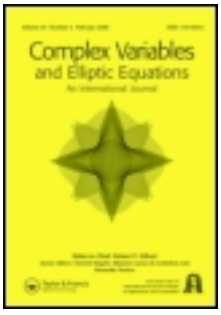

Complex Variables and Elliptic Equations: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov20

New invariants of CR manifolds and a criterion for finite mappings to be diffeomorphic

Peter Ebenfelt [a] & Linda P. Rothschild [a]
[a] Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112, USA

Available online: 18 Nov 2010

To cite this article: Peter Ebenfelt & Linda P. Rothschild (2009): New invariants of CR manifolds and a criterion for finite mappings to be diffeomorphic, Complex Variables and Elliptic Equations: An International Journal, 54:3-4, 409-423

To link to this article: http://dx.doi.org/10.1080/17476930902763827

Complex Variables and Elliptic Equations
Vol. 54, Nos. 3–4, March–April 2009, 409–423

New invariants of CR manifolds and a criterion for finite mappings to be diffeomorphic

Peter Ebenfelt* and Linda P. Rothschild

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112, USA

Communicated by S.G. Krantz

(*Received 10 January 2009; final version received 12 January 2009*)

We introduce a new sequence of invariants for a generic (CR) submanifold of complex space. We provide a sufficient condition, in terms of these invariants, which guarantees that all finite CR mappings are locally diffeomorphic.

Keywords: invariants; CR manifolds; finite mappings; diffeomorphism; biholomorphisms

AMS Subject Classifications: 32V40; 32H35

1. Introduction

The main purpose of this note is to introduce a new sequence of invariants for a smooth generic (CR) submanifold of complex space (Theorem 1.1) and use these to provide a new sufficient condition for a finite holomorphic mapping to be locally biholomorphic (Theorem 1.2). The latter result generalizes a previous result of the authors from [1], which in turn extended classical results by Poincaré, Alexander, Pinchuk, and others (see [1] for more detail). We shall also show that for a rigid hypersurface the condition we give for a mapping to be locally biholomorphic is also necessary. Our results are proved in the context of formal manifolds and formal mappings and, hence, also yield analogous results for smooth CR mappings, provided that finite is interpreted in a suitable way [1]. For simplicity, we state our results in the introduction for holomorphic mappings only.

To formulate our results more precisely, we need to introduce some notation and terminology. Recall that a real smooth submanifold M of codimension d in $\mathbb{C}^N$ is called *generic* at a point $p \in M$ if there is a smooth function $\rho: (\mathbb{C}^N, p) \to \mathbb{R}^d$ such that M is locally given near p by $\rho = 0$ and

$$\partial\rho_1 \wedge \cdots \wedge \partial\rho_d \neq 0$$

at p. A generic submanifold M is a CR manifold of CR dimension $n := N - d$. Let $\mathbb{C}[\![\chi]\!]$ denote the formal power series ring in n variables $\chi = (\chi_1, \ldots, \chi_n)$, and observe that it is

*Corresponding author. Email: pebenfel@math.ucsd.edu

ISSN 1747–6933 print/ISSN 1747–6941 online

DOI: 10.1080/17476930902763827
http://www.informaworld.com

also an algebra over $\mathbb{C}$. Given M and $p \in M$, we shall associate to the pair (M, p) an increasing sequence of subalgebras (without identity), henceforth referred to as the *essential algebras* of M:

$$A_M^{(1)}(p) \subset A_M^{(2)}(p) \subset \cdots \subset A_M^{(k)}(p) \subset \cdots \subset \mathbb{C}[\![\chi]\!]. \tag{1.1}$$

The precise definition of the essential algebras will be given in Section 3. One of our main results, Theorem 1.1 below, guarantees that these algebras are biholomorphic invariants of M at p; more generally, this result describes how the essential algebras transform under finite holomorphic mappings. Recall that (a germ of) a holomorphic mapping $H: (\mathbb{C}^N, p) \to (\mathbb{C}^N, \tilde{p})$ is said to be *finite* if the (germ of a) complex subvariety $H^{-1}(\tilde{p})$ equals $\{p\}$. (In Section 3, we shall give an equivalent algebraic definition of this concept that carries over immediately to formal mappings.)

THEOREM 1.1 *Let $M, \tilde{M}$ be smooth generic submanifolds of the same dimension in $\mathbb{C}^N$ with $p \in M$ and $\tilde{p} \in \tilde{M}$, and $H: (\mathbb{C}^N, p) \to (\mathbb{C}^N, \tilde{p})$ a finite holomorphic mapping sending M into $\tilde{M}$. Then, there is an algebra homomorphism $\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$ such that*

$$A_M^{(k)}(p) \subset \psi\left(A_{\tilde{M}}^{(k)}(\tilde{p})\right) \tag{1.2}$$

for every $k \geq 1$. Moreover, if H is a local biholomorphism at p, then ψ is an isomorphism and

$$A_M^{(k)}(p) = \psi\left(A_{\tilde{M}}^{(k)}(\tilde{p})\right) \tag{1.3}$$

for every $k \geq 1$.

Theorem 1.1 will follow from the slightly more general result Theorem 3.1, in which the homomorphism ψ is described explicitly in terms of the mapping H. We should mention here that the ideal generated by the union of all the essential algebras (or, equivalently, by $A_M^{(k)}(p)$ for k sufficiently large) is the essential ideal $I_M(p)$ (see Section 3 for the definition) of M at p. The essential ideal is a biholomorphic invariant of M that was introduced independently by D'Angelo in [2] and Baouendi and Rothschild in [3]. It also played a role implicitly in the work of Diederich-Fornæss (see e.g. [4]) and others. The essential algebras introduced here provide a strictly finer set of invariants than the essential ideal alone, as is illustrated by Example 3.2.

Next, we recall that a generic submanifold M is said to be of *finite type* at p (in the sense of Kohn and Bloom–Graham) if the (complex) Lie algebra $\mathfrak{g}_M$ generated by all smooth $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(p) = \mathbb{C}T_pM$. The generic submanifold is said to be *essentially finite* at p if the essential ideal $I_M(p)$ has finite codimension in the ring of formal power series $\mathbb{C}[\![\chi]\!]$, i.e. if the complex vector space $\mathbb{C}[\![\chi]\!]/I_M(p)$ is finite dimensional (see also Section 3). Our second result is a sufficient condition, expressed in terms of the essential algebras, for a finite holomorphic mapping to be a local biholomorphism.

THEOREM 1.2 *Let M be a smooth generic submanifold through $0 \in \mathbb{C}^N$ and assume that M is essentially finite and of finite type at 0. Let $A_M^{(k)} \subset \mathbb{C}[\![\chi]\!]$, $k \geq 1$, be the essential algebras of M. Assume that any algebra homomorphism $\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$ that satisfies $A_M^{(k)} \subset \psi(\mathbb{C}[\![\chi]\!])$, for all $k \geq 1$, is surjective. Then any finite holomorphic mapping $H: (\mathbb{C}^N, p) \to (\mathbb{C}^N, \tilde{p})$ sending M into another smooth generic submanifold $\tilde{M}$ (through $\tilde{p}$) of the same dimension is a local formal biholomorphism.*

In Theorem 1.2 of the paper [1], the authors showed that if M is a generic submanifold, finitely non-degenerate and of finite type at $p \in M$, then the conclusion of Theorem 1.2 holds, i.e. if H is a finite holomorphic mapping sending M into another generic submanfold $\tilde{M}$ of the same dimension, then H is a local biholomorphism at p. Since the notion of finite non-degeneracy does not play any role in this article, except in this comparison with Theorem 1.2 in [1], we shall refer the reader to e.g. [1] or the book [5] for the definition. We mention here only that, as a consequence of Proposition 11.8.27 in [5], a generic submanifold M is finitely non-degenerate at p if and only if the essential algebra $A_M^{(k)}(p)$ equals $\mathbb{C}[\![\chi]\!]_0$ for k sufficiently large; here, $\mathbb{C}[\![\chi]\!]_0$ denotes the subalgebra of formal power series without constant term (or the maximal ideal if we think of $\mathbb{C}[\![\chi]\!]$ as a ring). Finite non-degeneracy implies essential finiteness and any algebra homomorphism $\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$ that satisfies $A_M^{(k)} \subset \psi(\mathbb{C}[\![\chi]\!])$, for all $k \geq 1$, is surjective. Thus, Theorem 1.2 above implies Theorem 1.2 in [1], and Example 4.3 below shows that Theorem 1.2 in this article is a strictly stronger result. The reader is referred to [1] for more information about previous results along these lines.

In Section 5, we consider rigid essentially finite hypersurfaces and show (Theorem 5.1) that for such manifolds the condition on the essential algebras in Theorem 1.2 is also necessary for the conclusion to hold. However, we also give an example (Example 5.2) showing that this condition is not necessary in general.

Finally, in Section 6 we show that manifolds for which the conclusion of Theorem 1.2 holds play a special role in the theory of essentially finite manifolds of finite type. Indeed, any manifold of the latter type can be realized as a branched cover over one of the former type.

2. Formal manifolds, formal mappings and preliminaries

We shall assume, without loss of generality, that the points p and $\tilde{p}$ that appear in the introduction are both the origin in $\mathbb{C}^N$. We shall consider M and $\tilde{M}$ as formal generic submanifolds through 0 in $\mathbb{C}^N$ and H as a formal holomorphic mapping $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$.

We begin by recalling some definitions; the reader is referred to [6] for further definitions and properties related to formal generic submanifolds and their mappings. As said above, we let $\mathbb{C}[\![x]\!] = \mathbb{C}[\![x_1, \ldots, x_k]\!]$ be the ring of formal power series in $x = (x_1, \ldots, x_k)$ with complex coefficients. Suppose that $\rho = (\rho_1, \ldots, \rho_d) \in \mathbb{C}[\![Z, \zeta]\!]^d$, where $Z = (Z_1, \ldots, Z_N)$ and $\zeta = (\zeta_1, \ldots, \zeta_N)$, satisfies the reality condition $\rho(Z, \zeta) = \bar{\rho}(\zeta, Z)$, where $\bar{\rho}$ is the formal series obtained from ρ by replacing each coefficient in the series by its complex conjugate. If, in addition, the series ρ satisfies the condition $\rho(0) = 0$, and

$$\partial_Z \rho_1(0) \wedge \cdots \wedge \partial_Z \rho_d(0) \neq 0,$$

then we say that ρ defines a *formal* (*real*) *generic submanifold* M of $\mathbb{C}^N$ through 0 of codimension d (and dimension $2N - d$). We shall refer to $n := N - d$ as the *CR dimension of* M. This definition is motivated by the fact that if in addition the components of ρ are convergent power series, then the equations $\rho(Z, \bar{Z}) = 0$ define a real-analytic submanifold M of $\mathbb{C}^N$ through 0. Also, if M is a smooth generic submanifold in $\mathbb{C}^N$ through 0, then the Taylor series at 0 of a smooth defining function $\rho(Z, \bar{Z})$ of M near 0, with $\bar{Z}$ formally replaced by ζ, defines a formal generic submanifold through 0 (which by a slight abuse of notation will still be denoted by M).

For a formal generic submanifold $M \subset \mathbb{C}^N$, we shall use the notation $\mathbb{C}T_0M$, $T_0^{1,0}M$, and $T_0^{0,1}M$ for the vector spaces of all complex tangent vectors, all $(1,0)$ tangent vectors, and all $(0,1)$ tangent vectors, respectively, at 0. Recall that M is said to be of *finite type* at 0 (in the sense of Kohn and Bloom–Graham) if the Lie algebra $\mathfrak{g}_M$ generated by all formal $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(0) = \mathbb{C}T_0M$.

Let $H(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a formal holomorphic (or simply formal) mapping, i.e. $H \in \mathbb{C}[\![Z_1, \ldots, Z_N]\!]^N$ such that each component of $H(Z) = (H_1(Z), \ldots, H_N(Z))$ has no constant term. If M and $\tilde{M}$ are formal real submanifolds of $\mathbb{C}^N$ defined by formal series $\rho(Z,\zeta) = (\rho_1(Z,\zeta), \ldots, \rho_d(Z,\zeta))$ and $\tilde{\rho}(Z,\zeta) = (\tilde{\rho}_1(Z,\zeta), \ldots, \tilde{\rho}_d(Z,\zeta))$, respectively, then we say that the formal mapping H, as above, maps M into $\tilde{M}$, denoted $H\colon (M,0) \to (\tilde{M},0)$, if

$$\tilde{\rho}(H(Z), \bar{H}(\zeta)) = c(Z,\zeta)\rho(Z,\zeta),$$

for some $d \times d$ matrix $c(Z,\zeta)$ of formal power series. Recall that a formal mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is called finite if

$$\dim_{\mathbb{C}} \mathbb{C}[\![Z]\!]/I(H(Z)) < \infty,$$

where $I(H(Z))$ denotes the ideal generated by the components of the mapping $H(Z) = (H_1(Z), \ldots, H_N(Z))$. The dimension of the vector space $\mathbb{C}[\![Z]\!]/I(H(Z))$ is called the multiplicity of the mapping H and is denoted mult (H). It is well known that a holomorphic mapping H is finite if and only if its formal Taylor series is a finite formal mapping, and mult (H) is then the number of distinct preimages of a generic point. The formal mapping H is said to be *CR transversal* to $\tilde{M}$ at 0 if

$$T_0^{1,0}\tilde{M} + dH(T_0^{1,0}\mathbb{C}^N) = T_0^{1,0}\mathbb{C}^N. \tag{2.1}$$

It will be convenient to choose *normal coordinates*, $Z = (z,w)$ and $\zeta = (\chi,\tau)$ with $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_d)$, $\chi = (\chi_1, \ldots, \chi_n)$ and $\tau = (\tau_1, \ldots, \tau_d)$, in $\mathbb{C}^N \times \mathbb{C}^N$ for M at 0. By this we mean a formal change of coordinates $Z = Z(z,w)$ and $\zeta = \bar{Z}(\chi,\tau)$ with $Z(z,w)$ a formal invertible mapping $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ such that

$$\rho(Z(z,w), \bar{Z}(\chi,\tau)) = a(z,w,\chi,\tau)(w - Q(z,\chi,\tau)),$$

where $a(z,w,\chi,\tau)$ is an invertible $d \times d$ matrix of formal power series, and the vector valued $Q \in \mathbb{C}[\![z,\chi,\tau]\!]^d$ satisfies

$$Q_j(0,\chi,\tau) = Q_j(z,0,\tau) = \tau_j, \quad j = 1, \ldots, d. \tag{2.2}$$

(See Chapter IV.2 of [5] for the existence of such coordinates.) Here, and in what follows, we use matrix notation and the convention that the variables $z \in \mathbb{C}^n$, $w \in \mathbb{C}^d$ are column vectors; for instance, A^t denotes the transpose of a matrix A and, hence, $Q(z,\chi,\tau) = (Q^1(z,\chi,\tau), \ldots, Q^d(z,\chi,\tau))^t$ is a column vector. For convenience, we shall simply say that $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d$ are normal coordinates for M at 0 and M is defined by the equation

$$w = Q(z,\bar{z},\bar{w}). \tag{2.3}$$

As said above, let M and $\tilde{M}$ be formal generic submanifolds of codimension d through 0 in $\mathbb{C}^N$, and $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ a formal mapping sending M to $\tilde{M}$. If we write

$H = (F, G)$ in normal coordinates $(\tilde{z}, \tilde{w}) \in \mathbb{C}^n \times \mathbb{C}^d$ for $\tilde{M}$ at 0, then the fact that H sends $(M, 0)$ into $(\tilde{M}, 0)$ means that

$$G = \tilde{Q}(F, \bar{F}, \bar{G}), \quad \text{or equivalently,} \quad \bar{G} = \bar{\tilde{Q}}(\bar{F}, F, G), \tag{2.4}$$

where $F = F(z, w)$, $G = G(z, w)$, $\bar{F} = \bar{F}(\chi, \tau)$, $\bar{G} = \bar{G}(\chi, \tau)$, whenever $(z, w; \chi, \tau)$ satisfies

$$w = Q(z, \chi, \tau), \quad \text{or equivalently,} \quad \tau = \bar{Q}(\chi, z, w). \tag{2.5}$$

In normal coordinates, CR transversality of H to $\tilde{M}$ at 0 is equivalent to the $d \times d$ matrix $G_w(0) := (\partial G/\partial w)(0)$ being invertible and a CR transversal mapping $H = (F, G)$ is finite if and only if the mapping $\mathbb{C}^n \ni z \mapsto F(z, 0) \in \mathbb{C}^n$ is finite.

3. The essential algebras

In this section, we shall first recall the notions of the essential ideal, essential finiteness and essential type for a formal generic submanifold M through 0 in $\mathbb{C}^N$. We shall then introduce the sequence of *essential algebras* of M at 0 mentioned in the introduction. We shall also study how these algebras transform under formal mappings, and at the end of the section prove Theorem 1.1.

We choose normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ for M at 0 as described in Section 2, so that M is defined by (2.3), and expand the power series

$$Q(z, \chi, \tau) = (Q^1(z, \chi, \tau), \ldots, Q^d(z, \chi, \tau))^t$$

at $\tau = 0$ as a Taylor series in $z = (z_1, \ldots, z_n)$:

$$Q^j(z, \chi, 0) = \sum_{\alpha \in \mathbb{N}_+^n} q_\alpha^j(\chi) z^\alpha, \quad j = 1, \ldots, d. \tag{3.1}$$

We also write $q_\alpha(\chi) = (q_\alpha^1(\chi), \ldots, q_\alpha^d(\chi))^t$. Observe that all the power series $q_\alpha^j(\chi)$ have zero constant terms. Recall that if we consider $\mathbb{C}[\![\chi]\!]$ as a ring, then the *essential ideal* I_M of M at 0 is the ideal generated by all the $q_\alpha^j(\chi)$. This ideal is an invariant of M (cf. [2,3]), and M is said to be *essentially finite* at 0 if I_M has finite codimension in $\mathbb{C}[\![\chi]\!]$. Its codimension is then called the *essential type* of M at 0 and is denoted by $\mathrm{Ess}_0(M)$.

We may now define the sequence of *essential algebras* (without unit)

$$A_M^{(1)} \subset A_M^{(2)} \subset \cdots \subset A_M^{(k)} \subset \cdots \subset \mathbb{C}[\![\chi]\!] \tag{3.2}$$

of M at 0 (since our given point on M is taken to be 0, we shall suppress the dependence of the point in the notation) as follows. For every integer $k \geq 1$, we define the kth essential algebra $A_M^{(k)}$ of M at 0 to be

$$A_M^{(k)} := \mathbb{C}[\![q(\chi)]\!]_0, \tag{3.3}$$

where $q = (q_1, \ldots, q_m)$ is the m-tuple of formal power series

$$q(\chi) := \left(q_\alpha^j(\chi) \right)_{1 \leq j \leq \mathrm{d},\ |\alpha| \leq k} \tag{3.4}$$

for some ordering of the pairs of indices (j, α), $1 \leq j \leq d$ and $|\alpha| \leq k$ and $\mathbb{C}[\![x_1, \ldots, x_m]\!]_0$ denotes the formal power series in $x = (x_1, \ldots, x_m)$ without constant terms (which

coincides with the maximal ideal in $\mathbb{C}[\![x]\!]$ considered as a ring); here, m is the number of such pairs.

Recall that, for n_1, $n_2 \in \mathbb{N}$ and a formal mapping $R: (\mathbb{C}^{n_1}, 0) \to (\mathbb{C}^{n_2}, 0)$, one may associate an algebra homomorphism $\psi_R: \mathbb{C}[\![x_1, \ldots, x_{n_2}]\!] \to \mathbb{C}[\![y_1, \ldots, y_{n_1}]\!]$ by

$$\psi_R(h)(\chi) := (h \circ R)(\chi), \quad h \in \mathbb{C}[\![x]\!]. \tag{3.5}$$

Conversely, any algebra homomorphism $\psi: \mathbb{C}[\![x_1, \ldots, x_{n_2}]\!] \to \mathbb{C}[\![y_1, \ldots, y_{n_1}]\!]$ is of the form ψ_R with $R(y) = (R_1(y), \ldots, R_{n_2}(y))$ defined by

$$R_j = \psi(x_j). \tag{3.6}$$

We claim that the sequence $A_M^{(k)}$ is an invariant of M, in the sense that if there is a local invertible mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M to $\tilde{M}$, then there is an algebra isomorphism $\psi: \mathbb{C}[\![\tilde{\chi}]\!] \to \mathbb{C}[\![\chi]\!]$ such that $A_M^{(k)} = \psi(A_{\tilde{M}}^{(k)})$ for every $k \geq 1$. Indeed, we shall prove the following stronger result.

THEOREM 3.1 *Let $M, \tilde{M}$ be formal generic submanifolds of the same CR dimension through $0 \in \mathbb{C}^N$, and $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a formal holomorphic mapping sending M into $\tilde{M}$. Assume that H is CR transversal to $\tilde{M}$ at 0. Let (z, w) and $(\tilde{z}, \tilde{w})$ be normal coordinates for M at 0 and $\tilde{M}$ at 0, respectively, and let $H(z, w) = (F(z, w), G(z, w))$ in these coordinates. Let $\psi = \psi_R: \mathbb{C}[\![\tilde{\chi}]\!] \to \mathbb{C}[\![\chi]\!]$ be the algebra homomorphism associated to the mapping $R(\chi) := \bar{F}(\chi, 0)$ (see (3.5)). Then,*

$$A_M^{(k)} \subset \psi\left(A_{\tilde{M}}^{(k)}\right) \tag{3.7}$$

for every $k \geq 1$. Moreover, if H is invertible at 0, then ψ is an isomorphism and

$$A_M^{(k)} = \psi\left(A_{\tilde{M}}^{(k)}\right) \tag{3.8}$$

for every $k \geq 1$.

Proof If we substitute (2.5) in (2.4), then by using $G(z, 0) = 0$ (which follows easily from the normality of the coordinates) we obtain

$$G(z, Q(z, \chi, 0)) = \tilde{Q}(F(z, Q(z, \chi, 0)), \bar{F}(\chi, 0), 0). \tag{3.9}$$

Also, observe that we have the identity

$$q_\alpha(\chi) = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} Q}{\partial z^\alpha}(0, \chi, 0), \tag{3.10}$$

and similarly for $\tilde{q}_\alpha(\tilde{\chi})$.

We shall first show that $A_M^{(k)} \subset \psi(A_{\tilde{M}}^{(k)})$ for every k . If we differentiate (3.9) with respect to z_i, we obtain

$$\begin{aligned} &\frac{\partial G}{\partial z_i}(z, Q(z, \chi, 0)) + \frac{\partial G}{\partial w}(z, Q(z, \chi, 0)) \frac{\partial Q}{\partial z_i}(z, \chi, 0) \\ &\quad = \frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z, \chi, 0)), \bar{F}(\chi, 0), 0) \left(\frac{\partial F}{\partial z_i}(z, Q(z, \chi, 0)) + \frac{\partial F}{\partial w}(z, Q(z, \chi, 0)) \frac{\partial Q}{\partial z_i}(z, \chi, 0) \right), \end{aligned} \tag{3.11}$$

which we shall write as

$$\left(\frac{\partial G}{\partial w}(z, Q(z,\chi,0)) - \frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z,\chi,0)), \bar{F}(\chi,0), 0)\frac{\partial F}{\partial w}(z, Q(z,\chi,0))\right)$$
$$\times \frac{\partial Q}{\partial z_i}(z,\chi,0) = \frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z,\chi,0)), \bar{F}(\chi,0), 0)\frac{\partial F}{\partial z_i}(z, Q(z,\chi,0)) - \frac{\partial G}{\partial z_i}(z, Q(z,\chi,0)). \tag{3.12}$$

Since H is assumed to be CR transversal, $\partial G/\partial w(0)$ is invertible, and by normality of the coordinates $\partial \tilde{Q}/\partial \tilde{z}(0) = 0$. It follows that the $d \times d$ matrix

$$\frac{\partial G}{\partial w}(z, Q(z,\chi,0)) - \frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z,\chi,0)), \bar{F}(\chi,0), 0)\frac{\partial F}{\partial w}(z, Q(z,\chi,0))$$

of formal power series is invertible, and we can therefore rewrite (3.12) as follows:

$$\frac{\partial Q}{\partial z_i}(z,\chi,0) = \left(\frac{\partial G}{\partial w}(z, Q(z,\chi,0)) - \frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z,\chi,0)), \bar{F}(\chi,0), 0)\frac{\partial F}{\partial w}(z, Q(z,\chi,0))\right)^{-1}$$
$$\times \left(\frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z,\chi,0)), \bar{F}(\chi,0), 0)\frac{\partial F}{\partial z_i}(z, Q(z,\chi,0)) - \frac{\partial G}{\partial z_i}(z, Q(z,\chi,0))\right). \tag{3.13}$$

If we finally evaluate at $z = 0$, use again the fact that $G(z,0) = 0$ (which implies $\partial^{|\alpha|}G/\partial z^\alpha(0) = 0$ for all α) and the fact that $Q(0,\chi,0) = 0$, we obtain

$$\frac{\partial Q}{\partial z_i}(0,\chi,0) = \left(\frac{\partial G}{\partial w}(0) - \frac{\partial \tilde{Q}}{\partial \tilde{z}}(0, \bar{F}(\chi,0), 0)\frac{\partial F}{\partial w}(0)\right)^{-1}\frac{\partial \tilde{Q}}{\partial \tilde{z}}(0, \bar{F}(\chi,0), 0)\frac{\partial F}{\partial z_i}(0). \tag{3.14}$$

This shows that $A_M^{(1)}$ is generated by elements of the form $f(\bar{F}(\chi,0))$, where $f \in A_{\tilde{M}}^{(1)}$, and hence we have $A_M^{(1)} \subset \psi(A_{\tilde{M}}^{(1)})$. Thus, let us assume that $A_M^{(k)} \subset \psi(A_{\tilde{M}}^{(k)})$ holds for $k \le k_0$. To prove that it holds for $k = k_0 + 1$, choose a multi-index α with $|\alpha| = k_0 + 1$. Let β be a multi-index with $|\beta| = k_0$ and $i \in \{1, \ldots, n\}$ such that α equals the multi-index obtained by adding 1 to the ith component of β. Thus, if we apply $\partial^{|\beta|}/\partial z^\beta$ to the lth equation in (3.13) (i.e. the one which on its left-hand side has $\partial Q^l/\partial z_i$) and then set $z = 0$, we obtain $\partial^{|\alpha|}Q^l/\partial z^\alpha(0,\chi,0)$ on the left. On the right, we obtain (by using the fact that $\partial^{|\gamma|}G/\partial z^\gamma(0) = 0$ for any γ) a finite sum of terms, each of the form

$$c\frac{\partial^{|\gamma_1|}\tilde{Q}^{k_1}}{\partial \tilde{z}^{\gamma_1}}\cdots\frac{\partial^{|\gamma_p|}\tilde{Q}^{k_p}}{\partial \tilde{z}^{\gamma_p}}\frac{\partial^{|\sigma_1|}Q^{l_1}}{\partial z^{\sigma_1}}\cdots\frac{\partial^{|\sigma_q|}Q^{l_q}}{\partial z^{\sigma_q}}f(\bar{F}(\chi)), \tag{3.15}$$

where c is a complex constant, the derivatives of $\tilde{Q}$ are evaluated at $(0, \bar{F}(\chi,0), 0)$, the derivatives of Q at $(0,\chi,0)$, and the multi-indices γ_i, σ_j satisfy

$$|\gamma_1| + \cdots + |\gamma_p| \le k_0 + 1, \quad |\sigma_1| + \cdots + |\sigma_q| \le k_0. \tag{3.16}$$

The function f is an element of $A^{(1)}_{\tilde{M}}$; in fact, f is a matrix entry of some power of the $d \times d$ matrix

$$\left(\frac{\partial G}{\partial w}(0) - \frac{\partial \tilde{Q}}{\partial \tilde{z}}(0, \bar{F}(\chi, 0), 0)\frac{\partial F}{\partial w}(0)\right)^{-1}. \tag{3.17}$$

Hence, by (3.16) and our induction hypothesis, the terms in (3.15) are all in $A^{(k_0+1)}_{\tilde{M}}$, and then so is $\partial^{|\alpha|}Q/\partial z^{\alpha}(0, \chi, 0)$. This proves that $A^{(k_0+1)}_{M} \subset \psi(A^{(k_0+1)}_{\tilde{M}})$ and, by induction, $A^{(k)}_{M} \subset \psi(A^{(k)}_{\tilde{M}})$ for all $k \geq 1$.

To prove the last part of the theorem, we observe that if H is invertible at 0, then (since $G(z, 0) = 0$) we must have

$$\det\left(\frac{\partial F}{\partial z}(0)\right) \neq 0 \tag{3.18}$$

and hence $z \to F(z, 0)$ is invertible. This implies that the homomorphism ψ is an isomorphism. To complete the proof, we must prove the inclusion $\psi(A^{(k)}_{\tilde{M}}) \subset A^{(k)}_{M}$ under the assumption (3.18). We go back to Equation (3.11) (which we write as a matrix equation by using all $i = 1, 2, \ldots, n$) and observe that under this assumption the $n \times n$ matrix

$$\frac{\partial F}{\partial z}(z, Q(z, \chi, 0)) + \frac{\partial F}{\partial w}(z, Q(z, \chi, 0))\frac{\partial Q}{\partial z}(z, \chi, 0) \tag{3.19}$$

is invertible. Thus, we may solve for the term

$$\frac{\partial \tilde{Q}}{\partial \tilde{z}}(F(z, Q(z, \chi, 0)), \bar{F}(\chi, 0), 0)$$

in (3.11) rather than the term $\partial Q/\partial z(z, \chi, 0)$ as above. Now, proceeding as above, differentiating this new identity with respect to z (and making use of the invertibility of (3.19) and then setting $z = 0$, we conclude, by an induction analogous to the one above, that $\psi(A^{(k)}_{\tilde{M}}) \subset A^{(k)}_{M}$ for all $k \geq 1$. The details of this are left to the reader. This completes the proof. ■

We now show that Theorem 1.1 is a consequence of Theorem 3.1.

Proof of Theorem 1.1 We assume, without loss of generality, that p and $\tilde{p}$ are both 0. Since $\tilde{M}$ is of finite type at 0 and H is finite, it follows from Theorem 1.1 in [1] that H is CR transversal to $\tilde{M}$ at 0. Now, by a slight abuse of notation, we let M, $\tilde{M}$, and H denote the formal generic submanifolds and formal mapping associated with the smooth submanifolds and holomorphic mapping, respectively, given in Theorem 1.1. Since H is CR transversal to $\tilde{M}$ at 0, the conclusion of Theorem 1.1 now follows from Theorem 3.1. ■

We conclude this section by giving a simple example showing that the sequence of essential algebras is a finer invariant than the essential ideal.

Example 3.2 Consider the two hypersurfaces $M_1, M_2 \subset \mathbb{C}^2$ given near 0 by the equations

$$\begin{aligned} M_1&: \quad \operatorname{Im} w = |z|^4 \\ M_2&: \quad \operatorname{Im} w = |z|^4 + |z|^6. \end{aligned} \tag{3.20}$$

A simple calculation shows that the essential ideals in both cases are the same, namely $I_{M_1} = I_{M^2} = \mathfrak{m}^2$, where $\mathfrak{m}$ denotes the maximal ideal in $\mathbb{C}[\![\chi]\!]$, but the essential algebras are different (not isomorphic in the algebra of power series). For M_1 we have

$$A^{(1)}_{M_1} = 0, \quad A^{(2)}_{M_1} = \langle \chi^2 \rangle, \quad A^{(k)}_{M_1} = A^{(2)}_{M_1}, \quad k \geq 3,$$

where $\langle f_1, \ldots, f_k \rangle$ denotes the algebra generated by $f_1, \ldots, f_k$. For M_2, we have

$$A^{(1)}_{M_2} = 0, \quad A^{(2)}_{M_2} = \langle \chi^2 \rangle, \quad A^{(3)}_{M_2} = \langle \chi^2, \chi^3 \rangle, \quad A^{(k)}_{M_2} = A^{(3)}_{M_2}, \quad k \geq 4.$$

Clearly, $A^{(3)}_{M_2}$ is not isomorphic to $A^{(3)}_{M_1} = A^{(2)}_{M_1}$.

4. Proof of Theorem 1.2

Theorem 1.2 is a consequence of the following more general result concerning formal manifolds and formal mappings.

THEOREM 4.1 *Let M be a formal generic submanifold through $0 \in \mathbb{C}^N$ and assume that M is essentially finite and of finite type at 0. Let $A^{(k)}_M \subset \mathbb{C}[\![\chi]\!]$, $k \geq 1$, be the essential algebras of M. Assume that any algebra homomorphism $\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$ that satisfies $A^{(k)}_M \subset \psi(\mathbb{C}[\![\chi]\!])$, for all $k \geq 1$, is surjective. Then, any formal finite holomorphic mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into another formal generic submanifold $\tilde{M}$ (through 0) of the same dimension is a local formal biholomorphism.*

Proof of Theorem 4.1 Let M, H and $\tilde{M}$ be as above. Let (z, w) and $(\tilde{z}, \tilde{w})$ be normal coordinates for M and $\tilde{M}$, respectively, at 0 and write $H(z, w) = (F(z, w), G(z, w))$. Recall that mult (H) denotes the multiplicity of H. It follows from Proposition 2.3 in [1] that $\tilde{M}$ is of finite type at 0. It then follows from Theorem 5.1 in [1] that $\tilde{M}$ is essentially finite and H is CR transversal to $\tilde{M}$ at 0. Lemma 5.2 in [1] then implies that $m_H := \mathrm{mult}(F) = \mathrm{mult}(H)$. If m_H were ≥ 2, then the algebra homomorphism $\psi_R: \mathbb{C}[\![\tilde{\chi}]\!] \to \mathbb{C}[\![\chi]\!]$, where ψ_R is defined by (3.5) with $R(\chi) = \bar{F}(\chi, 0)$, would not be surjective (because any $f \in \mathbb{C}[\![\chi]\!]_0$ with non-zero image in $\mathbb{C}[\![\chi]\!]/I(\bar{F}(\chi, 0))$ would not be in the image, and there is at least one of these if $m_H \geq 2$). But, since $A^{(k)}_M \subset \psi_R(\mathbb{C}[\![\tilde{\chi}]\!])$ by Theorem 3.1, this would contradict the hypothesis in the theorem, and hence $m_H = \mathrm{mult}(H) = 1$. Thus, H is a formal biholomorphism. ■

We conclude this section by giving two examples. Example 4.2 shows that the conclusion of Theorem 4.1 fails for many essentially finite generic submanifolds of finite type. More precisely, for any essentially finite generic submanifolds of finite type $\tilde{M} \subset \mathbb{C}^N$ and any integer k, there is an essentially finite generic submanifolds of finite type $M \subset \mathbb{C}^N$ and a finite map H with $\mathrm{mult}(H) = k$ such that $H(M) \subset \tilde{M}$.

Example 4.2 Let $\tilde{M} \subset \mathbb{C}^N$ be a generic submanifold of codimension d through 0 given by $\mathrm{Im}\, w = \tilde{\phi}(z, \bar{z}, \mathrm{Re}\, w)$, where $z \in \mathbb{C}^n$, $w \in \mathbb{C}^d$ and $\phi(z, \bar{z}, s)$ is an $\mathbb{R}^d$-valued function. Let k be a positive integer and $F: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ any finite holomorphic mapping with $\mathrm{mult}(F) = k$. Define $M \subset \mathbb{C}^N$ by $\mathrm{Im}\, w = \tilde{\phi}(F(z), \overline{F(z)}, \mathrm{Re}\, w)$ and consider the mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ given by $H(z, w) = (F(z), w)$. It is clear that $H(M) \subset \tilde{M}$ and $\mathrm{mult}(H) = k$. If $\tilde{M}$ is essentially finite and of finite type, then so is M.

Our second example shows that there are generic submanifolds that are not finitely non-degenerate for which the assumptions (and hence the conclusions) of Theorem 4.1 hold.

Example 4.3 Let $M \subset \mathbb{C}^2$ be the real-analytic hypersurface defined by

$$\operatorname{Im} w = |z|^4 + |z|^6(1 + \phi_1(z, \bar{z})) + (\operatorname{Re} w)\phi_2(z, \bar{z}, \operatorname{Re} w), \tag{4.1}$$

where ϕ_1, ϕ_2 are real-valued real-analytic functions that satisfy $\phi_1(z, 0) = \phi_2(z, 0, s) = 0$. Observe that M is essentially finite but not finitely non-degenerate at 0. It is easy to see that the first three essential algebras $A_M^{(1)}, A_M^{(2)}, A_M^{(3)}$ coincide with those of the hypersurface M_2 in Example 3.2. To see that the condition on the algebras in Theorem 4.1 is satisfied, we shall suppose it is not. Recall that any algebra homomorphism $\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$ is of the form $\psi = \psi_R$, where $R(\chi)$ is a formal power series such that $R(0) = 0$. The fact that $A_M^k \subset \psi(\mathbb{C}[\![\chi]\!])$ means that

$$\chi^2 = (g_1 \circ R)(\chi), \quad \chi^3 = (g_2 \circ R)(\chi),$$

for some formal power series g_1 and g_2. Now, since ψ is assumed to be not surjective, we must have $R'(0) = 0$. We leave it to the reader to show that this is impossible.

5. Rigid hypersurfaces

In this section, we shall show that the condition imposed on the essential algebras in Theorem 4.1 is also necessary for the conclusion of that theorem to hold in the special case of rigid hypersurfaces. Thus, in this section M will be a formal hypersurface (i.e. of codimension $d = 1$) through 0 in $\mathbb{C}^{n+1}$. Recall that M is said to be rigid if there are normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for M at 0 such that M is defined by an equation of the form

$$\operatorname{Im} w = \phi(z, \bar{z}), \tag{5.1}$$

where $\phi(z, \chi)$ is a real power series, i.e.

$$\overline{\phi(\bar{z}, \bar{\chi})} = \phi(\chi, z).$$

The fact that the coordinates (z, w) are normal for M at 0 means that $\phi(z, 0) = \phi(0, \chi) = 0$. Hence, M is rigid if and only if M is also defined by (2.3) with

$$Q(z, \chi, \tau) = \tau + 2i\phi(z, \chi). \tag{5.2}$$

Also, recall that, for hypersurfaces, essential finiteness implies finite type. Our main result in this section is the following, which provides a converse to Theorem 4.1 for rigid hypersurfaces.

THEOREM 5.1 *Let M be a formal rigid hypersurface through $0 \in \mathbb{C}^N$ and assume that M is essentially finite at 0. Let $A_M^{(k)} \subset \mathbb{C}[\![\chi]\!]$, $k \geq 1$, be the essential algebras of M. Then the following are equivalent*:

(i) *Any formal finite holomorphic mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into another formal hypersurface $\tilde{M}$ (through 0) is a local formal biholomorphism.*
(ii) *Every algebra homomorphism $\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$ satisfying $A_M^{(k)} \subset \psi(\mathbb{C}[\![\chi]\!])$, for all $k \geq 1$, is surjective.*

Proof The implication (ii) $\Rightarrow$ (i) follows directly from Theorem 4.1. To prove (i) $\Rightarrow$ (ii), assume that there exists a non-surjective homomorphism

$$\psi: \mathbb{C}[\![\chi]\!] \to \mathbb{C}[\![\chi]\!]$$

such that $A_M^{(k)} \subset \psi(\mathbb{C}[\![\chi]\!])$ for every $k \geq 1$. Let $R: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be the formal holomorphic mapping (3.6) such that

$$\psi(f)(\chi) = (f \circ R)(\chi).$$

Since the essential ideal I_M, which is generated by $A_M^{(k)}$ for k large enough, has finite codimension in $\mathbb{C}[\![\chi]\!]$, it follows that R must be a finite mapping. Also, since ψ is non-surjective, R cannot be a formal biholomorphism. We claim that the formal finite mapping $(z, w) \mapsto (\bar{R}(z), w)$ sends M into another formal rigid hypersurface $\tilde{M}$ in $\mathbb{C}^{n+1}$. This will complete the proof. To this end, we observe that, by (5.2), the power series $q_\alpha(\chi)$ that generate $A_M^{(k)}$ satisfy $q_\alpha(\chi) = 2i\phi_\alpha(\chi)$, where

$$\phi(z, \chi) = \sum_\alpha \phi_\alpha(\chi) z^\alpha. \tag{5.3}$$

Since ϕ is real, we also have

$$\phi(z, \chi) = \sum_\alpha \bar{\phi}_\alpha(z) \chi^\alpha. \tag{5.4}$$

Let V_M be the vector space in $\mathbb{C}[\![\chi]\!]$ generated by finite linear combinations of the power series $\phi_\alpha(\chi)$, for $\alpha \in \mathbb{Z}_+^n$, or equivalently generated by finite linear combinations of the power series

$$\frac{\partial^{|\alpha|}\phi}{\partial z^\alpha}(0, \chi), \quad \forall \alpha \in \mathbb{Z}_+^n.$$

If we denote by U_M the vector space in $\mathbb{C}[\![z]\!]$ generated by finite linear combinations of

$$\frac{\partial^{|\alpha|}\phi}{\partial \chi^\alpha}(z, 0), \quad \forall \alpha \in \mathbb{Z}_+^n,$$

then the reality of ϕ implies that

$$U_M = \{f(z) : \bar{f}(\chi) \in V_M\}. \tag{5.5}$$

Let $P_m \subset \mathbb{C}[\![\chi]\!]$ be the vector space of all polynomials of degree at most m, and $j_0^m : \mathbb{C}[\![\chi]\!] \to P_m$ the jet or truncation operator which maps a power series to its m-jet at 0. It is easy to see that we can find, inductively for every $m \geq 1$, a sequence of integers $0 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq \cdots$ (with $k_\infty := \lim_{m\to\infty} k_m = \dim_{\mathbb{C}} V_M$, where $k_\infty = \infty$ is allowed) and elements $g_k \in V_A$, $k \leq k_\infty$, such that the polynomials $j_0^m g_k$, $k = 1, \ldots, k_m$, form a basis for $j_0^m V_M$ for every $m \geq 1$, and such that $j_0^m g_k = 0$ for $k \geq k_m + 1$. Clearly, any $f \in V_M$ can be written

$$f = \sum_{k=1}^{k_\infty} c_k g_k, \tag{5.6}$$

with $c_k \in \mathbb{C}$. Observe that the possibly infinite series in (5.6) converges in $\mathbb{C}[\![\chi]\!]$ equipped with its usual topology by the fact that only a finite number of the g_k have a non-zero coefficient for a given monomial χ^α (in fact, only those for which $k \leq |\alpha|$); recall that a sequence of formal power series $f_k(\chi) = \sum c_{\alpha,k} \chi^\alpha$ converges in $\mathbb{C}[\![\chi]\!]$ if and only if every

sequence of coefficients $\{c_{\alpha,k}\}_{k=1}^{\infty}$, for fixed α, converges in the complex plane. In particular, the series ϕ_α can be written

$$\phi_\alpha = \sum_{k=1}^{k_\infty} c_{\alpha,k} g_k, \tag{5.7}$$

with $c_{\alpha,k} \in \mathbb{C}$. It follows that we have

$$\begin{aligned}\phi(z,\chi) &= \sum_\alpha \sum_{k=1}^{k_\infty} c_{\alpha,k} g_k(\chi) z^\alpha \\ &= \sum_{k=1}^{k_\infty} g_k(\chi) h_k(z),\end{aligned} \tag{5.8}$$

where

$$h_k(z) = \sum_\alpha c_{\alpha,k} z^\alpha. \tag{5.9}$$

Hence, we have

$$\frac{\partial^{|\alpha|}\phi}{\partial\chi^\alpha}(z,0) = \sum_{k=1}^{k_\infty} \frac{\partial^{|\alpha|} g_k}{\partial\chi^\alpha}(0) h_k(z). \tag{5.10}$$

For $|\alpha| \le m$, we have, by construction, $\partial^{|\alpha|} g_k/\partial\chi^\alpha(0) = 0$ for $k \ge k_m + 1$, and the collection of vectors $(\partial^{|\alpha|} g_k/\partial\chi^\alpha(0))_{k=1}^{k_m} \in \mathbb{C}^{k_m}$, for $|\alpha| \le m$, have rank k_m (i.e. they span $\mathbb{C}^{k_m}$). It follows that there are k_m multi-indices $\alpha_1, \ldots, \alpha_{k_m}$ with $|\alpha_j| \le m$ such that the $k_m \times k_m$-matrix $(\partial^{|\alpha_j|} g_k/\partial\chi^{\alpha_j}(0))_{j,k=1}^{k_m}$ is invertible. Hence, we can solve the collection of equations in (5.10), with $\alpha = \alpha_j$ and $j = 1, \ldots, k_m$, for $h_k(z)$, with $k \le k_m$, and obtain

$$h_k(z) = \sum_{|\alpha| \le m} d_{k,\alpha} \frac{\partial^{|\alpha|}\phi}{\partial\chi^\alpha}(z,0), \quad k \le k_m. \tag{5.11}$$

Moreover, since $\phi(z,\chi)$ is real, as observed in (5.5), we also have that

$$\frac{\partial^{|\alpha|}\phi}{\partial\chi^\alpha}(z,0) = \sum_{k=1}^{k_\infty} e_{\alpha,k} \bar{g}_k(z). \tag{5.12}$$

We conclude that

$$\phi(z,\chi) = \sum_{k,l=1}^{k_\infty} b_{kl} g_k(\chi) \bar{g}_l(z), \tag{5.13}$$

for some $\mathbb{C}$-valued double sequence b_{kl}. Again, observe that the possibly infinite double sum in (5.13) converges in $\mathbb{C}[\![\chi]\!]$. Moreover, the reality of $\phi(z,\chi)$ implies that $b_{kl} = \overline{b_{lk}}$.

Now, since each element of V_M belongs to $A_M^{(k)}$, for k large enough, we conclude, by our assumption, that there are $\tilde{g}_k \in \mathbb{C}[\![\chi]\!]$ such that $g_k = \psi(\tilde{g}_k) = \tilde{g}_k \circ R$. Define the real formal power series

$$\tilde{\phi}(z,\chi) = \sum_{k,l=1}^{k_\infty} b_{kl} \tilde{g}_k(\chi) \bar{\tilde{g}}_l(z). \tag{5.14}$$

We then have

$$\phi(z, \chi) = \tilde{\phi}(\bar{R}(z), R(\chi)). \tag{5.15}$$

Thus, if we define the formal rigid hypersurface $\tilde{M}$ through 0 in $\mathbb{C}^{n+1}$ by

$$\operatorname{Im} w = \tilde{\phi}(z, \bar{z}), \tag{5.16}$$

then it follows from (5.15) that the finite formal mapping $(z, w) \mapsto (\bar{R}(z), w)$ sends M to $\tilde{M}$. This completes the proof. ■

We should point out that the condition (ii) in Theorem 5.1 is not necessary for (i) to hold (although it is always sufficient by Theorem 4.1) unless the hypesurface is rigid as is shown by the following example.

Example 5.2 Let $M \subset \mathbb{C}^2$ be the real-analytic hypersurface given by $\operatorname{Im} w = |z|^4 + (\operatorname{Re} w)|z|^2$. Observe that M is essentially finite at 0 (with $\operatorname{Ess}_0(M) = 2$) and the essential algebras coincide with those of M_1 in Example 3.2. It is easy to check that condition (ii) in Theorem 5.1 does not hold for M. (Consider the homomorphism induced by $\chi \mapsto \chi^2$.) However, we claim that (i) holds. To see this, note that if $\tilde{M} \subset \mathbb{C}^2$ is a formal hypersurface through 0 and H is a formal finite non-invertible mapping sending M into $\tilde{M}$, then $\tilde{M}$ is finitely non-degenerate (i.e. $\operatorname{Ess}_0(\tilde{M}) = 1$) and H is CR transversal with $\operatorname{mult}(H) = m_H = 2$ by Proposition 2.3, Theorem 5.1 and Lemma 5.2, all in [1]. Suppose that such a hypersurface $\tilde{M}$ and mapping H exist. Then, by choosing normal coordinates for $\tilde{M}$ at 0 and writing $H = (F, G)$, we see that H must be of the form

$$F(z, w) = az^2 + O(|z|^3, |w|), \quad G(z, w) = cw + O(|zw|, |w|^2), \tag{5.17}$$

where $a \in \mathbb{C}$ and $c \in \mathbb{R}$ (as is easy to check) with $a \neq 0$ and $c \neq 0$. The equation for $\tilde{M}$ is $\operatorname{Im} \tilde{w} = \tilde{\phi}(\tilde{z}, \bar{\tilde{z}}, \operatorname{Re} \tilde{w})$, where $\tilde{\phi}(\tilde{z}, 0, \tilde{s}) = \tilde{\phi}(0, \bar{\tilde{z}}, \tilde{s}) = 0$. If H sends M into $\tilde{M}$, then we have (after complexifying) the identity

$$\begin{aligned} &\operatorname{Im} G(z, s + i(z^2\chi^2 + sz\chi)) = \tilde{\phi}(F(z, s + i(z^2\chi^2 + sz\chi)), \\ &\bar{F}(\chi, s - i(z^2\chi^2 + sz\chi)), \operatorname{Re} G(z, s + i(z^2\chi^2 + sz\chi))). \end{aligned} \tag{5.18}$$

By Taylor expanding both sides and observing that there is a non-zero term $z\chi s$ on the left but none on the right, we conclude that the existence of such a mapping H and hypersurface $\tilde{M}$ is a contradiction and, hence, condition (i) in Theorem 5.1 holds, as claimed.

6. Concluding remarks

We conclude this article by pointing out that essentially finite formal generic submanifolds M of finite type through 0 in $\mathbb{C}^N$ that further satisfy the condition:

(*) *Any finite formal holomorphic mapping* $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ *sending* M *into another formal generic submanifold* $\tilde{M}$ (*through* 0) *of the same dimension is a local formal biholomorphism*,

are in some sense the simplest manifolds of this type and deserve to be studied further. For brevity, we shall refer to formal manifolds that satisfy (*) as *simple*. If M is a real-analytic manifold that satisfies (*) with 'formal holomorphic' replaced by 'holomorphic (or convergent)', then we shall say that M is *analytically simple*. The following result justifies this terminology, in some sense, by showing that any essentially finite generic submanifold of finite type through 0 in $\mathbb{C}^N$ can be realized as a branched cover over one that is simple.

THEOREM 6.1 *Let M be a formal (resp. real-analytic) generic submanifold through $0 \in \mathbb{C}^N$ and assume that M is essentially finite and of finite type at 0. Then, precisely one of the following holds*:

(i) *M is simple (resp. analytically simple).*
(ii) *There exist an integer $k \geq 2$, a finite formal (resp. convergent) mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ with $\mathrm{mult}(H) = k$, and a formal (resp. real-analytic) generic submanifold $\tilde{M}$ which is essentially finite, of finite type, and simple (resp. analytically simple) such that H sends M into $\tilde{M}$.*

Moreover, if (ii) *holds, then*

$$\mathrm{Ess}_0(M) = k\mathrm{Ess}_0(\tilde{M}). \tag{6.1}$$

Proof We shall prove the statement of the theorem in the case where M is formal. (The analytic case is completely analogous.) If M is not simple, then by definition there exist a number $k \geq 2$, a formal finite mapping H with mult $(H) = k$, and a formal generic submanifold $\tilde{M}$ of the same dimension as M such that $H(M) \subset \tilde{M}$. The formal manifold $\tilde{M}$ is of finite type at 0 by Proposition 2.3 in [1]. Theorem 5.1 in [1] then implies that H is CR transversal, $\tilde{M}$ is essentially finite at 0, and the identity (6.1) holds. In particular, $\mathrm{Ess}_0(\tilde{M}) < \mathrm{Ess}_0(M)$. If $\tilde{M}$ is not simple, then we can repeat the procedure above with M replaced by $\tilde{M}$. Since the essential type decreases at every step, the process must terminate in a simple manifold after a finite number of steps since every manifold with essential type 1 is simple, again by Theorem 5.1 in [1]. ■

Acknowledgements

P. Ebenfelt is supported in part by DMS-0701070. L.P. Rothschild is supported in part by DMS-0701121.

References

[1] P. Ebenfelt and L.P. Rothschild, *Transversality of CR mappings*, Am. J. Math. 128 (2006), pp. 1313–1343.
[2] J.P. D'Angelo, *The notion of formal essential finiteness for smooth real hypersurfaces*, Indiana Univ. Math. J. 36 (1987), pp. 897–903.
[3] M.S. Baouendi and L.P. Rothschild, *Germs of CR maps between real analytic hypersurfaces*, Invent. Math. 93 (1988), pp. 481–500.
[4] K. Diederich and J.E. Fornæss, *Pseudoconvex domains with real-analytic boundary*, Ann. Math. 107 (1978), pp. 371–384.

[5] M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild, *Real Submanifolds in Complex Space and their Mappings*, Princeton Mathematical Series, 47, Princeton University Press, Princeton, NJ, 1999.

[6] M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild, *Rational dependence of smooth and analytic CR mappings on their jets*, Math. Ann. 315 (1999), pp. 205–249.