Eigendistribution Expansions on Heisenberg Groups

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§1. Introduction. Let N be a connected, simply connected nilpotent Lie group. N acts on the space $C_c^\infty(N)$ of compactly supported C^∞ functions by $f^g(x) = f(g^{-1}xg)$ for $x, g \in N$. A distribution Θ on N is called invariant if $\Theta(f^g) = \Theta(f)$ for all $f \in C_c^\infty(N)$ and $g \in N$. For any irreducible unitary representation π of N, the operator

$$\pi(f) = \int_N \pi(x)f(x)\, dx, \qquad f \in C_c^\infty(N)$$

is of trace class, and the map $f \to \text{trace } \pi(f)$ is an invariant distribution Θ_π called the *global character* of π [1]. Furthermore, Θ_π is tempered; *i.e.* Θ_π extends to the Schwartz space $\mathcal{S}(N)$, defined by the exponential map identification of N with Euclidean space. Hence there is a natural class of invariant tempered distributions on N.

In this paper we consider the converse question: can an arbitrary tempered invariant distribution on N be expanded in terms of characters of irreducible unitary representations? We give an affirmative answer in the case where N is locally isomorphic to a product of Heisenberg groups.

To illustrate the type of result obtained, let N be the 3-dimensional Heisenberg group with Lie algebra $\mathfrak{n}$ spanned by x, y, t, such that $[x, y] = t$. Let x^*, y^*, t^* be a corresponding basis for $\mathfrak{n}^*$, the real dual space of $\mathfrak{n}$, so that (x, y, t) is the linear coordinate on $\mathfrak{n}^*$. For $f \in \mathcal{S}(N)$ let $\hat{f}$ denote Euclidean Fourier transform of f. Then any infinite dimensional irreducible unitary representation of N has global character Θ_s given by

$$\Theta_s(f) = \tfrac{1}{2}\, |s|^{-1} \iint \hat{f}(x, y, s)\, dx\, dy$$

for some $s \in \mathbf{R}$, $s \neq 0$. (See §3 for more details.) Any finite dimensional irreducible unitary representation is one dimensional with global character of the form

$$\Theta_{x,y}(f) = \hat{f}(x, y, 0)$$

for some $(x, y) \in \mathbf{R}^2$. Then for any $f \in \mathcal{S}(N)$, $|s|\Theta_s(f) \in \mathcal{S}(\mathbf{R})$ and $\Theta_{x,y}(f) \in \mathcal{S}(\mathbf{R}^2)$.

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We may now state our main result for this case. Let $h(x, y) \in C_c^\infty(\mathbf{R}^2)$ with $\iint h(x, y)dxdy = 1$. If Θ is an invariant tempered distribution on N, then there is a distribution S_Θ on $\mathbf{R}^2$ such that

$$\Theta(f) = \tilde{\Theta}(2h(x, y)\,|s|\,\Theta_s(f)) + S_\Theta(\Theta_{x,y}(f))$$

for all $f \in \mathcal{S}(N)$, where $\tilde{\Theta}$ is the Euclidean Fourier transform of Θ. In particular, this proves that if $f \in \mathcal{S}(N)$ is annihilated by every global character on N, then f is annihilated by every invariant tempered distribution on N.

§2. Preliminaries on characters and distributions. We first recall the Kirillov theory of irreducible unitary representations and their global characters [4]. Let $\mathfrak{n}$ be the Lie algebra of N. The exponential map Exp: $\mathfrak{n} \to N$ is a real analytic diffeomorphism, equivariant for the adjoint actions of N on $\mathfrak{n}$ and N. Thus we may view functions and distributions on N as functions and distributions on $\mathfrak{n}$, and conjugation-invariance on N corresponds to Ad(N)-invariance on $\mathfrak{n}$. Hence Haar measure on N determines a volume element on $\mathfrak{n}$. One may also identify the Schwartz space $\mathcal{S}(N)$ and its dual $\mathcal{S}'(N)$, the tempered distributions, with the corresponding spaces $\mathcal{S}(\mathfrak{n})$ and $\mathcal{S}'(\mathfrak{n})$.

Let $\mathfrak{n}^*$ be the real linear dual of $\mathfrak{n}$ and Ad* the co-adjoint representation of N on $\mathfrak{n}^* : [\mathrm{Ad}^*(g)\lambda](x) = \lambda(\mathrm{Ad}(g^{-1})x)$, $\lambda \in \mathfrak{n}^*$. $\mathfrak{n}^*$ has the unique volume element which makes the Euclidean Fourier transform, $f \to \tilde{f}$,

$$\tilde{f}(\lambda) = \int f(x)e^{i\lambda(x)}\,dx,$$

an isometry from $L^2(\mathfrak{n})$ onto $L^2(\mathfrak{n}^*)$. The Fourier transform carries $\mathcal{S}(N)$ onto $\mathcal{S}(\mathfrak{n}^*)$ and thus $\mathcal{S}'(N)$ onto $\mathcal{S}'(\mathfrak{n}^*)$ by $\tilde{\Theta}(\tilde{f}) = \Theta(f)$ for $f \in \mathcal{S}(N)$ and $\Theta \in \mathcal{S}'(\mathfrak{n})$. The Fourier transform is also N-equivariant.

Kirillov [4] identifies the space $\hat{N}$ of irreducible unitary representations of N with the space of Ad*(N)-orbits on $\mathfrak{n}^*$. In this identification the class $[\pi_\mathcal{O}] \in \hat{N}$ corresponding to an orbit $\mathcal{O} \subset \mathfrak{n}^*$ has global character

$$\Theta_\mathcal{O}(f) = c_\mathcal{O} \int_\mathcal{O} \tilde{f}(\lambda)\,d\lambda,$$

where $d\lambda$ is an invariant measure on $\mathcal{O}$. Each $\Theta_\mathcal{O}$ is an invariant, tempered distribution.

We always assume that distributions take values in a semi-complete locally convex space so that we can use the Schwartz Kernel Theorem to carry our results over from one Heisenberg group to a product of Heisenberg groups. For this application the quasi-complete locally convex space involved is the space of distributions of a subgroup.

Recall that a distribution Θ vanishes on an open set U if $\Theta(f) = 0$ for all $f \in C_c^\infty(U)$. Then supp Θ, the support of Θ, is the complement of the largest open set on which Θ vanishes. If Σ is a closed submanifold of $\mathbf{R}^n$, any distribution Θ_Σ on Σ extends to a distribution Θ on $\mathbf{R}^n$ defined by $\Theta(f) = \Theta_\Sigma(f|\Sigma)$, where

$f \mid \Sigma$ is the restriction of f. We shall not distinguish between Θ and Θ_Σ. If D is a differential operator on $\mathbf{R}^n$, the distribution $D\Theta$ is defined as

$$D\Theta(f) = \Theta(D^t f),$$

where D^t is the adjoint of D. Suppose Σ is a submanifold of $\mathbf{R}^n$ and $\{s_i\}$ a coordinate system such that Σ is given by $s_1 = s_2 = \cdots = s_m = 0$. If Θ is a distribution with supp $\Theta \subset \Sigma$, then Θ has a unique representation as an (infinite) sum of terms of the form $(\partial^I/\partial s_I)\Theta_I$, where each Θ_I is a distribution on Σ, and I runs through all distinct multi-indices $(i_1, i_2, \cdots, i_m)$, $i_j \geqq 0$ [8, Ch. III, §10, Théorème XXXVI].

If h is a smooth (not necessarily completely supported) function on $\mathbf{R}^n$, then $h\Theta$ is defined by

$$h\Theta(f) = \Theta(hf).$$

Note that these definitions reverse the usual order of differentiation; *i.e.* $hD\Theta(f) = \Theta(D^t(hf))$.

§3. **The Heisenberg Group.** Let N be the Heisenberg group of dimension $2n + 1$. We view N as $\mathbf{R} \oplus \mathbf{C}^n$ with group law

$$(w, z)(w', z') = (w + w' + \tfrac{1}{2} \operatorname{Im} z\bar{z}', z + z').$$

Thus N has Lie algebra $\mathfrak{n}$ with basis $\{x_1, \cdots, x_n; y_1, \cdots, y_n; t\}$ in which the exponential map Exp: $\mathfrak{n} \to N$ is given by

$$\operatorname{Exp}\left(\sum u^i x_i + \sum v^i y_i + wt\right) = (w, u + iv).$$

The only nonzero brackets in this basis are $[x_i, y_i] = t = -[y_i, x_i]$.

Let $\{x_1^*, \cdots, x_n^*; y_1^*, \cdots, y_n^*; t^*\}$ denote the dual basis of $\mathfrak{n}^*$. Then $(x_1, \cdots, x_n; y_1, \cdots, y_n; t) = (x, y, t)$ is the corresponding linear coordinate on $\mathfrak{n}^*$. It is standard, and we will see in a moment, that the co-adjoint representation Ad^* of N on $\mathfrak{n}^*$ has orbits

(3.1a) the hyperplanes $\{(x, y, s) : \; x, y \in \mathbf{R}^n, s \neq 0 \text{ fixed}\}$

and

(3.1b) the one point sets with coordinates $(x, y, 0)$, $\; x, y \in \mathbf{R}^n$.

We denote the corresponding irreducible unitary representation classes on N by $[\pi_s]$ and $[\pi_{x,y}]$. Their global (distribution) characters are multiples of integration of Fourier transform over the orbits, as follows [7].

(3.2a) $$\Theta_s = \Theta_{\pi_s} : f \mapsto c^{-1} |s|^{-n} \int \hat{f}(x, y, s)\, dx\, dy, \qquad c = n!\, 2^n,$$

and

(3.2b) $$\Theta_{x,y} = \Theta_{\pi_{x,y}} : f \mapsto \hat{f}(x, y, 0).$$

Now we can state our main result.

3.3 Theorem. *Let Θ be an invariant tempered distribution on N with values in a locally convex topological vector space E. Then Θ is an E-valued distribution combination of the Θ_t and the $\Theta_{x,y}$, as follows.*

Choose $h \in C_c^\infty(\mathbf{R}^{2n})$ with $\int h(x, y)dxdy = 1$. Then there is an E-valued distribution S on $\mathbf{R}^{2n}$ such that

$$\Theta(f) = \tilde{\Theta}(n!2^n h(x, y)\,|t|^n \Theta_t(f)) + S(\Theta_{x,y}(f)) \text{ for } f \in \mathcal{S}(N). \tag{3.4}$$

In particular,

3.5. Corollary. *If $f \in \mathcal{S}(N)$ is annihilated by the characters of the irreducible unitary representations of N, then f is annihilated by every invariant tempered distribution on N.*

Harish-Chandra [2, Theorem 2] and S. Helgason [3, Proposition 2.6] proved theorems of the following type. If a Lie group G acts as Lie transformation group on a manifold M, and if there is a smooth section Σ to the orbits, then every G-invariant distribution on M consists of integration over the orbits followed by a distribution on Σ. If $\tilde{\Theta}$ were supported in the set $t \neq 0$ (resp. were a distribution on the hyperplane $t = 0$) in $\mathfrak{n}^*$, this would give Theorem 3.3 directly. As will be seen below, the proof of Theorem 3.3 is a modification of that idea. It consists of examining the difference

$$f \mapsto \Theta(f) - \tilde{\Theta}(n!2^n h(x, y)|t|^n \Theta_t(f))$$

of Θ and the corresponding distribution combination of orbital integrals on the set $t \neq 0$ in $\mathfrak{n}^*$.

We now set about proving Theorem 3.3. We first need the standard facts on Ad*:

3.6. Lemma. Ad*(N) *acts on the coordinates (x, y, t) of $\mathfrak{n}^*$ by*
Ad*(Exp (rx_i)) : $y_i \to y_i - rt$, *all other coordinates fixed;*
Ad*(Exp (ry_i)) : $x_i \to x_i + rt$; *all other coordinates fixed;*
Ad*(Exp (rt)) : *all coordinates fixed.*
In particular the Ad*(N) = *orbits are given by* (3.1), *and the induced action on* $C^\infty(\mathfrak{n}^*)$ *is*

$$ad^*(x_i) = -t\frac{\partial}{\partial y_i}, \qquad ad^*(y_i) = t\frac{\partial}{\partial x_i} \quad \text{and} \quad ad^*(t) = 0. \tag{3.7}$$

Proof. If $x, z \in \mathfrak{n}$ and $\lambda \in \mathfrak{n}^*$ then $[\mathrm{Ad}^*(\mathrm{Exp}\,(rx))\lambda](z) = \lambda(\mathrm{Ad}(\mathrm{Exp}\,(-rx))z) = \lambda(z - r[x, z])$ because $[\mathfrak{n}, \mathfrak{n}]$ is central in $\mathfrak{n}$. Take $x = t$: then $\mathrm{Ad}^*(\mathrm{Exp}\,(rt))\lambda = \lambda$. Take $x = x_i$ and let z run over our basis of $\mathfrak{n}$: then $[\mathrm{Ad}^*(\mathrm{Exp}\,(rx_i))\lambda](z) = \lambda(z)$ for $z \neq y_i$, $= \lambda(y_i - rt)$ for $z = y_i$. Now $\mathrm{Ad}^*(\mathrm{Exp}\,(rx_i))$ sends x_j^* to x_j^*, y_j^* to y_j^* and t^* to $t^* - ry_i^*$. That gives the action on coordinates as claimed. If $f \in C^\infty(\mathfrak{n})$ now

$$[ad^*(x_i)f](\lambda) = \frac{d}{dr} f(Ad^*(\mathrm{Exp}\,(-rx))\lambda)|_{r=0} = -t\frac{\partial f}{\partial y_i}(\lambda).$$

The calculation for y_i is similar. Q.E.D.

We now fix
(3.8a) a locally convex topological vector space E, and
(3.8b) an E-valued distribution $T : C_c^\infty(\mathfrak{n}^*) \to E$.
Eventually T will be $\tilde{\Theta}$, which is $\mathrm{Ad}^*(N)$-invariant by the $\mathrm{Ad}(N)$-invariance of Θ, so we now assume that T is invariant:
(3.9a) if $f \in C_c^\infty(\mathfrak{n}^*)$ and $g \in N$ then $T(f \circ \mathrm{Ad}^*(g)) = T(f)$.
Differentiating and using (3.7), we phrase this invariance as infinitesmal invariance [2, 3]:

(3.9b) $$t\frac{\partial T}{\partial x_i} = 0 \quad \text{and} \quad t\frac{\partial T}{\partial y_i} = 0 \quad \text{for} \quad 1 \leqq i \leqq n.$$

Now we are ready to start the reduction of T. Choose

(3.10a) $$h \in C_c^\infty(\mathbf{R}^{2n}) \quad \text{with} \quad \int h(x, y)\, dx\, dy = 1$$

and define a distribution $T' : C_c^\infty(\mathfrak{n}^*) \to E$ by

(3.10b) $$T'(f) = T(f') \quad \text{where} \quad f'(x, y, t) = h(x, y) \int_{\mathbf{R}^{2n}} f(u, v, t)\, du\, dv.$$

3.11. Lemma. T' is invariant (3.9) *and $T - T'$ is supported in the hyperplane $t = 0$ of $\mathfrak{n}^*$.*

Proof. Observe $\int \{f(x, y, t) - f'(x, y, t)\} dx\, dy = 0$ because $\int h(x, y) dx\, dy = 1$. Thus there exist p_i, $q_i \in C_c^\infty(\mathfrak{n}^*)$ with $f - f' = \sum \partial p_i/\partial x_i + \sum \partial q_i/\partial y_i$. Now $(T - T')(f) = T(f - f') = T(\sum \partial p_i/\partial x_i + \sum \partial q_i/\partial y_i) = \sum (\partial T/\partial x_i)(p_i) + \sum (\partial T/\partial y_i)(q_i)$.

If the support supp (f) does not meet $(t = 0)$, the same holds for f', p_i and q_i, and we have

$$(T - T')(f) = \sum \left(t\frac{\partial T}{\partial x_i}\right)\left(\frac{1}{t}\, p_i\right) + \sum \left(t\frac{\partial T}{\partial y_i}\right)\left(\frac{1}{t}\, q_i\right),$$

which vanishes by (3.9). Thus $T - T'$ is supported in $(t = 0)$.

Invariance of T' is just translation invariance of the Lebesgue integral in the definition (3.10b) of f'. Q.E.D.

As consequence of Lemma 3.11, we have E-valued distributions T_k on the hyperplane $(t = 0)$ such that $T - T' = \sum_{k=0}^{\infty} (\partial^k/\partial t^k)(T_k)$; see Schwartz [8].

3.12. Lemma. If $\ell > 0$ there is a vector $e_\ell \in E$ such that $T_\ell(f) = (\int f(x, y, 0) dx\, dy) e_\ell$ for all $f \in C_c^\infty(\mathfrak{n}^*)$.

Proof. Since T and T' are invariant we have (3.9b)

$$0 = \left[\left(t\frac{\partial}{\partial x_i}\right)(T - T')\right](f) = \left[\left(t\frac{\partial}{\partial x_i}\right)\sum \frac{\partial^k T_k}{\partial t^k}\right](f) = \sum T_k\left\{\left(\frac{\partial^k}{\partial t^k}\left(t\frac{\partial f}{\partial x_i}\right)\right)\Big|_{t=0}\right\}$$
$$= \sum T_k\left\{\left(t\frac{\partial^k}{\partial t^k}\frac{\partial f}{\partial x_i} + k\frac{\partial^{k-1}}{\partial t^{k-1}}\frac{\partial f}{\partial x_i}\right)\Big|_{t=0}\right\} = \sum_{k=1}^{\infty} T_k\left\{\left(k\frac{\partial^{k-1}}{\partial t^{k-1}}\frac{\partial f}{\partial x_i}\right)\Big|_{t=0}\right\}.$$

Fix $\ell \geqq 1$ and choose $a \in C_c^\infty(\mathbf{R})$ with $a(t) = 1$ for $|t| < 1$. If $q \in C_c^\infty(\mathbf{R}^{2n})$ we define $f = f_q$ by $f(x, y, t) = a(t)t\ell^{-1}q(x, y)/\ell!$. Then

$$T_k\left\{\left(k\frac{\partial^{k-1}}{\partial t^{k-1}}\frac{\partial f}{\partial x_i}\right)\bigg|_{t=0}\right\} = 0 \quad \text{for} \quad k \neq \ell, \quad \text{and} \quad T_\ell\left\{\left(\ell\frac{\partial^{\ell-1}}{\partial t^{\ell-1}}\frac{\partial f}{\partial x_i}\right)\bigg|_{t=0}\right\}$$

$= T_\ell(\partial q/\partial x_i)$. Thus $\partial T_\ell/\partial x_i = 0$. Similarly $\partial T_\ell/\partial y_i = 0$. If $e^* \in E$, now the scalar distribution $e^* \circ T_\ell$ is killed by the $\partial/\partial x_i$ and the $\partial/\partial y_i$, so there is a constant $c(\ell, e^*)$ such that $e^*T_\ell(q) = c(\ell, e^*)\int q(x, y)\, dx\, dy$. Thus T_ℓ has the required form.

3.13. Lemma. *If $\ell > 0$ then $T_\ell = 0$.*

Proof. Recall $\int\{f(x, y, t) - f'(x, y, t)\}dx\, dy = 0$. We differentiate under the integral to see $\int(\partial^k/\partial t^k)\{f(x, y, t) - f'(x, y, t)\}dx\, dy = 0$. Lemma 3.12 then implies Q.E.D.

$$\left(\frac{\partial^k}{\partial t^k}T_k\right)(f - f') = 0 \quad \text{for} \quad k \geqq 1. \tag{3.14}$$

Also observe $(f')' = f'$, so that

(3.15) $T'(f') = T(f') = T''(f)$.

Now compute using (3.15)

$$T(f) - T'(f) = T(f - f') = T'(f - f') + \sum \frac{\partial^k T_k}{\partial t^k}(f - f').$$

Using (3.14) we conclude

$$T(f) = T'(f) + T_0(f - f'). \tag{3.16}$$

Now $S : f \mapsto T_0(f - f')$ is an E-valued distribution on $(t = 0)$.
Since $S = T - T' = \sum (\partial^k/\partial t^k)(T_k)$, Lemma 3.12 gives us

$$S(f) = T_0(f) + \sum_{k=1}^{\infty}\left(\int \frac{\partial^k f}{\partial t^k}(x, y, 0)\, dx\, dy\right)e_k\,. \tag{3.17}$$

Fix $\ell \geqq 1$ and $a \in C_c^\infty(\mathbf{R})$ with $a(t) = 1$ for $|t| < 1$. If $q \in C_c^\infty(\mathbf{R}^{2n})$ define $f = f_q \in C_c^\infty(\mathfrak{n}^*)$ by $f(x, y, t) = t\ell a(t)q(x, y)$. Then $S(f) = T_0(f) = 0$ because S and T_0 are distributions on $(t = 0)$, where f vanishes. If $k \neq \ell$ then $(\partial^k f/\partial t^k)(x, y, 0) = 0$, and $(\partial\ell f/\partial t\ell)(x, y, 0) = \ell! q(x, y)$. Now (3.17) says $e_\ell = 0$, so $T_\ell = 0$. Q.E.D.

In summary, we have proved

3.18. Theorem. *Let T be an E-valued invariant distribution on $\mathfrak{n}^*$. Let $h \in C_c^\infty(\mathbf{R}^{2n})$ with $\int h(x, y)dxdy = 1$, and define an E-valued invariant distribution T' on $\mathfrak{n}^*$ by*

$$T'(f) = T(f') \quad \textit{where} \quad f'(x, y, t) = h(x, y)\int f(u, v, t)\, du\, dv.$$

Then there is an E-valued distribution S on the hyperplane $(t = 0)$ *in* $\mathfrak{n}^*$ *such that* $T = T' + S$.

The proof of Lemma 3.13 shows $S(h) = 0$.

The formulae (3.2) for the global characters of the irreducible unitary representations of N give us

$$T'(f) = T(n!2^n h(x, y)|t^n|\tilde{\Theta}_t(f)) \tag{3.19a}$$

and

$$S(f) = S(\tilde{\Theta}_{x,y}(f)). \tag{3.19b}$$

Thus Theorem 3.18 expresses on E-valued invariant distribution T on $\mathfrak{n}^*$ as a distribution combination of Fourier transforms of global characters.

$$T(f) = T'(n!2^n h(x, y)|t|^n\tilde{\Theta}_t(f)) + S(\tilde{\Theta}_{x,y}(f)). \tag{3.20}$$

If we take $T = \tilde{\Theta}$ where Θ is an E-valued invariant tempered distribution on N, and if we take $f = \tilde{\varphi}$ where $\varphi \in \mathcal{S}(N)$, then (3.20) becomes

$$\Theta(\varphi) = \tilde{\Theta}(n!2^n h(x, y)|t|^n\Theta_t(\varphi)) + S(\Theta_{x,y}(\varphi)). \tag{3.21}$$

That agrees with (3.4). Theorem 3.3 is proved.

§4. Products of Heisenberg Groups. Let $N = N_1 \times \cdots \times N_k$ where N_i is the Heisenberg group of dimension $2n_i + 1$. The corresponding Lie algebras and their linear duals are

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k \quad \text{and} \quad \mathfrak{n}^* = \mathfrak{n}_1^* \oplus \cdots \oplus \mathfrak{n}_k^*.$$

In the manner of §3 we choose a basis

$$\{x_{i,1}, \cdots, x_{i,n_i}; y_{i,1}, \cdots, y_{i,n_i}; t_i\}$$

of $\mathfrak{n}_i$ in which the only nonzero brackets are $[x_{i,j}, y_{i,j}] = t_i = -[y_{i,j}, x_{i,j}]$, obtaining a linear coordinate

$$(x_{(i)}, y_{(i)}, t_i) = (x_{i,1}, \cdots, x_{i,n_i}; y_{i,1}, \cdots, y_{i,n_i}; t_i)$$

on $\mathfrak{n}_i^*$. Then $(x_{(1)}, y_{(1)}, t_1; \cdots; x_{(k)}, y_{(k)}, t_k)$ serves as linear coordinate on $\mathfrak{n}^*$. In that coordinate system, the Ad*(N)-orbits on $\mathfrak{n}^*$ are grouped into families for multi-indices $I = (i_1, \cdots, i_m)$, where $0 \leqq m = |I| \leqq k$ and $1 \leqq i_1 < \cdots < i_m \leqq k$, as follows. The orbits in the family for I are the subsets of $\mathfrak{n}^*$ given by

$$\begin{cases} \text{if } \ell \in I \text{ then } t_\ell = \text{nonzero constant and } x_{(\ell)}, y_{(\ell)} \text{ variable over } \mathbf{R}^{n_\ell} \\ \text{if } \ell \notin I \text{ then } t_\ell = 0 \text{ and } x_{(\ell)}, y_{(\ell)} \text{ are constant.} \end{cases} \tag{4.1a}$$

A cross section to this family of orbits is the submanifold of $\mathfrak{n}^*$ given by

$$\Sigma_I = \Sigma_1 \oplus \cdots \oplus \Sigma_k \quad \text{where} \quad \begin{cases} \text{if } \ell \varepsilon I, & \Sigma_\ell = \{\nu \varepsilon \mathfrak{n}_\ell^*: x_{(\ell)} = y_{(\ell)} = 0, \quad t_\ell \neq 0\} \\ \text{if } \ell \varepsilon I, & \Sigma_\ell \text{ is the hyperplane } t_\ell = 0 \text{ in } \mathfrak{n}_\ell^*. \end{cases} \tag{4.1b}$$

If $\lambda \varepsilon \mathfrak{n}^*$ we write $[\pi_\lambda]$ for the corresponding unitary representation class and Θ_λ for its global character.

If Σ is a submanifold of $\mathfrak{n}^*$ and U is a distribution on Σ, then we sometimes write $\int_\Sigma h(\lambda) dU(\lambda)$ for $U(h)$.

Now we can state the main result of this §4.

4.2. Theorem. *If Θ is an invariant tempered distribution on N, then there are distributions U_I on Σ_I such that*

$$\Theta(f) = \sum_{m=0}^{k} \sum_{|I|=m} \int_{\Sigma_I} \{\prod_{i \varepsilon I} |t_i(\lambda)|^{n_i}\} \cdot \Theta_\lambda(f)\, dU_I(\lambda) \quad \textit{for} \quad f \varepsilon \mathcal{S}(N). \tag{4.3}$$

In particular,

4.4. Corollary. *If $f \varepsilon \mathcal{S}(N)$ is annihilated by the characters of the irreducible unitary representations of N, then f is annihilated by every invariant tempered distribution on N.*

It will be clear from the proof that Theorem 4.2 and Corollary 4.4 remain true *(i)* for distributions with values in a quasi-complete locally convex topological vector space and *(ii)* with N replaced by any Lie quotient group *e.g.* for a product of Heisenberg, vector and torus groups. {For *(ii)* note that the linear dual of the Lie algebra of N/D is the annihilator of the Lie algebra of D in $\mathfrak{n}^*$.}

In the manner of §3 we first prove

4.5. Proposition. *Let T be an invariant distribution on $\mathfrak{n}^*$. Then there are distributions S_I on Σ_I such that*

$$T(f) = \sum_{m=0}^{k} \sum_{|I|=m} \int_{\Sigma_I} \{\prod_{i \varepsilon I} |t_i(\lambda)|^{n_i}\} \cdot \tilde{\Theta}_\lambda(f)\, dS_I(\lambda) \quad \textit{for} \quad f \varepsilon C_c^\infty(\mathfrak{n}^*). \tag{4.6}$$

Proof. If $k = 1$ this follows from Theorem 3.18 in its formulation (3.2). Now let $k > 1$ and let E be the space of distributions on $\mathfrak{n}_1^* \oplus \cdots \oplus \mathfrak{n}_{k-1}^*$. The Schwartz Kernel Theorem [9, Proposition 25] represents T as an $\mathrm{Ad}^*(N_k)$-invariant distribution $T_k : C_c^\infty(\mathfrak{n}_k^*) \to E$ whose values are $\mathrm{Ad}^*(N_1 \times \cdots \times N_{k-1})$-invariant, under

$$T_\nu(f_k)(\varphi) = T(\varphi \otimes f_k) \quad \text{for} \quad \varphi \varepsilon C_c^\infty(\mathfrak{n}_1^* \oplus \cdots \oplus \mathfrak{n}_{k-1}^*) \quad \text{and} \quad f_k \varepsilon C_c^\infty(\mathfrak{n}_k^*).$$

Choose $h_k \varepsilon C_c^\infty(\mathbf{R}^{2n_k})$ with $\int h_k(x_{(k)}, y_{(k)}) dx_{(k)} dy_{(k)} = 1$. According to Theorem 3.18, $T_k = T_k' + T_k''$ where T_k'' is an E-valued distribution on the hyperplane $t_k = 0$ in $\mathfrak{n}_k^*$ and $T_k'(f_k) = T_k(h_k(x_{(k)}, y_{(k)}) \int f_k(u, v, t_k) du\, dv)$. The values of T_k' are $\mathrm{Ad}^*(N_1 \times \cdots \times N_{k-1})$-invariant, and the same invariance follows for $T_k'' = T_k - T_k'$.

By induction on k, we have distributions $A_J(f_k)$ and $B_J(f_k)$ on $\Sigma_1 \oplus \cdots \oplus \Sigma_{k-1}$, as J runs through the multi-indices $(j_1, \cdots, j_m)$ with $1 \leqq j_1 < \cdots < j_m \leqq k-1$, such that

$$T_k'(f_k)(\varphi) = \sum_{m=0}^{k-1} \sum_{|J|=m} \int \Psi_\lambda(\varphi)\, dA_J(f_k)(\lambda)$$

and

$$T_k''(f_k)(\varphi) = \sum_{m=0}^{k-1} \sum_{|J|=m} \int \Psi_\lambda(\varphi)\, dB_J(f_k)(\lambda)$$

where $\Psi_\lambda(\varphi)$ denotes the Riemann integral of φ over the orbit $\mathrm{Ad}^*(N_1 \times \cdots \times N_{k-1})(\lambda)$. Glancing back at (4.1) and the definitions of T_k' and T_k'', the Schwartz Kernel Theorem gives us distributions C_I on Σ_I such that

$$T(f) = \sum_{m=0}^{k} \sum_{|I|=m} \int_{\Sigma_I} \Phi_\lambda(f)\, dC_I(\lambda) \tag{4.7a}$$

where $\Phi_\lambda(f)$ is the integral of f over the orbit $\mathrm{Ad}^*(N)(\lambda)$. If $\lambda \in \Sigma_I$ then ([7], [5], [6])

$$\tilde{\theta}_\lambda(f) = (n!\,2^n)^{-1}\{\prod_{i\in I} |t_i(\lambda)|^{-n_i}\}\Phi_\lambda(f), \qquad n = \sum_{i\in I} n_i\,. \tag{4.7b}$$

Combine (4.7a) and (4.7b) to obtain (4.6) with $S_I = (n!2^n)C_I$. Q.E.D.

Theorem 4.2 follows from Proposition 4.5 with $T = \tilde{\theta}$.

5. Some open questions. The problems studied here for products of Heisenberg groups are still unresolved for general nilpotent Lie groups. That is, can an invariant tempered distribution on a nilpotent Lie group be expanded, in some sense, in terms of the characters of the irreducible unitary representations of that group? Since the cross-sections for the co-adjoint representation are in general more complicated than those of the Heisenberg group, one cannot expect to obtain an exact analog of Theorem 4.2. However, it still makes sense to ask whether a function which is annihilated by all characters is annihilated by all invariant tempered distributions.

More generally, consider a nilpotent Lie group N and a unipotent representation on N on some Euclidean space $\mathbf{R}^m$. It is known that any orbit of N on $\mathbf{R}^m$ carries an invariant, tempered measure. Thus each orbit determines an invariant tempered distribution on $\mathbf{R}^m$. Again one may ask to what extent these distributions determine all N-invariant tempered distributions on $\mathbf{R}^m$. Here one of us is optimistic and the other is pessimistic.

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