COMMUN. IN PARTIAL DIFFERENTIAL EQUATIONS, 13(11), 1441-1466 (1988)

EXTENSION OF HOLOMORPHIC FUNCTIONS IN GENERIC WEDGES AND THEIR WAVE FRONT SETS

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[1]Supported by NSF Grant DMS 8603176.
[2]Supported by NSF Grant DMS 8601260.

§0. Introduction

In this paper we consider holomorphic functions defined in open wedges whose edges are a smooth generic submanifold M of $\mathbf{C}^{n+\ell}$ of codimension ℓ. (Precise definitions are given below.) If such functions have tempered growth at M, their boundary values are CR (Cauchy-Riemann) distributions on M. The holomorphic extendability of such distributions has been extensively studied, beginning with the pioneering work of Hans Lewy [10]. A microlocal approach to the question of extendability was introduced in Baouendi-Chang-Treves [2], where the hypoanalytic wave front set of a CR distribution was defined, in analogy with the analytic wave front set of a distribution on R^n (See Hörmander[8]). A main tool in the microlocal approach is the use of the FBI (Fourier-Bros-Iagolnitzer) transform (see [6] and Sjöstrand [12] for the case of R^n). A slightly different version, called the mini-FBI transform was used in Baouendi-Rothschild-Treves [4] for rigid CR structures, and then in Baouendi-Rothschild [3] and Treves [14] in the general case.

In the present work we consider holomorphic functions in generic wedges but with no growth conditions assumed at M. Therefore no distribution boundary value is assumed to exist in this case. (However, if the edge M is a real analytic manifold a hyperfunction approach following Sato-Kawai-Kashiwara[11] would be possible, but is not discussed here.) We define a wave front set for such a holomorphic function and characterize it by an FBI transform similar to the mini-FBI transform for CR functions mentioned above. If the holomorphic function has tempered growth, our wave front agrees with the hypoanalytic wave front set of its boundary value. It seems that the notion of wave front set discussed in this paper is related to the one defined in Kashiwara-Schapira [9].

We now give some precise definitions. Let M be a smooth submanifold of $\mathbf{C}^{n+\ell}$, of real dimension $2n+\ell$ $(n \geq 0,\ \ell \geq 0)$. It is called *generic* if it is locally defined (near $p_0 \in M$) by

$$\rho_j(z,\bar{z}) = 0, \qquad 1 \leq j \leq \ell, \tag{0.1}$$

where the ρ_j are smooth real functions such that the complex differentials $\partial\rho_1, \dots, \partial\rho_\ell$ are linearly independent. [Here $\partial\rho_j = \sum_{k=1}^{n+\ell} \frac{\partial\rho_j}{\partial z_k} dz_k$]. We write $\rho = (\rho_1, \dots, \rho_\ell)$.

If $\rho^1 = (\rho_1^1, \dots, \rho_\ell^1)$ is another set of defining functions for M near p_0, then we clearly have

$$\rho^1(z, \bar{z}) = A(z, \bar{z})\rho(z, \bar{z}) \tag{0.2}$$

where A is an $\ell \times \ell$ invertible matrix of real, smooth functions.

If $\mathcal{O}$ is a small neighborhood of p_0 in $\mathbf{C}^{n+\ell}$, and $\Gamma \subset \mathbf{R}^\ell \setminus \{0\}$ is an open convex cone we set

$$\mathcal{W}(\mathcal{O}, \rho, \Gamma) = \{z \in \mathcal{O}; \rho(z, \bar{z}) \in \Gamma\}. \tag{0.3}$$

The set defined by (0.3) is an open subset of $\mathbf{C}^{n+\ell}$ whose boundary contains $M \cap \mathcal{O}$. Such a set is called a *wedge* of *edge* M in the *direction* Γ.

Note that definition (0.3) is independent of the choice of ρ in the following sense. If ρ^1 is another vector of defining functions for M satisfying (0.2), then for every Γ_1,

$$\Gamma_1 \Subset A(0)\Gamma,$$

there exists $\mathcal{O}_1$, a neighborhood of p_0 in $\mathbf{C}^{n+\ell}$ such that

$$\mathcal{W}(\mathcal{O}_1, \rho^1, \Gamma_1) \subset \mathcal{W}(\mathcal{O}, \rho, \Gamma). \tag{0.4}$$

We shall consider holomorphic functions in wedges of edge M. Thanks to (0.4) we may freely change defining functions ρ_j.

We may assume $p_0 = 0$. It is easy to see that there exist holomorphic coordinates (z, w) in $\mathbf{C}^{n+\ell}$, $z \in \mathbf{C}^n$, $w \in \mathbf{C}^\ell$, and defining functions for M of the form

$$\rho = Im\ w - \varphi(z, \bar{z}, s), \qquad s = Re\ w, \tag{0.5}$$

with φ smooth, valued in $\mathbf{R}^\ell$, and

$$\varphi(0) = 0,\ \varphi'(0) = 0,\ \varphi''_{ss}(0) = 0. \tag{0.6}$$

The paper is organized as follows. In Sections 1 and 2 we consider the case of a totally real edge M i.e. $n = 0$. We define the FBI transform of a holomorphic function defined in a wedge $\mathcal{W}$ and its wave front set, and we give microlocal necessary and sufficient conditions for the extendability of these functions to larger wedges or to a full neighborhood in $\mathbf{C}^{n+\ell}$ of a point $p_0 \in M$, (Theorems 1, 2, and 3). In Section 3 we extend the results to the case of a generic edge with $n > 0$ (Theorems 4, 5 and 6).

In Section 4 we show that if every CR function defined near $p_0 \in M$ extends to a wedge, then for every neighborhood U of p_0 in M there exists a fixed wedge $\mathcal{W}$ such that any CR function on U extends to $\mathcal{W}$ (Theorem 7). In addition, we prove under this hypothesis that the mini-FBI transforms satisfy uniform estimates.

In Section 5 we explore the relationship between extendability of holomorphic functions in wedges and holomorphic extendability of CR functions. For hypersurfaces holomorphic extendability of CR functions to one side is equivalent to extendability of holomorphic functions from the other side to that side. We consider the analogue of this question in higher codimension and show (Theorem 8) that extendability of CR functions to a wedge always implies extendability of holomorphic functions from another wedge to that wedge. However, we show by an example that the converse is not true in general. Theorem 10 gives a sufficient condition for the converse to hold.

§1. Wave front and FBI transforms of a holomorphic function defined in a wedge with a totally real edge

In this section we consider the case $n = 0$, $\ell > 0$. The generic manifold M is then called a *totally real submanifold* of $\mathbf{C}^\ell$. Near the origin M is given by

$$Im\ w = \varphi(s), \qquad s = Re\ w, \tag{1.1}$$

where φ is a smooth function defined in B_{δ_0}, the ball of radius δ_0 in $\mathbf{R}^\ell$, valued in $\mathbf{R}^\ell$ and satisfying (0.6).

Let $0 < \delta < \delta_0$ satisfying

$$\sup_{|s|<\delta} |\varphi(s)| < \delta \tag{1.2}$$

(which is possible since $\varphi'(0) = 0$) and Γ an open convex cone in $\mathbf{R}^\ell \setminus \{0\}$. We define the wedge $\mathcal{W}_\delta(\Gamma)$ with edge M by

$$\mathcal{W}_\delta(\Gamma) = \{w \in \mathbf{C}^\ell : |Re\ w| < \delta, |Im\ w| < \delta, Im\ w - \varphi(Re\ w) \in \Gamma\}. \tag{1.3}$$

We also set

$$M_\delta = \{w \in \mathbf{C}^\ell : |Re\ w| < \delta, Im\ w - \varphi(Re\ w) = 0\}.$$

By (1.2) M_δ is contained in the boundary of $\mathcal{W}_\delta(\Gamma)$.

If $\gamma \in \Gamma$ satisfies

$$0 < |\gamma| < \delta - \sup_{|s|<\delta} |\varphi(s)| \tag{1.4}$$

we put

$$M_{\delta,\gamma} = \{w \in \mathbf{C}^\ell : Im\ w = \varphi(Re\ w) + \gamma,\ |Re\ w| < \delta\}. \tag{1.5}$$

Then we have

$$M_{\delta,\gamma} \subset \mathcal{W}_\delta(\Gamma). \tag{1.6}$$

For h holomorphic in $\mathcal{W}_\delta(\Gamma)$, written $h \in \mathcal{H}(\mathcal{W}_\delta(\Gamma))$, and $\delta' < \delta$, we define the FBI transform $F_{\delta',\gamma}(w, \sigma)$ of h by

$$F_{\delta',\gamma}(w,\sigma) = \int_{M_{\delta',\gamma}} e^{i(w-\widetilde{w})\sigma - |\sigma|(w-\widetilde{w})^2} \Delta(w - \widetilde{w}, \sigma) h(\widetilde{w}) d\widetilde{w} \tag{1.7}$$

with $\sigma \in \mathbf{C}^\ell$, $w \in \mathbf{C}^\ell$,

$$\Delta(w,\sigma) = \det \frac{\partial \theta}{\partial \sigma}(w,\sigma),\ \theta = \sigma + i\langle\sigma\rangle w \qquad \text{and} \qquad d\widetilde{w} = d\widetilde{w}_1 \wedge \cdots \wedge d\widetilde{w}_\ell.$$

We show first that $F_{\delta',\gamma}$ is independent of the choice of δ' and γ modulo a function of exponential decrease in σ.

(1.8) PROPOSITION. *There exists δ_1, $0 < \delta_1 < \delta$, such that for all δ', δ'', $0 < \delta'' \le \delta' \le \delta_1$ and for all γ', $\gamma'' \in \Gamma$ with $|\gamma'|$, $|\gamma''|$ both less than $\frac{\delta''^2}{2}$, there exist C and $r > 0$, with r independent of γ' and γ'' such that*

$$|F_{\delta',\gamma'}(w,\sigma) - F_{\delta'',\gamma''}(w,\sigma)| \le Ce^{-r|\sigma|} \tag{1.9}$$

for $|w| < r$, $\sigma \in \mathbb{C}^\ell$ with $|Im\ \sigma| < K|Re\ \sigma|$, K independent of γ', γ''. Furthermore, if h is bounded in $\mathcal{W}_\delta(\Gamma)$, then the constant C in (1.9) is also independent of γ', γ''.

Proof: We write

$$F_{\delta',\gamma'} - F_{\delta'',\gamma''} = (F_{\delta',\gamma'} - F_{\delta',\gamma''}) + (F_{\delta',\gamma''} - F_{\delta'',\gamma''}) \tag{1.10}$$

and we shall prove that (1.9) holds for both terms on the right hand side of (1.10). To estimate the first term, let

$$D = \{w + i(\mu\gamma' + (1-\mu)\gamma'');\ \ w \in M_{\delta'}, \qquad 0 < \mu < 1\}.$$

D is an $(\ell+1)$ dimensional manifold contained in $\mathcal{W}_\delta(\Gamma)$ (recall that Γ is convex). Note that

$$\partial D = M_{\delta',\gamma'} \cup M_{\delta',\gamma''} \cup E,$$

with

$$E = \{w + i\mu\gamma' + (1-\mu)\gamma'',\ \ w \in \partial M_{\delta'}, \qquad 0 < \mu < 1\}.$$

Let α be the ℓ-form defined by

$$\alpha = e^{i(w-\tilde{w})\sigma - |\sigma|(w-\tilde{w})^2} h(\tilde{w}) d\tilde{w}_1 \wedge \cdots \wedge d\tilde{w}_\ell.$$

Since h is holomorphic we have

$$d\alpha = 0 \qquad \text{in} \qquad \mathcal{W}_\delta(\Gamma).$$

By Stokes' Theorem applied to $\int_D d\alpha$ we obtain

$$F_{\delta',\gamma'}(w,\sigma) - F_{\delta',\gamma''}(w,\sigma) = \int_E \alpha.$$

It suffices to estimate the quantity

$$Q = |e^{i(w-\widetilde{w})\sigma - |\sigma|(w-\widetilde{w})^2}|$$

for $\widetilde{w} \in E$. When $w = 0$ and $\sigma \in \mathbf{R}^\ell$ we have

(1.11)

$$Q = \exp[-|\sigma|(-\varphi(\widetilde{s})\frac{\sigma}{|\sigma|} - \mu\gamma'\cdot\frac{\sigma}{|\sigma|} - (1-\mu)\gamma''\cdot\frac{\sigma}{|\sigma|} + \widetilde{s}^2 - (\varphi(\widetilde{s}) + \mu\gamma' + (1-\mu)\gamma'')^2)]$$

with $|\widetilde{s}| = \delta'$. If δ_1 is small enough we obtain

$$Q \le e^{-|\sigma|\frac{\delta'^2}{4}},$$

and therefore for $|w| < r$ and $|Im\ \sigma| < K|Re\ \sigma|$, with r and K independent of γ', γ'',

$$|Q| \le e^{-|\sigma|\frac{\delta'^2}{16}}.$$

The second term in the right hand side of (1.10) is easier to estimate. We have

$$\begin{aligned} &F_{\delta',\gamma''}(w,\sigma) - F_{\delta'',\gamma''}(w,\sigma) \\ (1.12) \qquad &= \int_{M_{\delta',\gamma''}\setminus M_{\delta'',\gamma''}} e^{i(w-\widetilde{w})\sigma - |\sigma|(w-\widetilde{w})^2} \Delta(w-\widetilde{w},\sigma) h(\widetilde{w}) d\widetilde{w}; \end{aligned}$$

here again it suffices to estimate the exponent. Details are left to the reader.

We shall now define the analytic wave front set of a holomorphic function defined in a wedge. Let M be a totally real manifold of $\mathbf{C}^\ell$ defined by (0.1) and $\pi : M \to \mathbf{C}^\ell$ the embedding of M in $\mathbf{C}^\ell$. If T_p^*M denotes the cotangent space of M at $p \in M$, then any $\sigma \in T_p^*M$ can be uniquely written

$$\sigma = 2i\sum_{j=1}^{\ell} \lambda_j \pi^*(\partial\rho_j(p)), \tag{1.13}$$

where π^* is the pull back of 1-forms on $\mathbb{C}^\ell$ to M. This easily follows from the independence of $\partial\rho_1, \ldots, \partial\rho_\ell$ and the identity

$$0 \equiv \pi^*(d\rho_j) = \pi^*(\partial\rho_j) + \overline{\pi^*(\partial\rho_j)}.$$

(1.14) Definition: Let $\mathcal{W}(\mathcal{O}, \rho, \Gamma)$ be the wedge defined by (0.3) with totally real edge M, and $h \in \mathcal{H}(\mathcal{W}(\mathcal{O}, \rho, \Gamma))$. We say that a cotangent vector $(p, \sigma) \in T^*M \setminus \{0\}$ is *not in the analytic wave front set of* h (written $(p, \sigma) \notin WFh$ or $\sigma \notin WF_p h$) if there exist open convex cones Γ_j in $\mathbb{R}^\ell$ and $\mathcal{O}' \Subset \mathcal{O}$, $p \in \mathcal{O}'$, and $h_j \in \mathcal{H}(\mathcal{W}(\mathcal{O}', \rho, \Gamma_j))$ satisfying

$$\bigcap_{j=1}^{r} \Gamma_j \cap \Gamma \neq \emptyset, \tag{1.15}$$

$$h = \sum_j h_j \quad \text{in} \quad \mathcal{W}(\mathcal{O}', \rho, \bigcap_j \Gamma_j \cap \Gamma), \tag{1.16}$$

and for each j, $j = 1, \ldots, r$, there exists $\gamma_j \in \Gamma_j$ with

$$\lambda \cdot \gamma_j < 0 \tag{1.17}$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is defined by (1.13).

The reader can easily check that Definition (1.14) is independent of the choice of the defining functions ρ_j; it suffices to use the invariance of the wedge $\mathcal{W}(\mathcal{O}, \rho, \Gamma)$ under the change of defining functions as discussed in §1 (see in particular (0.4)).

(1.18) Remark: For a cone Γ we let $\Gamma^0 = \{\tau \in \mathbb{R}^\ell \setminus \{0\};\ \tau \cdot \gamma > 0$ for all $\gamma \in \Gamma\}$. If, in addition, Γ is open, then $\Gamma^0 = \{\tau : \tau \cdot \gamma \geq 0$ all $\gamma \in \Gamma\}$. It is therefore clear that if h is as in Definition (1.14) and if $(p, \sigma) \in WFh$, and λ is defined by (1.13) then $\lambda \in \Gamma^0$.

§2. Main results for a wedge with a totally real edge

If M is given by (1.1) and $p = 0$, then (1.13) becomes

$$\sigma = \sum_{j=1}^{\ell} \lambda_j ds_j$$

which we use to identify $T_0^* M$ with $\mathbf{R}^\ell$ and σ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The following connects the definition of WFh with the exponential decrease of the FBI transform introduced in §1.

THEOREM 1. *Let h be holomorphic in a wedge $\mathcal{W}_\delta(\Gamma)$ defined by (1.3) with totally real edge, and $\sigma_0 \in \mathbf{R}^\ell \setminus \{0\}$. The following are equivalent.*

(i) $(0, \sigma_0) \notin WFh$.

(ii) *There exists δ_1, $0 < \delta_1 < \delta$, such that for all $\delta' < \delta_1$ and all $\gamma \in \Gamma$ with $|\gamma| < \frac{\delta'^2}{2}$ there exist $C > 0$ and $r > 0$, r independent of γ, such that*

$$|F_{\gamma,\delta'}(w,\sigma)| \le Ce^{-r|\sigma|} \tag{2.1}$$

for $|w| < r$ and $\sigma \in \mathcal{C}$, a conic neighborhood of σ_0, in $\mathbf{C}^\ell \setminus \{0\}$ independent of γ.

It should be noted that by using Proposition (1.8), condition (ii) could be weakened by assuming that (2.1) holds for some $\gamma \in \Gamma$.

We first need the following, which shows that (i) implies (ii) for $\sigma_0 \notin \Gamma^0$ (see Remark (1.18)).

(2.2) LEMMA. *Let h be as in Theorem 1 and $\sigma_0 \notin \Gamma^0$. Then (ii) holds.*

Proof: By Proposition (1.8), it suffices to show (2.1) for $\gamma = \gamma_0$ with $\gamma_0 \in \Gamma$, $|\gamma_0| < \frac{\delta'^2}{2}$, and δ_1 as in the Proposition. We may choose γ_0 so that $\sigma_0 \cdot \gamma_0 + |\gamma_0|^2 < 0$, which exists by choosing $\sigma_0 \cdot \gamma_0 < 0$, $|\gamma_0|$ sufficiently small. It suffices to estimate $|e^{i(w-\widetilde{w})\sigma-|\sigma|(w-\widetilde{w})^2}|$ when $w=0$ and $\sigma=\lambda\sigma_0$, $\lambda>0$, $|\sigma_0|=1$. We obtain

$$e^{-\lambda[-\varphi(s)\cdot\sigma^0-\sigma^0\gamma_0+s^2-|\varphi(s)+\gamma_0|^2]}<e^{\lambda(\sigma_0\gamma_0+|\gamma_0|^2)}. \tag{2.3}$$

If $|w|$ is sufficiently small, and if $|\sigma-\sigma_0|$ is small, with $|\sigma|=1$, then (2.1) holds.

We shall now prove Theorem 1. First assume $\sigma_0\notin WF_0h$. Since (1.15), (1.16) and (1.17) hold we have $h=\sum h_j$ in $\mathcal{W}_{\delta'}(\cap\Gamma_j\cap\Gamma)$ for some $0<\delta'<\delta$. It suffices to prove (2.1) for each h_j. By Proposition (1.8) it suffices to prove the estimate for some (γ_j,δ_j), $\gamma_j\in\Gamma_j$, $|\gamma_j|<\frac{\delta_j^2}{2}$. Since $\sigma_0\cdot\gamma_j<0$ for some $\gamma_j\in\Gamma_j$, we obtain (2.1) by Lemma (2.2) and Proposition (1.8).

Next we assume (ii) holds. We will show $\sigma_0\notin WF_0h$. For $\gamma\in\Gamma$, $|\gamma|\leq\delta-\sup_{|s|<\delta}|\varphi(s)|$, we define h_γ on M_δ by

$$h_\gamma(w)=h(w+i\gamma). \tag{2.4}$$

For $f:M_\delta\to\mathbf{C}$ we may define its FBI transform as in [3] (where it is called the mini-FBI transform) by

$$I_{\delta'}(f;w,\sigma)=\int_{M_{\delta'}}e^{i(w-\widetilde{w})\sigma-|\sigma|(w-\widetilde{w})^2}\Delta(w-\widetilde{w},\sigma)f(\widetilde{w})d\widetilde{w}. \tag{2.5}$$

Then we have

$$F_{\delta',\gamma}(w,\sigma)=I_{\delta'}(h_\gamma;w-i\gamma,\sigma). \tag{2.6}$$

By the inverse formula for the FBI transform of functions (see [3]), we have

$$h_\gamma(s+i\varphi(s))=\frac{1}{(2\pi)^\ell}\int_{\mathbf{R}^\ell}F_{\delta',\gamma}(s+i\varphi(s)+i\gamma,A(s)\sigma)\det A(s)d\sigma \tag{2.7}$$

for $|s|<\delta'$ where $A(s)={}^t(I+i\varphi'(s))$.

We shall decompose the right hand side of (2.7) into a sum of integrands. We may assume $\sigma_0 \in \Gamma^0$, since otherwise there is nothing to prove by the definition of the wave front. Let $\mathcal{E}_1 \Subset \mathbb{R}^\ell \setminus \Gamma^0$ and $\mathcal{E}_2 \Subset \mathcal{C}$, $\sigma^0 \in \mathcal{E}_2$, and choose $\mathcal{C}_j$, $j = 1, \dots, m$ so that

$$(2.8) \qquad \mathbb{R}^\ell \setminus (\mathcal{E}_1 \cup \mathcal{E}_2) \subset \bigcup_{j=1}^{m} \mathcal{C}_j,$$

with $\mathcal{C}_j$ closed, strictly convex, and $\sigma_0 \notin \mathcal{C}_j$ for all j. Then $\bigcap_{j=1}^{m} \mathcal{C}_j^\circ \cap \Gamma \neq \emptyset$, if $\mathcal{E}_1$ is chosen sufficiently large.

For $|\gamma| < \frac{r}{4}$

$$\frac{1}{(2\pi)^\ell} \int_{\sigma \in \mathcal{E}_1 \cup \mathcal{E}_2} F_{\delta',\gamma}(s + i\varphi(s) + i\gamma, A(s)\sigma) \det(A(s)) d\sigma$$

extends to a holomorphic function

$$(2.9) \qquad h_\gamma^0(w) = \frac{1}{(2\pi)^\ell} \int_{\sigma \in \mathcal{E}_1 \cup \mathcal{E}_2} F_{\delta',\gamma}(w + i\gamma, A(s)\sigma)(\det A(s)) d\sigma$$

for $|w| < \frac{r}{2}$. Then $h_0(w) = h_\gamma^0(w - i\gamma)$ is still holomorphic in a neighborhood of the origin.

Now let

$$(2.10) \quad h_\gamma^j(s + i\varphi(s)) = \frac{1}{(2\pi)^\ell} \int_{\sigma \in \mathcal{B}_j} F_{\gamma,\delta'}(s + i\varphi(s) + i\sigma, A(s)\sigma) \det A(s) d\sigma.$$

where $\mathcal{B}_j = \mathcal{C}_j \cap (\mathbb{R}^\ell \setminus (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \bigcup_{k<j} \mathcal{C}_k))$. We shall show that h_γ^j extends to be holomorphic in a wedge. Here the argument is similar to that given in §6 of [3]. Indeed it suffices to estimate

$$(2.11) \qquad |e^{i(w-\widetilde{w})\cdot A(s)\sigma - |A(s)\cdot\sigma|(w-\widetilde{w})^2}|,$$

where $w = s + i\varphi(s) + it$, $t \in \mathcal{C}_j^0$. For this, we observe that for $\widetilde{w} = \widetilde{s} + i\varphi(\widetilde{s})$ and $w = s + i\varphi(s)$ the term $|Re(w - \widetilde{w})A(s)| < \frac{|Re(w-\widetilde{w})^2|}{2}$ for δ' small. Since

$A(s)\sigma = \sigma + \mathcal{O}(s)\cdot\sigma$ and $t\cdot\sigma > 0$, we conclude the desired holomorphy i.e. that $h^j_\gamma(w+i(t-\gamma))$ is defined and holomorphic for $t \in \mathcal{C}^0_j$. Now let $h_j(s+i\varphi(s)+it) = h^j_\gamma(s+i\varphi(s)+i(t-\gamma))$ which is holomorphic in $\mathcal{W}_{\delta''}(\mathcal{C}^0_j)$. For $t \in \bigcap_{j=1}^{m} \mathcal{C}^0_j \cap \Gamma$, we have

$$h(s+i\varphi(s)+it) = \sum h_j(s+i\varphi(s)+it).$$

This completes the proof of Theorem 1.

The following is a corollary of the proof of Theorem 1.

THEOREM 2. *Let $h \in \mathcal{H}(\mathcal{W}_\delta(\Gamma))$. Then the following are equivalent:*

(i) *h extends holomorphically near 0,*

(ii) $WF_0 h = \emptyset$,

(iii) *there exists δ_1, $0 < \delta_1 < \delta$, such that for all δ', $0 < \delta' < \delta_1$, all $\gamma \in \Gamma$, $\gamma < \frac{\delta'^2}{2}$ there exist C and $r > 0$, with r independent of γ, such that $|F_{\delta',\gamma}(w,\sigma)| \le Ce^{-r|\sigma|}$, for $|w| < r$ and σ in a conic neighborhood of $\mathbb{R}^\ell$ in $\mathbb{C}^\ell$ independent of γ.*

Proof: That (i) implies (ii) follows immediately from the definition of wave front, and (ii) implies (iii) is contained in Theorem 1. Finally, (iii) implies (i) is proved as in the proof of (ii) implies (i) in Theorem 1, for the case where there are no $\mathcal{C}_j$.

The following result addresses the question of extending $h \in \mathcal{H}(\mathcal{W}_\delta(\Gamma))$ to a function holomorphic in a larger wedge.

THEOREM 3. *Let $h \in \mathcal{H}(\mathcal{W}_\delta(\Gamma))$ and $\mathcal{C}$ a closed strictly convex cone satisfying $\mathcal{C} \subset \Gamma^0$. Then the following are equivalent.*

(i) *For all open convex cones $\Gamma' \Subset \mathcal{C}^0 + \Gamma$ there exists δ_1, $0 < \delta_1 < \delta$, such that h extends holomorphically to $\mathcal{W}_{\delta_1}(\Gamma')$.*

(ii) $WF_0 h \subset \mathcal{C}$.

Proof: In order to prove that (i) implies (ii) we take $\Gamma' \Subset \mathcal{C}^0$ and observe that by (i) and Remark (1.18) that

$$WF_0 h \subset (\Gamma')^0.$$

Since $\mathcal{C} \subset (\Gamma')^0$ and $(\Gamma')^0$ is arbitrarily close to $\mathcal{C}$, (ii) follows.

Now suppose (ii) holds. It suffices to assume $\Gamma' = \Gamma_1 + \Gamma''$, $\Gamma_1 \Subset \mathcal{C}^0$ and $\Gamma'' \Subset \Gamma$. By the arguments of Theorem 1, since $h \in \mathcal{H}(\mathcal{W}_\delta(\Gamma))$ there exist δ_1, r_2 and C_2 positive such that (2.1) holds with these constants for all $\sigma \notin \Gamma_2$, where $\Gamma^0 \Subset \Gamma_2 \Subset (\Gamma'')^0$. Now by (ii) we may find δ_3, $r_3 > 0$, and $C_3 > 0$ such that (2.1) holds with these constants for $\sigma \notin \Gamma_3$, for Γ_3 satisfying $\mathcal{C} \Subset \Gamma_3 \Subset \Gamma_1^0$. Hence we may find δ_1, r_1, C_1 such that (2.1) holds for all $\sigma \notin \Gamma_2 \cap \Gamma_3$. Now we use the inversion formula (2.7), breaking the integral into two parts:

$$\int_{\mathbf{R}^\ell \setminus \Gamma_2 \cap \Gamma_3} + \int_{\Gamma_2 \cap \Gamma_3}.$$

Since the first integral is holomorphic in a fixed neighborhood of the origin uniformly in γ, we conclude that h extends as a holomorphic function in $\mathcal{W}_{\delta_1}(\Gamma_4)$ for any strictly convex cone Γ_4 satisfying

$$\Gamma_4 \Subset (\Gamma_2 \cap \Gamma_3)^0.$$

Since $(\Gamma_2 \cap \Gamma_3)^0 = \Gamma_2^0 + \Gamma_3^0 \Supset \Gamma'' + \Gamma_1$, we are done.

(2.12) Remark: If $h \in \mathcal{H}(\mathcal{W}_\delta(\Gamma))$ has tempered growth near M i.e. $|h(w)| \leq C \operatorname{dist}(w, M)^{-N}$ for some $N \geq 0$, then the boundary value of h is defined as a CR distribution on M, denoted by $f = bh$. The hypoanalytic wave front set of f, as defined in [2] and [3] coincides with WFh given in this paper. Indeed, both wavefront sets are characterized by the exponential decrease of the FBI transform. By reducing to the case where h is bounded, we may pass to the limit as $\gamma \to 0$ and conclude the result by Proposition (1.8).

§3. Holomorphic functions in wedges with generic edges

Let M be a generic manifold defined by (0.5) with $\varphi : B^{2n}_{\eta_0} \times B^{\ell}_{\delta_0} \to \mathbf{R}^{\ell}$ satisfying (0.6). For $z \in B^{2n}_{\eta_0}$ let M^z be the submanifold of $\mathbf{C}^{\ell}$ defined by

$$M^z = \{w \in \mathbf{C}^{\ell} : Re\ w \in B^{\ell}_{\delta_0}, \qquad Im\ w - \varphi(z, \bar{z}, Re\ w) = 0\}.$$

For each fixed z, M^z is totally real. By analogy to §1, we fix η and δ small enough such that

$$\sup_{\substack{|z|<\eta \\ |s|<\delta}} |\varphi(z, \bar{z}, s)| < \delta. \tag{3.1}$$

As in (1.3) we define, for $|z| < \eta$, the wedge $\mathcal{W}^z_\delta(\Gamma)$ by

$$\mathcal{W}^z_\delta(\Gamma) = \{w \in \mathbf{C}^{\ell}, |Re\ w| < \delta, |Im\ w| < \delta, Im\ w - \varphi(z, \bar{z}, Re\ w) \in \Gamma\}. \tag{3.2}$$

Similarly we write

$$M^z_\delta = \{w \in \mathbf{C}^{\ell} : |Re\ w| < \delta, Im\ w - \varphi(z, \bar{z}, Re\ w) = 0\}. \tag{3.3}$$

The open subset of $\mathbf{C}^{n+\ell}$, $\mathcal{W}_{\delta,\eta}(\Gamma)$, defined by

$$\mathcal{W}_{\delta,\eta}(\Gamma) = \bigcup_{|z|<|\eta} \{z\} \times \mathcal{W}^z_\delta(\Gamma) \tag{3.4}$$

is a *wedge of edge M in the direction* Γ.

For $z \in B_\eta$, $\gamma \in \Gamma$ with

$$0 < |\gamma| < \delta - \sup_{\substack{|z|<\eta \\ |s|<\delta}} |\varphi(z, \bar{z}, s)|, \tag{3.5}$$

we define $M^z_{\delta,\gamma}$ by the analogue of (1.5).

For $h \in \mathcal{H}(\mathcal{W}_{\delta,\eta}(\Gamma))$, $0 < \delta' < \delta$, $\gamma \in \Gamma$ satisfying (3.5) and $z \in B_\eta^{2n}$ we define its FBI transform $F_{\delta',\gamma}(z,w,\sigma)$ by the right hand side of (1.7) where $M_{\delta',\gamma}$ is replaced by $M_{\delta',\gamma}^z$, and $h(\widetilde{w})$ by $h(z,\widetilde{w})$. The reader should be cautioned that the FBI transform of h, though an entire function in w, is not holomorphic in z. An analogue of Proposition (1.8) is still valid provided z is sufficiently small. The only difference in the proof is in estimating the exponent in (1.11) and similarly in (1.12)). Here $\varphi(z,\overline{z},0)$, $\varphi_s'(z,\overline{z},0)$ and $\varphi_{ss}''(z,\overline{z},0)$ do not vanish but can be made arbitrarily small by taking $|z|$ small.

The analogue of Definition (1.14) is the following.

(3.6) Definition: Let $h \in \mathcal{H}(\mathcal{W}_{\delta,\eta}(\Gamma))$ and $\sigma_0 \in \mathbf{R}^\ell \setminus 0$. We say that $(0,\sigma_0) \notin WFh$ if there exist open convex cones Γ_j in $\mathbf{R}^\ell$, $0 < \delta' < \delta$, $0 < \eta' < \eta$ and $H_j(z,w)$ smooth functions defined in $\mathcal{W}_{\delta',\eta'}(\Gamma_j)$, holomorphic in $w \in \mathcal{W}_{\delta'}^z(\Gamma_j)$ for $|z| < \eta'$, satisfying

$$(3.7)\qquad \begin{aligned} &\bigcap_{j=1}^{r} \Gamma_j \cap \Gamma \neq \emptyset \\ &h = \sum H_j \qquad \text{in } \mathcal{W}_{\delta',\eta'}\Big(\bigcap_j \Gamma_j \cap \Gamma\Big), \end{aligned}$$

and for each j, $j = 1, \dots, r$, there exists $\gamma_j \in \Gamma_j$ with

$$\sigma_0 \cdot \gamma_j < 0.$$

We now state the analogue of Theorem 1.

THEOREM 4. *Let $h \in \mathcal{H}(\mathcal{W}_{\delta,\eta}(\Gamma))$. The following hold.*

(i) $WF_0 h \subset \Gamma^0$.

(ii) *If $\sigma_0 \in \mathbf{R}^\ell \setminus \{0\}$, then $(0,\sigma_0) \notin WFh$ if and only if there exists δ_1, $0 < \delta_1 < \delta$, such that for all δ' with $0 < \delta' < \delta_1$, there exist $r > 0$, η' satisfying $0 < \eta' < \eta$, and $\mathcal{C}$, a conic neighborhood of σ_0 in $\mathbf{C}^\ell$ with the following:*

For every $\gamma \in \Gamma$, $|\gamma| < \frac{\delta'^2}{2}$, there exists C such that

$$|F_{\delta',\gamma}(z,w,\sigma)| \leq Ce^{-r|\sigma|} \tag{3.8}$$

for $|z| < \eta'$, $|w| < r$, and $\sigma \in \mathbb{C}$.

(iii) *If h has tempered growth at M then $WFh = WFf$, where f is the CR distribution boundary value of h on M, and WFf is its hypoanalytic wave front set.*

Proof: The proof of (i) is similar to that of Remark (1.18), the proof of (ii) follows that of Theorem 1, and (iii) is the analogue of Remark (2.12).

The following is the analogue of Theorem 2.

THEOREM 5. *If h is as in Theorem 4 then the following are equivalent:*

(i) *h extends holomorphically near 0,*

(ii) *$WF_0 h = \emptyset$,*

(iii) *For every $\sigma_0 \in \mathbb{R}^\ell \setminus \{0\}$ condition (3.8) of Theorem 4 holds.*

Proof: The proof that (i) implies (ii) implies (iii) proceeds as in Theorem 2.

The proof of (iii) implies (i) proceeds similarly to that of (ii) implies (i) in Theorem 1, but with some significant additions. For $\gamma \in \Gamma$, satisfying (3.5), we define h_γ by

$$h_\gamma(z,w) = h(z, w + i\gamma).$$

Clearly h_γ is defined on M, for $|z| < \eta$ and $|s| < \delta$. Moreover the restriction of h_γ to M is a CR function.

The FBI transform $F_{\delta',\gamma}(z,w,\sigma)$ is in fact the mini-FBI transform, as defined in [3], of χh_γ, where χ is the characteristic function of the subset of M defined by $|s| < \delta'$. It is proved in [3] that if the mini-FBI transform of a CR

function f satisfies estimates of the form (3.8) then f extends to be holomorphic in a neighborhood of the origin. By carefully following the proof one can show that the domain of extendability depends only on the domain of definition of f and the constant r in (3.8), and not on f. Therefore in this case the domain of extendability of h_γ is independent of γ. By choosing $|\gamma|$ sufficiently small we obtain that h extends holomorphically in a neighborhood of 0.

The following is the analogue of Theorem 3.

THEOREM 6. *Let h be as in Theorem 4 and $\mathcal{C}$ a closed strictly convex cone satisfying $\mathcal{C} \subset \Gamma^0$. Then the following are equivalent.*

(i) *For all open convex cones $\Gamma' \Subset \mathcal{C}^0 + \Gamma$ there exist δ_1, η_1, $0 < \delta_1 < \delta$, $0 < \eta_1 < \eta$, such that h extends holomorphically to $\mathcal{W}_{\delta_1,\eta_1}(\Gamma')$.*

(ii) *$WF_0 h \subset \mathcal{C}$.*

§4. Extendability of CR functions to a fixed wedge and uniform estimates.

The remainder of this paper deals with the connections between extension of holomorphic functions defined in wedges with edge M and holomorphic extension of CR functions defined on M. We first give some results on extension of CR functions which will be used in what follows.

THEOREM 7. *Let M be a generic manifold and $p_0 \in M$. Suppose that for any continuous CR function f defined on M near p_0 there is a wedge of edge M to which f extends holomorphically. Then for every open neighborhood U of $p_0 \in M$ there is a wedge $\mathcal{W}$ (of the form (0.3)) such that if f is a continuous CR function defined on U, then f extends holomorphically to $\mathcal{W}$.*

If, in addition, $\mathcal{C}$ is a convex closed cone for which $WF_{p_0} f \subset \mathcal{C}$ for all CR functions f defined in U, then for every $\Gamma \Subset \mathcal{C}^0$ there exists a wedge $\mathcal{W}(\mathcal{O}, \rho, \Gamma)$ to which every f extends holomorphically.

We need the following.

(4.1) LEMMA. *One can find a sequence* $\{\Gamma_j\}$ *of open strictly convex cones in* $\mathbb{R}^\ell \setminus \{0\}$ *such that if* Γ *is any closed strictly convex cone in* $\mathbb{R}^\ell \setminus \{0\}$ *then* $\Gamma \subset \Gamma_j$ *for some* j.

Proof of Lemma: Let Γ be a closed strictly convex cone. Suppose first that $\Gamma \subset \{\xi_1 > 0\}$. Then for some $m > 0$

$$\Gamma \subset \{(\xi_1, \xi') : \xi_1 > \frac{1}{m}|\xi'|\} = \Gamma^m.$$

Now if Γ is arbitrary, there is a rational rotation σ_k (i.e. the entries of the matrix of σ_k are rational) such that $\sigma_k(\Gamma) \subset \Gamma^m$. Now the desired sequence is obtained by applying all rational rotations σ_k to the sequence of cones Γ^m.

We shall now prove Theorem 7. We may assume that M is given by $\rho = 0$, where ρ is given by (0.5) with $z \in B^{2n}_{\eta_0}$, $s \in B^\ell_{\eta_0}$ and $p_0 = 0$. We take U to be parametrized by $z \in B^{2n}_\eta$ and $s \in B^\ell_\delta$, with $0 < \eta < \eta_0$, and $0 < \delta < \delta_0$. For a continuous CR function f on U we define its FBI transform (or mini-FBI transform in the sense of [3]) by

$$F_{\delta'}(f, z, w, \sigma) = \int_{M^z_{\delta'}} e^{i(w-\tilde{w})\sigma - |\sigma|(w-\tilde{w})^2} \Delta(w - \tilde{w}, \sigma) f(z, \tilde{w}) d\tilde{w} \tag{4.2}$$

with $0 < \delta' < \delta$, and $M^z_{\delta'}$ defined by (3.3).

Let $\{\Gamma_k\}$ be the sequence constructed in Lemma (4.1) and let E be the Banach space of CR functions continuous and bounded in U with norm

$$\|f\|_E = \sup_{(z,w)\in U} |f(z, w)|. \tag{4.3}$$

For any triple (j, k, m) of positive integers with $\frac{1}{j} < \delta$, $\frac{1}{m} < \rho$, we define a subset E_{jkm} of E by

(4.4)

$$E_{jkm} = \{f \in E : \|f\|_E \le m, \text{ and } |F_{\frac{1}{j}}(f, z, w, \sigma)| \le m e^{-\frac{1}{m}|\sigma|}$$
$$\text{for all } z, w, \sigma \text{ with } |z| \le \frac{1}{m}, |w| \le \frac{1}{m}, |Im\ \sigma| \le \frac{1}{m}|Re\ \sigma| \text{ and } Re\ \sigma \notin \Gamma_k\}$$

(4.5) LEMMA. *Each E_{jkm} is closed in E and*

$$E = \bigcup E_{jkm}. \tag{4.6}$$

Proof: We first show (4.6). By assumption, for each $f \in E$ there exists a wedge $\mathcal{W}_{\delta'',\rho''}(\Gamma_1)$, defined by (3.4), to which f extends holomorphically. Hence by [3], there exist j, m, k such that $f \in E_{jkm}$. Indeed if Γ is any closed strictly convex cone in $\mathbb{R}^\ell \setminus \{0\}$ for which Γ_1^0 is contained in the interior of Γ, then we may apply Lemma (4.1) and take k so that $\Gamma \subset \Gamma_k$. Then, if j is chosen sufficiently large, we may find m so that $f \in E_{jkm}$. This proves (4.6).

To show E_{jkm} is closed let $\{f_q\}$ be a sequence in E_{jkm} with limit $f \in E$. It is clear by passing to the limit that the inequality in (4.4) still holds for f. This proves Lemma (4.5).

We may now complete the proof of Theorem 7. By the Baire Category Theorem, at least one of the E_{jkm}, say $E_{j_0k_0m_0}$, has nonempty interior. Since $E_{jkm} = -E_{jkm}$ for all j, k, m, it follows that the zero function belongs to the interior of $E_{j_0k_0m_0}$. Hence $E = \bigcup_{p=1}^{\infty} pE_{j_0k_0m_0}$. Hence, for every $f \in E$ there exists an integer p such that

$$\frac{f}{p} \in E_{j_0k_0m_0}.$$

Now choose an open strictly convex cone $\Gamma'' \Subset \Gamma_{k_0}^0$. Then by [3], f extends to $\mathcal{W}_{\delta'',\eta''}(\Gamma'')$, where δ'', η'' are independent of f.

For the last statement of the Theorem it suffices to take $\Gamma_k = \Gamma'$, where $\Gamma \Subset \Gamma' \Subset \mathcal{C}^0$, dropping the index k in (4.6). This completes the proof of Theorem 7.

(4.7) Remark: In the course of the proof of Theorem 7 we have obtained the following uniform estimates. There exist a strictly convex open cone Γ and positive constants δ, r such that if f is a continuous CR function defined on U, then its FBI transform satisfies the estimate

$$|F_\delta(f, z, w, \sigma)| \leq C_f e^{-r|\sigma|}$$

for $|z| < r$, $|w| < r$, $Re\ \sigma \notin \Gamma$, $|Im\ \sigma| \leq r|Re\ \sigma|$.

It is shown in [3] that if a generic manifold M is of finite type and semi-rigid at $p_0 \in M$ (in the sense of [3]) then any CR function defined near p_0 extends holomorphically to a wedge of edge M. Therefore, combining this result with Theorem 7 we obtain

(4.8) COROLLARY. *Let M be a generic semi-rigid manifold of finite type at p_0. Then for every open neighborhood U of p_0 there exists a fixed wedge $\mathcal{W}$ of edge M to which every continuous CR function on U extends holomorphically.*

(4.9) Example: Let $n = 1$, $\ell = 2$, and M given by $Im\ w_1 = -|z|^4$ and $Im\ w_2 = |z|^2$. We claim that for any open U, $0 \in U \subset M$, and any $R > 0$, there exists a wedge $\mathcal{W}_{\delta,\eta}(\Gamma)$ such that

$$\Gamma \cap \{\sigma : \sigma_2 > R|\sigma_1|\} \neq \emptyset \tag{4.10}$$

to which every CR function on U extends. Indeed, since M is of finite type and rigid, by Theorem III.3 of [4] and the first part of Theorem 7, there exists a wedge $\mathcal{W}_{\delta_1,\eta_1}(\Gamma_1)$ to which every CR function on U extends. On the other hand, by Theorem III.1 of [4] it follows that if f is a CR function defined near 0 in M we have

$$WF_0 f \subset \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 \setminus \{0\} : \sigma_2 > 0 \text{ or } \sigma_2 = 0 \text{ and } \sigma_1 < 0\} = S. \tag{4.11}$$

Therefore, by Theorem II.2 of [4] we have $WF_0 f \subset S \cap \Gamma_1^0$. Now let $\mathcal{C}$ be the closure of $S \cap \Gamma_1^0$ in $\mathbb{R}^2 \setminus \{0\}$. Then $\mathcal{C}$ is a closed strictly convex cone contained in the half plane $\sigma_2 \geq 0$. Our claim then follows by applying the second part of Theorem 7.

§5. Relation between extension of holomorphic functions in wedges and extension of CR functions

The first result of this section shows that holomorphic extendability of all CR functions implies extendability of all holomorphic functions defined in a wedge. More precisely we have:

THEOREM 8. *Let M be a generic manifold defined by $\rho = 0$, where ρ is given by (0.5). Assume there exists U, an open neighborhood of 0 in M, and a wedge $\mathcal{W}_{\delta,\eta}(\Gamma)$ to which every continuous CR function f on U extends holomorphically. Then if h is holomorphic in a wedge $\mathcal{W}_{\delta_1,\eta_1}(\Gamma_1)$ such that $M \cap \{|z| < \eta_1, |Re\ w| < \delta_1\} \supset U$ and if Γ' is an open cone, $\Gamma' \Subset \Gamma_1 + \Gamma$, then h extends to a wedge of the form $\mathcal{W}_{\delta',\eta'}(\Gamma')$.*

Proof: Let $0 < \delta' < \delta_1$ and $\gamma \in \Gamma_1$ with $|\gamma| < \frac{\delta'^2}{2}$. As in §3, we write,

$$F_{\delta',\gamma}(z,w,\sigma) = F_{\delta'}(h_\gamma, z, w, \sigma), \tag{5.1}$$

where $F_{\delta'}$ is defined by (4.2) and, for $(z,w) \in M$,

$$h_\gamma(z,w) = h(z, w + i\gamma).$$

Note that h_γ is a CR function defined on U. By the assumption, h_γ extends to a wedge $\mathcal{W}_{\delta,\rho}(\Gamma)$. Therefore, by Remark (4.7), for every Γ'', with $\Gamma'' \subset$ int Γ^0, we have

$$|F_{\delta'',\gamma}(z,w,\sigma)| \le C_\gamma e^{-r|\sigma|},$$

with δ'' and r independent of γ, for $|z| \le r$, $|w| \le r$, $|Im\ \sigma| < r|Re\ \sigma|$, $Re\ \sigma \notin \Gamma''$. Hence, by Theorem 4 (ii), $WF_0 h \cap \Gamma'' = \emptyset$, which implies $WF_0 h \subset \Gamma^0$ and therefore by Theorem 4 (i),

$$WF_0 h \subset \Gamma^0 \cap \Gamma_1^0.$$

We now apply Theorem 6 with $\mathcal{C} = \Gamma^0 \cap \Gamma_1^0 = (\Gamma + \Gamma_1)^0$ to complete the proof of Theorem 8.

The following corollary is straightforward.

(5.2) COROLLARY. *Under the assumptions of Theorem 8 if* $\Gamma + \Gamma_1 = \mathbb{R}^\ell \setminus \{0\}$ *then* h *extends holomorphically in a neighborhood of* 0 *in* $\mathbb{C}^{n+\ell}$.

(5.3) Example: Let M be as in Example (4.9). We claim that if $\Gamma_1 = \{\sigma \in \mathbb{R}^2 \setminus \{0\} : \epsilon\sigma_2 < -|\sigma_1|\}$ then any holomorphic function in a wedge of the form $\mathcal{W}_{\delta_1,\eta_0}(\Gamma_1)$ extends as a holomorphic function in a neighborhood of 0 in $\mathbb{C}^3$. Indeed, it suffices to apply the Corollary and the claim in Example (4.9) with $R = \frac{1}{\epsilon}$.

For generic semi-rigid manifolds of finite type as defined in [3] we have the following result.

THEOREM 9. *Let* M *be a generic semi-rigid manifold of finite type at* 0 *defined by* $\rho = 0$, *where* ρ *is given by (0.5). Then for every open neighborhood* U *of* 0 *in* M *there exists a strictly convex open cone* $\Gamma \subset \mathbb{R}^\ell$ *such that if* h *is holomorphic in a wedge* $W_{\delta_1,\eta_1}(\Gamma_1)$, *with* $U \subset M \cap \{|z| < \eta_1, |Re\, w| < \delta_1\}$, *and if* Γ^1 *is any open cone,* $\Gamma' \Subset \Gamma_1 + \Gamma$, *then* h *extends to a wedge of the form* $W_{\delta',\eta'}(\Gamma')$. *In particular if* M *is a hypersurface of finite type at* 0, *then there exists a side of* M *such that every function* h *holomorphic on that side of* M *extends holomorphically to a neighborhood of* 0 *in* $\mathbb{C}^{n+1}$.

Proof: Theorem 9 is an immediate consequence of Corollary (4.8) and Theorem 8. The hypersurface case is a known result.

In general, extension of all holomorphic functions defined in a wedge does not imply holomorphic extension of all CR functions to some wedge; that is, the converse of Theorem 8 does not hold in general. This is shown by the following example which was introduced by Trépreau [13] to show that there exist CR manifolds for which not every CR function can be decomposed as a sum of boundary values of holomorphic functions in wedges. (However, if such a decomposition always holds on M then the converse of Theorem 8 is true (Theorem 9).)

(5.4) Example: Let $n = 1$, $\ell = 2$ and $M \subset \mathbf{C}^3$ defined by

$$Im\ w_1 - x^2 s_2 = 0 \qquad \text{and} \qquad Im\ w_2 + x^2 s_1 = 0 \tag{5.5}$$

with $s_j = Re\ w_j$, $j = 1,2$, and $z = x + iy \in \mathbf{C}$. Using a result of propagation of analytic singularities due to Hanges and Sjöstrand [7] one can check that if f is a CR function defined near the origin on M, given by (5.5), then

$$WF_0 f \text{ is either empty or all of } \mathbf{R}^\ell \setminus \{0\}. \tag{5.6}$$

On the other hand, by Corollary (2.16) of [3] we can find a CR function f of class C^1 which does not extend to any wedge of edge M. However, we claim that any $h \in \mathcal{H}(\mathcal{W}_{\delta,\rho}(\Gamma))$, where δ, ρ, Γ are arbitrary, extends to be holomorphic in some neighborhood of the origin in $\mathbf{C}^{n+\ell}$, as is proved by the following.

(5.7) LEMMA. *Let M be a generic manifold defined by $\rho = 0$, where ρ is given by (0.5). Assume that (5.6) holds for every CR function f on M, holds. Then any holomorphic function h in any wedge $\mathcal{W}_{\delta,\eta}(\Gamma)$ extends to be holomorphic in a neighborhood of* 0 *in* $\mathbf{C}^{n+\ell}$.

Proof: Let $U = M \cap \{|z| < \eta,\ |Re\ w| < \delta\}$ and E be the space of continuous CR functions on U which extend holomorphically to $\mathcal{W}_{\delta,\eta}(\Gamma)$ with bounded sup norm on $U \cup \mathcal{W}_{\delta,\eta}(\Gamma)$. E is clearly a Banach space for this norm. For any $f \in E$, $WF_0 f \neq \mathbf{R}^\ell \setminus \{0\}$, since $WF_0 f \subset \Gamma^0$, by Theorem 7 of [3]. By assumption (5.6) we must have $WF_0 f = \emptyset$ and hence f extends holomorphically in a neighborhood of 0 in $\mathbf{C}^{n+\ell}$. By using the Baire Category Theorem (as in the proof of Theorem 7) we conclude that there exists a fixed neighborhood $\mathcal{O}$ of 0 in $\mathbf{C}^{n+\ell}$ to which any function f in E extends holomorphically.

To complete the proof of the lemma, let $h \in \mathcal{H}(\mathcal{W}_{\delta,\eta}(\Gamma))$. Then if $\gamma \in \Gamma$ is sufficiently small, the CR function h_γ defined by (5.2) is in E (after shrinking δ, η, Γ if necessary.) Therefore we conclude that h_γ extends holomorphically to

0, and the desired conclusion follows. This proves the lemma and establishes the claim in Example (5.4).

A generic manifold M has the ***holomorphic decomposition property*** at $p_0 \in M$ if every CR function f defined on M near p_0 can be written as a finite sum

$$f = \sum_{i=1}^{r} bH_j, \tag{5.8}$$

where H_j is holomorphic in a wedge $\mathcal{W}_j$ with edge M, with tempered growth at M, and bH_j denotes its boundary value on M. (See [4] and [3] for classes of generic manifolds satisfying this property.)

The converse of Theorem 8 holds under the holomorphic decomposition property.

THEOREM 10. *Let M be a generic manifold defined by $\rho = 0$, ρ given by (0.5). Assume that the holomorphic decomposition property holds at 0. Assume also there exists a strictly convex open cone $\Gamma \subset \mathbf{R}^\ell \setminus \{0\}$ such that for every $\mathcal{W}_{\delta_1,\eta_1}(\Gamma_1)$, any $h \in \mathcal{H}(\mathcal{W}_{\delta_1,\eta_1}(\Gamma_1))$, and any $\Gamma' \Subset \Gamma_1 + \Gamma$, h extends to a wedge of the form $\mathcal{W}_{\delta_1',\eta_1'}(\Gamma')$. Then for any CR function f on M (near 0) and any $\Gamma'' \Subset \Gamma$, there exists a wedge $\mathcal{W}_{\delta'',\eta''}(\Gamma'')$ to which f extends.*

Proof: By assumption we may write f in the form (5.8) with $H_j \in \mathcal{H}(\mathcal{W}_{\delta_1,\eta_1}(\Gamma_j))$. Since any $\Gamma''' \Subset \Gamma$ satisfies $\Gamma''' \Subset \Gamma_j + \Gamma$, we conclude by the assumption that each H_j extends holomorphically to a wedge of the form $\mathcal{W}_{\delta_1',\eta_1'}(\Gamma''')$. Let $f_i = bH_j$. By Theorem 4 (i), $WF_0 H_j \subset (\Gamma''')^0$ and by Theorem 4 (iii), $WF_0 f_j = WF_0 H_j$. Using Theorem 7 of [3] we conclude that for any Γ'', with $\Gamma'' \Subset \Gamma'''$, each f_j extends to a wedge of the form $\mathcal{W}_{\delta'',\eta''}(\Gamma'')$, which completes the proof of the Theorem.

(5.9) Remark: The decomposition hypothesis holds for any hypersurface [1]. Therefore, if M is a hypersurface, then Theorems 8 and 10 combined give the known result that any CR function defined on M near p_0 extends to one

side if and only if any holomorphic function on the other side near p_0 extends to that side.

REFERENCES

[1] A. ANDREOTTI & C. D. HILL, *E. E. Levi convexity and the Hans Lewy problem*, I, II, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) **26** (1972), 325–362 and 747–806.

[2] M. S. BAOUENDI, C. H. CHANG & F. TREVES, *Microlocal hypo-analyticity and extension of CR functions*, J. Differential Geometry **18** (1983), 331–391.

[3] M. S. BAOUENDI & L. P. ROTHSCHILD, *Normal forms for generic manifolds and holomorphic extension of CR functions*, J. Differential Geometry **25** (1987), 431–467.

[4] M. S. BAOUENDI, L. P. ROTHSCHILD & F. TREVES, *CR structures with group action and extendability of CR functions*, Invent. Math. **82** (1985), 359–396.

[5] M. S. BAOUENDI & F. TREVES, *About the holomorphic extension of CR functions on real hypersurfaces in complex space*, Duke Math. J. **51** (1984), 77–107.

[6] J. BROS & D. IAGOLNITZER, *Support essentiel et structure analytique des distributions*, Semin. Goulaouic-Lions-Schwartz, Exp. 18, Ecole Polytechnique (1975–1976).

[7] N. HANGES & J. SJÖSTRAND, *Propagation of analyticity for a class of non-micro-characteristic operators*, Annals of Math. **116** (1982), 559–577.

[8] L. HÖRMANDER, "The Analysis of Linear Partial Differential Operators," I., Springer, Berlin-Heidelberg-New York, 1983.

[9] M. KASHIWARA & P. SCHAPIRA, *Microlocal Study of Sheaves*, Astérisque **128** (1985).

[10] H. LEWY, *On the local character of the solution of an atypical differential equation in three variables and a related problem for regular functions of two complex variables*, Ann. of Math 64 (1956), 514–522.

[11] M. SATO, T. KAWAI & M. KASHIWARA, *Microfunctions and pseudo-differential equations*, in "Lecture Notes in Math," Springer, Berlin, 1973, pp. 265–529.

[12] J. SJÖSTRAND, *Singularities analytiques microlocales*, Astérisque, No. 95, Soc. Math. France (1982), 1-166.

[13] J. M. TRÉPREAU, *personal communication.*

[14] F. TREVES, "Hypo-analytic structures." to appear

Received January 1988