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A Distribution – Theoretic Proof of Kirillov's Character Formula for Nilpotent Lie Groups

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Let G be a connected, simply connected nilpotent Lie group, and π an irreducible unitary representation of G. Dixmier [2] showed that the operator $\pi(f)=\int\limits_G \pi(x)f(x)\,dx$ is of trace class for every smooth function f of compact support on G; there is a well-defined distribution on G given by $f\to \operatorname{Tr}\pi(f)$. Kirillov [4] has given an explicit formula (1) for $\operatorname{Tr}\pi(f)$. In this note we give a new proof of Kirillov's formula, using the Bochner-Schwartz theorem for the Fourier transform of a positive definite distribution. Our proof is inspired by Schwartz's treatment [7] of representations of the Lorentz group. (See also [3].)

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the linear dual of $\mathfrak{g}$. G acts on $\mathfrak{g}$ by the adjoint representation Ad and on $\mathfrak{g}^*$ by the co-adjoint representation Ad* given by

$$\langle X, \mathrm{Ad}^*x\cdot Y\rangle=\langle \mathrm{Ad}\,x^{-1}\cdot X, Y\rangle \qquad Y\in\mathfrak{g}^*,\ X\in\mathfrak{g}.$$

Since $\exp\mathfrak{g}\to G$ is a diffeomorphism, the map $f\to f\circ\exp$ is an isomorphism between $\mathscr{D}(G)$ and $\mathscr{D}(\mathfrak{g})$, the spaces of smooth functions of compact support on G and $\mathfrak{g}$ respectively. Hence the spaces of distributions $\mathscr{D}'(\mathfrak{g})$ and $\mathscr{D}'(G)$ are isomorphic by the map $T\to T\circ\exp=t$, where $t(f)=T(f\circ\exp)$ for $T\in\mathscr{D}'(\mathfrak{g})$, $f\in\mathscr{D}(G)$.

Now let dX be a Lebesgue measure on $\mathfrak{g}$ and dx the corresponding Haar measure on G. $T\in\mathscr{D}'(\mathfrak{g})$ (resp. $T\in\mathscr{D}'(G)$) is called *central* if it is invariant under the adjoint representation of G (resp. inner automorphisms of G). T is called *positive definite* if for any $F\in\mathscr{D}(\mathfrak{g})$, $T(F*\tilde F)\geqq 0$, where $\tilde F$ is the function $X\to\overline{F(-X)}$ and convolution is defined by

$$F*\tilde F=\int\limits_{\mathfrak{g}} F(X)\,\tilde F(Y-X)\,dX$$

t is called *positive definite* if for any $f\in\mathscr{D}(G)$, $t(f*\tilde f)\geqq 0$, where $\tilde f$ is the function $x\to\overline{f(x^{-1})}$ and the convolution is defined by

$$f*\tilde f(x)=\int\limits_G f(x)\,\tilde f(y\,x^{-1})\,dx.$$

$T\in\mathscr{D}'(\mathfrak{g})$ is central if and only if the corresponding $t\in\mathscr{D}'(G)$ is central. For positive definite distributions one has

Schiffman's Theorem [6]. *Let T be a central distribution of $\mathfrak{g}$. Then T is positive definite as a distribution of $\mathfrak{g}$ if and only if $T\circ\exp=t$ is a positive definite distribution on G.*

If π is an irreducible unitary representation, the distribution $f \mapsto \mathrm{Tr}\,\pi(f)$ is called the *character* of π. Kirillov's character formula is given as follows.

Theorem [4, Theorem 7.4]. *Let G be a connected, simply connected nilpotent Lie group and π an irreducible unitary representation of G with character* $\mathrm{Tr}\,\pi$. *Then there is an* Ad* *G-orbit $\mathcal{O}$ in $\mathfrak{g}^*$ and an invariant measure $d\mu$ on $\mathcal{O}$ such that for any $f \in \mathcal{D}(G)$,*

$$\mathrm{Tr}\,\pi(f) = \int_{\mathcal{O}} f^* \, d\mu \tag{1}$$

where

$$f^*(Y) = \int_{\mathfrak{g}} f(\exp X)\, e^{i\langle X, Y\rangle}\, dX, \qquad Y \in \mathfrak{g}^*.$$

If t is the trace of a unitary representation of G, it follows that t is a positive definite, central distribution on G. Furthermore, if t is a character, then t is *extremal* in the following sense: if t_1 is any other positive definite, central distribution of G with $t - t_1$ also positive definite, then $t_1 = \lambda t$ for some $\lambda > 0$. (See [1, § 6.7].)

We shall need Schwartz's generalization of Bochner's theorem [8, vii, 9; 11, Theorem XVIII]:

If T is a distribution on $\mathbb{R}^n$, then T is positive definite if and only if T is tempered and $\hat{T}$ is a positive measure, where $\hat{T}$ is the Fourier transform defined by $\hat{T}(F) = T(\hat{F})$ with $\hat{F}(Y) = \int_{\mathbb{R}^n} F(X)\, e^{-iX \cdot Y}\, dX$, for $F \in \mathcal{D}(\mathbb{R}^n)$, $Y \in \mathbb{R}^n$.

A measure μ on $\mathfrak{g}^*$ will be called *invariant* if $\mu(E) = \mu(\mathrm{Ad}^*\, x \cdot E)$ for all $x \in G$ and all measurable $E \subset \mathfrak{g}^*$. The main step in proving the theorem is contained in the following.

Lemma. *Let π be an irreducible unitary representation of G with character t. Then the distribution T on $\mathfrak{g}$, corresponding to t, is tempered and $\hat{T}$ is a positive tempered invariant measure μ on $\mathfrak{g}^*$, whose support* $\operatorname{supp}\mu$, *is contained in the closure $\overline{\mathcal{O}}$ for some orbit $\mathcal{O}$ in $\mathfrak{g}^*$.*

Proof. T is a central positive definite distribution on $\mathfrak{g}^*$ by Schiffman's Theorem since t is positive definite, central on G. By the Bochner-Schwartz theorem $\hat{T}$ is a positive tempered measure μ on $\mathfrak{g}^*$. Since T is central, μ is Ad* G-invariant.

Since t is extremal among positive definite invariant distributions, it follows that μ is extremal among positive invariant measures; i.e. if μ_1 is any positive invariant measure on $\mathfrak{g}^*$ with $\mu_1 < \mu$, then $\mu_1 = k\mu$ for some constant k. Hence it follows (see Schwartz [7], pp. 72–73) that $\operatorname{supp}\mu \subset \overline{\mathcal{O}(x)}$ for some $x \in \mathfrak{g}^*$. ||

We may now prove the Theorem. Let T be the distribution on $\mathfrak{g}$ corresponding to $\mathrm{Tr}\,\pi$. Then T is also positive definite and central, and (1) is equivalent to the statement that $\hat{T}$ is an invariant measure μ such that $\operatorname{supp}\mu \subset \mathcal{O}$ for some orbit $\mathcal{O}$ in $\mathfrak{g}^*$. From the Lemma it follows that $\hat{T}$ is an invariant measure μ with $\operatorname{supp}\mu \subset \overline{\mathcal{O}}$ for some orbit $\mathcal{O}$. Since all the orbits of a unipotent representation on a finite dimensional vector space are closed [5, Part II, Chapter 1], $\mathcal{O} = \overline{\mathcal{O}}$ and the proof is complete.

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