbit $\mathcal{O}$ in $\mathfrak{g}^*$ and an invariant measure $d\mu$ on $\mathcal{O}$ such that for any $f \in \mathcal{D}(G)$,

$$\text{Tr} \pi(f) = \int_{\mathcal{O}} f^* \, d\mu \quad (1)$$

where

$$f^*(Y) = \int_{\mathfrak{g}} f(\exp X) e^{i\langle X, Y \rangle} \, dX, \quad Y \in \mathfrak{g}^*.$$ 

If $t$ is the trace of a unitary representation of $G$, it follows that $t$ is a positive definite, central distribution on $G$. Furthermore, if $t$ is a character, then $t$ is extremal in the following sense: if $t_1$ is any other positive definite, central distribution of $G$ with $t - t_1$ also positive definite, then $t_1 = \lambda t$ for some $\lambda > 0$. (See \cite[§ 6.7]{1}.)

We shall need Schwartz's generalization of Bochner's theorem \cite[vii, 9; 11, Theorem XVIII]{8}:

If $T$ is a distribution on $\mathbb{R}^n$, then $T$ is positive definite if and only if $T$ is tempered and $\hat{T}$ is a positive measure, where $\hat{T}$ is the Fourier transform defined by $\hat{T}(F) = T(\hat{F})$ with $\hat{F}(Y) = \int_{\mathbb{R}^n} F(X) e^{-ix \cdot Y} \, dX$, for $F \in \mathcal{D}(\mathbb{R}^n)$, $Y \in \mathbb{R}^n$.

A measure $\mu$ on $\mathfrak{g}^*$ will be called \textit{invariant} if $\mu(E) = \mu(\text{Ad}^* x \cdot E)$ for all $x \in G$ and all measurable $E \subset \mathfrak{g}^*$. The main step in proving the theorem is contained in the following.

\textbf{Lemma.} Let $\pi$ be an irreducible unitary representation of $G$ with character $t$. Then the distribution $T$ on $\mathfrak{g}_0$ corresponding to $t$, is tempered and $\hat{T}$ is a positive tempered invariant measure $\mu$ on $\mathfrak{g}^*$, whose support $\text{supp} \mu$, is contained in the closure $\overline{\mathcal{O}}$ for some orbit $\mathcal{O}$ in $\mathfrak{g}^*$.

\textbf{Proof.} $T$ is a central positive definite distribution on $\mathfrak{g}^*$ by Schifman's Theorem since $t$ is positive definite, central on $G$. By the Bochner-Schwartz theorem $\hat{T}$ is a positive tempered measure $\mu$ on $\mathfrak{g}^*$. Since $T$ is central, $\mu$ is $\text{Ad}^* G$-invariant.

Since $t$ is extremal among positive definite invariant distributions, it follows that $\mu$ is extremal among positive invariant measures; i.e. if $\mu_1$ is any positive invariant measure on $\mathfrak{g}^*$ with $\mu_1 < \mu$, then $\mu_1 = k \mu$ for some constant $k$. Hence it follows (see Schwartz \cite[pp. 72-73]{7}) that $\text{supp} \mu \subset \overline{\mathcal{O}}(x)$ for some $x \in \mathfrak{g}^*$.

We may now prove the Theorem. Let $T$ be the distribution on $\mathfrak{g}$ corresponding to $\text{Tr} \pi$. Then $T$ is also positive definite and central, and (1) is equivalent to the statement that $\hat{T}$ is an invariant measure $\mu$ such that $\text{supp} \mu \subset \mathcal{O}$ for some orbit $\mathcal{O}$ in $\mathfrak{g}^*$. From the Lemma it follows that $\hat{T}$ is an invariant measure $\mu$ with $\text{supp} \mu \subset \overline{\mathcal{O}}$ for some orbit $\mathcal{O}$. Since all the orbits of a unipotent representation on a finite dimensional vector space are closed \cite[Part II, Chapter 1]{5}, $\mathcal{O} = \overline{\mathcal{O}}$ and the proof is complete.

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