EMBEDDABILITY OF ABSTRACT CR STRUCTURES AND INTEGRABILITY OF RELATED SYSTEMS

by

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1. Introduction.

Let M be a smooth real manifold of dimension N, and $\mathcal{V}$ a subbundle of the complex tangent bundle, CTM, with dim $\mathcal{V} = n$. We shall say that $\mathcal{V}$ is *integrable* at a point $p_0 \in M$ if there exists a neighborhood Ω_0 of p_0 and smooth functions $\zeta_1, \ldots, \zeta_{N-n}$ defined on Ω_0 with linearly independent differentials and satisfying

$$(1.1) \qquad L\zeta_k = 0 \quad \text{in } \Omega_0, \quad k = 1, \ldots, N - n,$$

for all $L \in L_0$, where $L_0 = C^\infty(\Omega_0, \mathcal{V})$, the space of smooth sections of $\mathcal{V}$ over Ω_0. In this paper we shall give a criterion for local integrability.

We call $\mathcal{V}$ *formally integrable* if

$$(1.2) \qquad\qquad [\mathcal{V}, \mathcal{V}] \subset \mathcal{V},$$

i.e. if for any sections $L, L' \in L$, we have $[L, L'] \in L$, where $L = C^\infty(M, \mathcal{V})$. The Frobenius theorem then says that formal integrability implies integrability if $\mathcal{V}$ is real (resp. real analytic), i.e. if L has a basis of real (resp. real analytic) sections. In the general case it is easy to check by dimension that formal integrability is a necessary condition for integrability.

If, in addition, $\mathcal{V}$ satisfies

$$(1.3) \qquad\qquad \mathcal{V} \cap \overline{\mathcal{V}} = (0)$$

then $\mathcal{V}$ is called an *abstract* CR *bundle*, and M an *abstract* CR *manifold*. In this case we have $N = 2n + \ell$ with $\ell \geqslant 0$. We say that $\mathcal{V}$ is of *codimension* ℓ.

Key-words : Embeddability – CR structures – Complex Lie algebre.

A submanifold of $C^{n+\ell}$ is a *generic* CR *manifold* if it is locally given by $\rho_j = 0$, $j = 1, \ldots, \ell$, with ρ_j real valued, smooth, and satisfying $\partial\rho_1, \ldots, \partial\rho_\ell$ linearly independent. It can be easily shown that an abstract CR manifold is integrable at p_0 if and only if near p_0, M can be embedded as a generic CR manifold in $C^{n+\ell}$, with the image of $\mathcal{V}$ equal to the induced CR bundle i.e. the bundle whose sections are tangential, antiholomorphic vector fields.

For this reason an integrable CR structure is also called *embeddable* or *realizable*. The first example of a nonembeddable strictly pseudoconvex abstract hypersurface was given by Nirenberg [8]. (See also Jacobowitz-Treves [5]).

Our main result is the following :

THEOREM . — *Let* M *be a smooth manifold and* $\mathcal{V} \subset CTM$ *a subbundle satisfying*

$$[L, L] \subset L,$$

where $L = C^\infty(M, \mathcal{V})$. *Then* $\mathcal{V}$ *is locally integrable at* $p_0 \in M$ *if and only if there exist* $\Omega_0 \subset M$, *an open neighborhood of* p_0 *in* M, *and smooth complex vector fields* $R_1, \ldots, R_\ell$ *defined in* Ω_0 *spanning a complex Lie algebra i.e.*

$$(1.4) \qquad [R_i, R_j] = \sum_{k=1}^{\ell} a_{ijk} R_k \, , \, a_{ijk} \in C,$$

and satisfying

$$(1.5) \qquad [L_0, R_j] \subset L_0 \, , \, j = 1, \ldots, n,$$

with $L_0 = C^\infty(\Omega_0, \mathcal{V})$, *and for every* $p \in \Omega_0$

$$(1.6) \qquad \mathcal{V}_p + \overline{\mathcal{V}}_p + \mathcal{R}_p + \overline{\mathcal{R}}_p = CT_p\Omega_0 \, .$$

where $\mathcal{V}_p$ *is the fiber of* $\mathcal{V}$ *at* p, *and* $\mathcal{R}_p$ *is the span of the* R_j *at* p. *More precisely, if* $\mathcal{V}$ *is integrable, we may find* R_j *so that* $a_{ijk} = 0$ *for all* i, j, k *and replace (1.6) by*

$$(1.7) \qquad \mathcal{V}_p + \overline{\mathcal{V}}_p \oplus \mathcal{R}_p = CT_p\Omega_0 \, .$$

For an integrable structure, the existence of vector fields R_j satisfying conditions similar to (1.4) with $a_{ijk} = 0$, (1.5) and (1.6) was proved and used in Baouendi-Treves [2]. However, the proof we

give here is more natural to the embedding and is used to establish the result for the general case.

For the case where $\mathcal{V}$ is an abstract CR structure, the integrability result generalizes a theorem of Jacobowitz [4] where $\mathcal{V}$ is of codimension one, and a theorem of the authors and Treves [1] for the case where the R_j are real independent vector fields. As in [1], the proof of integrability depends, in the CR case, on the Newlander-Nirenberg theorem [6], and in the general case on a corollary of Nirenberg [7], (see also Hörmander [3] and Treves [9]) which states that $\mathcal{V}$ is integrable if $\mathcal{V} + \overline{\mathcal{V}} = CTM$; we reprove this result by methods in the spirit of this paper.

Remark. – Note that we do not require the vector fields R_j satisfying (1.4), (1.5) and (1.6) to be linearly independent at every point of Ω_0. However, when $\mathcal{V}$ is integrable, we may choose them linearly independent, and such that the subbundle $\mathcal{R}$ whose sections are spanned by them is totally real i.e.

$$\overline{\mathcal{R}} = \mathcal{R}.$$

2. Proof of the existence of the R_j.

We assume first that $\mathcal{V}$ is CR i.e. $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. Assume M is integrable at p_0, so that M may be regarded as a submanifold of $\mathbf{C}^{n+\ell}$ given by

$$(2.1) \qquad \rho_j = 0, \quad j = 1, \ldots, \ell$$

and $\partial \rho_1, \ldots, \partial \rho_\ell$ linearly independent.

By relabeling the coordinates in $\mathbf{C}^{n+\ell}$ we may take $(z, w) \in \mathbf{C}^{n+\ell}$, $w \in \mathbf{C}^\ell$, and assume that

$$(2.2) \qquad \rho_w = \begin{pmatrix} \dfrac{\partial \rho_1}{\partial w_1} & \cdots & \dfrac{\partial \rho_1}{\partial w_\ell} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \dfrac{\partial \rho_\ell}{\partial w_1} & \cdots & \dfrac{\partial \rho_\ell}{\partial w_\ell} \end{pmatrix}$$

is invertible near the origin. Similarly, we let

(2.3)
$$\rho_z = \begin{pmatrix} \dfrac{\partial \rho_1}{\partial z_1} & \cdots & \dfrac{\partial \rho_1}{\partial z_n} \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \dfrac{\partial \rho_\ell}{\partial z_1} & \cdots & \dfrac{\partial \rho_\ell}{\partial z_n} \end{pmatrix}$$

be an $\ell \times n$ matrix. Then a local basis for $C^\infty (M, \mathcal{V})$ is obtained

as $(L) = \begin{pmatrix} L_1 \\ \cdot \\ \cdot \\ \cdot \\ L_n \end{pmatrix}$, with

$$(L) = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

where we have written $\left(\dfrac{\partial}{\partial \bar{z}}\right)$ for $\begin{pmatrix} \dfrac{\partial}{\partial \bar{z}_1} \\ \cdot \\ \cdot \\ \dfrac{\partial}{\partial \bar{z}_n} \end{pmatrix}$ and similarly for $\dfrac{\partial}{\partial \bar{w}}$.

We have

(2.4) PROPOSITION. $- Set$ $(R) = \begin{pmatrix} R_1 \\ \cdot \\ \cdot \\ R_\ell \end{pmatrix}$ $where$

$$(R) = \left(\frac{\partial}{\partial w}\right) - {}^t\rho_w\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right).$$

Then the R_j are tangent to M, commute, and satisfy (1.5), and (1.7).

Proof. $-$ Since $R_j \rho_k = 0$ by construction, the R_j are tangent to M. To prove (1.7) we observe that since $N = 2n + \ell$, and the L_j, $\bar{L}_j$ and R_k are all linearly independent, the result holds by dimension.

For (1.4) and (1.5) we calculate $[L_j, R_k]$ and $[R_j, R_k]$. Each is again tangent to M, and from the form of the $L's$ and $R's$, they contain only $\dfrac{\partial}{\partial \overline{w}_k}$, and hence are antiholomorphic. Since the L_j form a basis for the tangential antiholomorphic vector fields to M, each $[L_j, R_k]$ and $[R_j, R_k]$ is a linear combination of the $L_j's$ with smooth coefficients. These coefficients must be zero, since neither commutator contains a term of the form $\dfrac{\partial}{\partial \overline{z}_p}$. This proves (1.4) (with $a_{ijk} = 0$) and (1.5), and hence Proposition (2.4).

□

We now assume that $\mathcal{V}$ is integrable but not necessarily CR. We shall construct the R_j by adding variables in order to reduce to the case of a CR bundle. Let Ω be a small neighborhood of p_0 in M. First choose a basis L_j of $C^\infty (\Omega, \mathcal{V})$ and coordinates (x, y, t, s) in Ω vanishing at p_0,

$$x, y \in \mathbf{R}^r, t \in \mathbf{R}^{n-r}, s \in \mathbf{R}^\ell$$

with $\ell = N - n - r$, such that

$$(2.5) \qquad L_j |_{p_0} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right), \quad 1 \leqslant j \leqslant r,$$

and

$$(2.6) \qquad L_{j+r} |_{p_0} = \frac{\partial}{\partial t_j}, \quad r + 1 \leqslant j \leqslant n.$$

We introduce $n - r$ new variables $t'_1, \ldots, t'_{n-r}$ and define new vector fields $\widetilde{L}_j$ in $\Omega' = \Omega \times \mathbf{R}^{n-r}$ by

$$\widetilde{L}_j = L_j, \quad 1 \leqslant j \leqslant r,$$

and for $r + 1 \leqslant j \leqslant n$, $\widetilde{L}_j$ is obtained from L_j by replacing $\dfrac{\partial}{\partial t_j}$ by $\dfrac{\partial}{\partial t_j} + i \dfrac{\partial}{\partial t'_j}$. Let $\mathcal{V}'$ be the bundle with sections spanned by the $\widetilde{L}_j$ on Ω'. If $\zeta_1, \ldots, \zeta_{r+\ell}$ is a set of independent solutions for $\mathcal{V}$, then $\zeta_1, \ldots, \zeta_{r+\ell}, t_1 + i t'_1, \ldots, t_{n-r} + i t'_{n-r}$ is a set of independent solutions for $\mathcal{V}'$. Since $\mathcal{V}' \cap \overline{\mathcal{V}}' = \{0\}$, we have proved

(2.7) LEMMA. $—\mathcal{V}'$ is an integrable CR bundle on Ω'.

Let $\tau_j = t_j + it'_j$, $\tau = (\tau_1, \ldots, \tau_{n-r})$ and $\zeta = (\zeta_1, \ldots, \zeta_{r+\ell})$. The mapping

$$(x, y, t, t', s) \longmapsto (\zeta(x, y, t, s), \tau)$$

is an embedding of Ω' onto a CR generic submanifold of $\mathbf{C}^{n+\ell}$. Therefore there exist real smooth functions $\rho_j(Z, \bar{Z})$ in $\mathbf{C}^{n+\ell}$ so that locally the image of Ω' is given by $\rho_j = 0$, $j = 1, \ldots, \ell$, with $\partial\rho_1, \ldots, \partial\rho_\ell$ linearly independent. Hence we have for $j = 1, \ldots, \ell$

$$(2.8) \qquad \rho_j(\zeta(x, y, t, s), \tau, \overline{\zeta(x, y, t, s)}, \bar{\tau}) \equiv 0$$

in Ω'.

We may assume that $\zeta(0) = 0$. If $Z_1, \ldots, Z_{n+\ell}$ are the variables in $\mathbf{C}^{n+\ell}$, we write τ_k for $Z_{k+r+\ell}$, $k = 1, \ldots, n-r$.

(2.9) LEMMA. — *We may assume that the ρ_j are independent of t'_k. Also we have for $j = 1, \ldots, \ell$ and $k = 1, \ldots, n-r$*

$$\frac{\partial\rho_j}{\partial\tau_k}(0) = 0.$$

Proof. — It suffices to differentiate (2.8) with respect to t_k and t'_k, and to use (2.6) and the fact that the ζ_j satisfy the equations

$$L_p \zeta_k = 0 \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant k \leqslant r + \ell.$$

This proves the lemma.   □

Since the ρ_j have independent complex differentials, the matrix

$$\begin{bmatrix} \rho_{1z_1} & \cdots & \rho_{1z_{\ell+r}} & \rho_{1\tau_1} & \cdots & \rho_{1\tau_{n-r}} \\ \rho_{\ell z_1} & \cdots & \rho_{\ell z_{\ell+r}} & \rho_{\ell\tau_1} & \cdots & \rho_{\ell\tau_{n-r}} \end{bmatrix}$$

has rank ℓ, therefore by Lemma (2.9) the submatrix

$$\left[\frac{\partial\rho_j}{\partial z_k}\right]_{1 \leqslant j \leqslant \ell, 1 \leqslant k \leqslant \ell+r}$$

must have rank ℓ at 0. Hence we may find new coordinates $(z,w) \in \mathbf{C}^r \times \mathbf{C}^\ell$ such that the matrix $\left[\dfrac{\partial \rho}{\partial w}\right]$ is invertible at 0. In these coordinates we may find a basis for $\mathcal{V}'$ in the form $(\widetilde{L}) = \begin{pmatrix} \widetilde{L}' \\ \widetilde{L}'' \end{pmatrix}$, where

$$(2.11) \qquad (\widetilde{L}') = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}} \, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

and

$$(2.12) \qquad (\widetilde{L}'') = \left(\frac{\partial}{\partial \bar{\tau}}\right) - {}^t\rho_{\bar{\tau}} \, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

where we use the notation conventions of § 2. Rectricting to $t' = 0$ we find a basis (L) for $\mathcal{V}$ given by $(L) = \begin{pmatrix} L' \\ L'' \end{pmatrix}$:

$$(2.13) \qquad (L') = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}} \, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

and

$$(2.14) \qquad (L'') = \left(\frac{\partial}{\partial t}\right) - {}^t\rho_t \, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right).$$

Now put

$$(R) = \left(\frac{\partial}{\partial w}\right) - {}^t\rho_w \, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right)$$

as before.

□

3. Proof of Integrability.

We now assume $\{R_j\}$ exist satisfying (1.4), (1.5), and (1.6) and prove $\mathcal{V}$ is integrable. First we give a new proof of the following result of Nirenberg [7].

(3.1) PROPOSITION. — *If $\mathcal{W}$ is a formally integrable subbundle of* **CTM** *for which*

(3.2) $$\mathcal{V} + \overline{\mathcal{V}} = CTM,$$

then $\mathcal{V}$ is locally integrable.

Proof. — Let Ω be a small neighborhood of $p_0 \in M$, and $V_1, V_2, \ldots, V_n$ be a commuting basis for $C^\infty(\Omega, \mathcal{V})$. After renumbering and multiplication by complex numbers we may assume $V_1, \ldots, V_r$ is a maximal set for which $V_1, \ldots, V_r, \overline{V}_1, \ldots, \overline{V}_r$ is linearly independent at p_0, and that these, together with $\mathrm{Re}\, V_j$, $j > r$, span the section of $CT\Omega$. Now let $\widetilde{\mathcal{V}}$ be the bundle over $\Omega \times R^{n-r}$ whose sections are spanned by $\widetilde{V}_j = V_j$, $1 \leqslant j \leqslant r$,

and $\widetilde{V}_j = V_j + i\dfrac{\partial}{\partial t_{r-j}}$, $j = r+1, \ldots, n$. Then $\widetilde{\mathcal{V}}$ satisfies the conditions of the Newlander-Nirenberg theorem [6] since

$$\widetilde{\mathcal{V}} \cap \overline{\widetilde{\mathcal{V}}} = (0).$$

Hence there exist n solutions $f_1(u,t), \ldots, f_n(u,t)$ for $\widetilde{\mathcal{V}}$, where (u) is a coordinate system near p_0 in Ω vanishing at p_0, and t is in a neighborhood of 0 in R^{n-r}. We may assume $f_j(0) = 0$, $j = 1, \ldots, n$.

We shall obtain solutions for $\mathcal{V}$ in the form

$$\zeta_k = F_k(f_1, \ldots, f_n),$$

where each $F_k(Z)$ is holomorphic and satisfies

(3.3) $\dfrac{\partial}{\partial t_j}[F_k(f_1(u,t), \ldots, f_n(u,t))] \equiv 0,$ $j = 1, \ldots, n-r.$

We shall prove that there exist $F_1, \ldots, F_r$ holomorphic satisfying (3.3) with linearly independent differentials. Indeed, for F holomorphic

(3.4) $\dfrac{\partial}{\partial t_j} F(f_1, \ldots, f_n) = \displaystyle\sum_{p=1}^{n} \dfrac{\partial f_p}{\partial t_j}\dfrac{\partial F}{\partial Z_p}(f_1, \ldots, f_n).$

Since we may choose a basis for $\widetilde{\mathcal{V}}$ taking vector fields with coefficients independent of the t_j, $\dfrac{\partial f_p}{\partial t_j}$ is again a solution for $\widetilde{\mathcal{V}}$. Hence there exists a holomorphic function H_{pj} such that

(3.5) $\quad \dfrac{\partial f_p}{\partial t_j} = H_{pj}(f_1, \ldots, f_n), \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant j \leqslant n - r.$

Substituting (3.4) and (3.5) into (3.3) we obtain the system

(3.6) $\quad \displaystyle\sum_{p=1}^{n} H_{pj}(Z) \dfrac{\partial F}{\partial Z_p}(Z) = 0, \quad j = 1, \ldots, n - r.$

Since $df_1, \ldots, df_n, d\overline{f_1}, \ldots, d\overline{f_n}$ are linearly independent we conclude that the matrix

$$\left(\dfrac{\partial f_p}{\partial t_j} \right), \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant j \leqslant n - r,$$

is of rank $n - r$. Therefore by (3.5) the same is true for the matrix (H_{pj}) at the origin. It follows by the Cauchy-Kovalevsky Theorem that there are $n - (n - r) = r$ linearly independent solution F_k of (3.6) near 0. Hence the functions

$$\zeta_k(u) = F_k(f_1(u,t), \ldots, f_n(u,t)), \quad 1 \leqslant k \leqslant r,$$

provide a system of solutions for $\mathcal{W}$, proving integrability.

$\square$

We may now complete the proof of the theorem. We assume we are given the R_j satisfying (1.4), (1.5) and (1.6). We let $S_1, \ldots, S_{\ell}$ be a basis for an abstract complex Lie algebra satisfying the same commutation relations as the R_j i.e.

(3.7) $\qquad\qquad [S_i, S_j] = \displaystyle\sum_{k=1}^{\ell} a_{ijk} S_k.$

By introducing local exponential coordinates on any corresponding connected complex Lie group we may find coordinates in an open neighborhood $\mathcal{O}$ of 0 in $\mathbf{C}^{\ell}$ near 0 in which we may represent the S_j as holomorphic vector fields with holomorphic coefficients i.e.

(3.8) $\qquad\qquad S_j = \displaystyle\sum_{k=1}^{\ell} a_{jk}(t) \dfrac{\partial}{\partial t_k}$

with $t_k = t'_k + i t''_k \in \mathbf{C}$ and the matrix (a_{jk}) is invertible. Now we let $R'_j = R_j + S_j$. We claim that the bundle $\widetilde{\mathcal{V}}$ over $\Omega \times \mathcal{O}$

spanned by $\mathcal{V}$, $\{R_j'\}_{1 < j < \ell}$ and $\left\{\dfrac{\partial}{\partial \bar{t}_k}\right\}_{1 < k < \ell}$ satisfies the condition of Proposition (3.1) for integrability.

Indeed, note that the S_j commute with $\dfrac{\partial}{\partial \bar{t}_j}$, as well as the R_j and L_0. Hence

$$(3.9) \qquad [R_i + S_i, R_j + S_j] = \Sigma\, a_{ijk}\,(R_k + S_k),$$

which proves that $\widetilde{\mathcal{V}}$ is formally integrable. Also, the span of the $\widetilde{R}_j$, $\overline{\widetilde{R}}_j$, $\dfrac{\partial}{\partial t_j}$ and $\dfrac{\partial}{\partial \bar{t}_j}$ is the same as that of the R_j, $\overline{R}_j$, $\dfrac{\partial}{\partial t_j}$ and $\dfrac{\partial}{\partial \bar{t}_j}$. Hence $\widetilde{\mathcal{V}}$ satisfies condition (3.2). By Proposition (3.1) there exist $N - n = N + 2\ell - (n + 2\ell)$ solutions $f_k(u, t', t'')$ which have linearly independent differentials.

Now let $\zeta_k(u) = f_k(u, 0, 0), k = 1, \ldots, N - n$. Since the coefficients of elements of L are independent of $(t'\, t'')$, it is clear that the ζ_k are solutions of (1.1). It suffices to check that the ζ_k have linearly independent differentials. This will follow if the matrix $\left(\dfrac{\partial f_L}{\partial u_k}\right)_{\substack{1 < j < N-n \\ 1 < k < N}}$ has rank $N - n$. By the linear independence of the f_k in the (u, t', t'') variables, it suffices to show that $\dfrac{\partial f_k}{\partial t_j'}$ and $\dfrac{\partial f_k}{\partial t_j''}$ are linear combinations of $\dfrac{\partial f_k}{\partial u_-}$. Since $\dfrac{\partial f_k}{\partial \bar{t}_i} = 0$ and $(R_j + S_j)\, f_k = 0$, $1 \leq j \leq \ell$, this follows, and hence the proof of the theorem is complete.

□

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