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Inventiones mathematicae

Germs of CR maps between real analytic hypersurfaces

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Table of contents

§1. Introduction and main results

This paper deals with various algebraic and analytic properties of smooth CR mappings H between two real analytic hypersurfaces M and M' in $\mathbb{C}^{n+1}$. One of the main results is that if M and M' are essentially finite at p_0 and $p_0'=H(p_0)$ respectively, and if H is of finite multiplicity at p_0, then H extends holomorphically to a neighborhood of p_0 in $\mathbb{C}^{n+1}$. Precise definitions are given below. Local holomorphic extension of CR mappings was first obtained in the diffeomorphic case when M and M' are strictly pseudo-convex by Lewy [19] and Pinčuk [21]. Their result was generalized by others, relaxing the conditions on M and M' (see Baouendi-Jacobowitz-Treves [4], Derridj [11], Diederich-Webster [13], Han [17]). The nondiffeomorphic case was considered by Bell [8] and Bedford-Bell [7] with restrictions on M and M'. A more general result in $\mathbb{C}^2$ has been obtained by the authors jointly with S. Bell [2, 3].

We recall the following definition introduced in [4]. Let M be a real analytic hypersurface in $\mathbb{C}^{n+1}$ containing the origin and defined locally by $\rho(Z, \bar{Z})=0$, $d\rho\neq 0$, $Z\in\mathbb{C}^{n+1}$, where ρ is a real valued analytic function, $\rho(0)=0$. We say that M is *essentially finite* at 0 if for any sufficiently small $Z\in\mathbb{C}^{n+1}\setminus\{0\}$ there exists an arbitrarily small $\Theta\in\mathbb{C}^{n+1}$ satisfying $\rho(Z, \Theta)\neq 0$, $\rho(0, \Theta)=0$. We can

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choose holomorphic coordinates Z such that $\rho(Z,0)=\alpha(Z)Z_{n+1}$, $\alpha(0)\neq 0$. We put $z=(Z_1,\ldots,Z_n)$, $w=Z_{n+1}$, and similarly replace $\bar{Z}$ by (ζ,τ), with $\zeta=(\zeta_1,\ldots,\zeta_n)$. We can write

$$\rho((z,0),(\zeta,0))=\sum_\alpha a_\alpha(z)\zeta^\alpha;$$

then M is essentially finite at 0 if and only if the germs of analytic functions $a_\alpha(z)$ have no common zero other than 0 near the origin. In this case we define the *essential type* of M at 0 as the positive integer given by

$$\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/(a_\alpha(z)), \tag{1.1}$$

where $(a_\alpha(z))$ is the ideal generated by all the coefficients $a_\alpha(z)$ in the ring of formal power series $\mathcal{O}[\![z]\!]$ in n indeterminates, and dimension is taken in the sense of vector spaces. One can easily check that the number defined by (1.1) is independent of the choice of the defining function ρ and the holomorphic coordinates (z, w).

If $H: M \to M'$ is a smooth CR mapping between two smooth real hypersurfaces in $\mathbb{C}^{n+1}$, there exist $(n+1)$ CR functions $j_1,\ldots,j_{n+1}$ defined on M such that $H=(j_1,\ldots,j_{n+1})$. On the other hand (see §2) if j is a smooth CR function defined on M near 0, there exists a formal (holomorphic) power series $J(Z)=\sum \alpha_\alpha Z^\alpha$ in $n+1$ indeterminates, such that if $U\ni u\mapsto Z(u)\in\mathbb{C}^{n+1}$ (U an open neighborhood of 0 in $\mathbb{R}^{2n+1}$, $Z(0)=0$) is a parametrization of M, then the Taylor series of $j(Z(u))$ at 0 is given by $J(Z(u))$. If we choose $Z=(z,w)$ as above, the mapping H is said to be of *finite multiplicity* at 0 if

$$\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/(J(z,0))<\infty \tag{1.2}$$

where $(J(z,0))$ is the ideal generated by $J_1(z,0),\ldots,J_{n+1}(z,0)$, the power series associated to the CR functions $j_1,\ldots,j_{n+1}$ as above. The number defined by the left hand side of (1.2) is an invariant independent of the choice of the holomorphic coordinates. This invariant is called the *multiplicity of* H at 0.

We now state our main results. Theorem 1 and its corollaries deal with holomorphic extension of CR mappings. Theorems 2 and 3 relate the essential finiteness of M with that of M' and give a formula relating the essential types and the multiplicity of the mapping.

Theorem 1. *Let $H: M\to M'$ be a smooth CR mapping, where M and M' are real analytic hypersurfaces in $\mathbb{C}^{n+1}$. Let $p_0\in M$ and $p'_0=H(p_0)$. If either one of the following conditions is satisfied, then H is the restriction of a holomorphic mapping from a neighborhood of p_0 in $\mathbb{C}^{n+1}$ into $\mathbb{C}^{n+1}$.*

(i) *The mapping H is of finite multiplicity at p_0, and M' is essentially finite at p'_0.*

(ii) *M is essentially finite at p_0 and H satisfies*

$$H'(\mathbb{C}T_{p_0}M)\not\subset \mathcal{V}'_{p'_0}\oplus\bar{\mathcal{V}}'_{p'_0}. \tag{1.3}$$

Here $\mathbb{C}T_{p_0}M$ is the complexified tangent space of M at p_0 and $\mathcal{V}'_{p'_0}$ the space of antiholomorphic vectors tangent to M' at p'_0.

An immediate corollary is the following.

Corollary 1. *Let $H: M'$ be a smooth* CR *mapping as in Theorem* 1, *and assume that M and M' are essentially finite at p_0 and p'_0 respectively. Then H extends holomorphically to a neighborhood of p_0 if either H is of finite multiplicity at p_0 or* (1.3) *holds.*

Theorem 1 was proved in the case where H is a diffeomorphism in [4]. Even in this case our result is slightly stronger since we assume only M or M' is essentially finite. For $n=1$, Theorem 1 was proved in [3].

Theorem 2. *Let $H: M \to M'$ be a smooth* CR *mapping, where M and M' are real analytic hypersurfaces in $\mathbb{C}^{n+1}$, $p_0 \in M$, $p'_0 = H(p_0)$. If H is of finite multiplicity at p_0 and M' is essentially finite at p'_0, then M is essentially finite at p_0.*

Theorem 3. *Let $H: M \to M'$ be a smooth* CR *mapping. If M is essentially finite at $p_0 \in M$ and* (1.3) *holds, then H is of finite multiplicity at p_0, M' is essentially finite at $H(p_0) = p'_0$ and the following holds.*

$$\text{ess type}_{p_0} M = (\text{mult } H)_{p_0} \cdot (\text{ess type}_{p'_0} M'), \tag{1.4}$$

where ess *type (essential type) and* mult *(multiplicity) are defined by* (1.1) *and* (1.2) *respectively.*

For proper holomorphic mappings our results combined with results on smoothness up to the boundary proved in Bell-Catlin [9] and Diederich-Fornaess [12], yield the following corollaries.

Corollary 2. *If $\mathscr{H}: D \to D'$ is a proper holomorphic mapping, where D and D' are bounded pseudoconvex domains in $\mathbb{C}^{n+1}$ with real analytic boundaries, then $\mathscr{H}$ extends holomorphically as a proper map from a neighborhood of $\bar{D}$ into a neighborhood of $\bar{D}'$.*

We also have

Corollary 3. *Let $\mathscr{H}$ be as in Corollary* 2, *but without the pseudoconvexity assumption. If, in addition, $\mathscr{H}$ is smooth up to the boundary of D, and, at every point $p_0 \in \partial D$, either $\mathscr{H}$ is of finite multiplicity or* (1.3) *holds, then the conclusion of Corollary* 2 *holds.*

Remark. When M and M' are hypersurfaces in $\mathbb{C}^2$ (case $n=1$) the multiplicity of H defined here is expressed in a different form from that given in [2, 3]. However, it follows from the results in [3] that these multiplicities are the same.

If $M \subset \mathbb{C}^{n+1}$ is essentially finite at p_0 then it is of finite type in the sense of Kohn [18] and Bloom-Graham [10]. (For $n=1$, M is essentially finite if and only if it is of finite type [4]). It is proved in [3] when $n=1$ that

$$\text{type}_{p_0} M = \text{type}_{p'_0} M' \cdot (\text{mult } H)_{p_0}. \tag{1.5}$$

It should be noted that (1.5) is no longer valid in general for $n>1$, as is shown by the following example. Let M, $M' \subset \mathbb{C}^3$ be respectively defined by $\operatorname{Im} w = |z_1|^2 + |z_2|^4$, $\operatorname{Im} w' = |z'_1|^2 + |z'_2|^2$, and $H = (z_1, z_2^2, w)$. Here $\text{type}_0 M$

$=\text{type}_0\, M' = (\text{mult}\, H)_0 = 2$. Even when $n=1$ the essential type of M is in general different from its type. Therefore (1.4) is new even when $n=1$.

Most of our results extend to generic CR manifolds of higher codimension. We do not address these questions in the present paper.

The organization of the paper is as follows.

Section 2 contains some results on the Taylor series of CR functions. In Sect. 3 we give a precise definition of finite multiplicity of CR mappings, some algebraic properties, and the proof of Theorem 2. Section 4 deals mainly with the proof of Theorem 3. Sections 5, 6, and 7 contain technical lemmas needed for the proofs, of Theorem 1 and Corollary 2, which are given in Sect. 8.

We take this opportunity to thank several people for help with some of the algebraic aspects of this paper. John D'Angelo pointed out the relevance of the paper of Eisenbud-Levine [14] as well as the connection with his own work [1]. David Eisenbud provided references for Lemmas (4.5) and (4.7). We are also grateful to Joseph Lipman for several conversations and helpful references concerning Lemmas (3.19) and (4.5).

The results contained in this work were announced in [5]. Upon completion of the present manuscript we received a preprint by K. Diederich and J.E. Fornaess entitled "Proper holomorphic mappings between real-analytic pseudoconvex domains in $\mathbb{C}^n$", which contains our Corollary 2 and a local holomorphic extendability result which seemingly is more restrictive than our Theorem 1. Their approach is different from ours.

§ 2. Taylor series of CR functions

Our proof of Theorem 1 requires studying the Taylor expansions of the components of H as formal power series. If M is a real analytic hypersurface in $\mathbb{C}^{n+1}$ containing the origin, we can find local holomorphic coordinates (z, w), $z=(z_1, \dots, z_n)\in\mathbb{C}^n$, $w\in\mathbb{C}$, such that M is given locally by

$$\text{(2.1)}\qquad \operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

where φ is real analytic, real valued, and satisfies

$$\text{(2.2)}\qquad \varphi(z, 0, 0) = \varphi(0, \zeta, 0) \equiv 0, \qquad \nabla\varphi(0) = 0.$$

Local coordinates on M are $(z, \bar{z}, s)$, with $s = \operatorname{Re} w$.

Let

$$\text{(2.3)}\qquad L_j = \frac{\partial}{\partial \bar{z}_j} - i\frac{\varphi_{\bar{z}_j}(z, \bar{z}, s)}{1 + i\varphi_s(z, \bar{z}, s)}\frac{\partial}{\partial s}, \qquad j = 1, \dots, n,$$

then $(L_1, \dots, L_n)$ is a local basis of the antiholomorphic tangent vector fields. Moreover $[L_j, L_k] = 0$, $1 \leq j, k \leq n$.

(2.4) **Proposition.** *Let M be a smooth hypersurface embedded in $\mathbb{C}^{n+1}$, $0\in M$, parametrized by $u \mapsto Z(u) \in \mathbb{C}^{n+1}$, with $u\in U$, an open neighborhood of 0 in $\mathbb{R}^{2n+1}$,*

$Z(0)=0$, $Z\in C^{\infty}(U)$. If j is a smooth CR function defined on U then there exists a unique power series in $n+1$ indeterminates

$$J(Z)\sim\sum_{\alpha}a_{\alpha}Z^{\alpha} \tag{2.5}$$

such that the Taylor series of j around 0 satisfies

$$j(u)\sim\sum_{\alpha}a_{\alpha}Z^{\alpha}(u). \tag{2.6}$$

In particular if the coordinates on $\mathbb{C}^{n+1}$ are written $z=(Z_1, \dots, Z_n)$, $w=Z_{n+1}$ and the parametrization of M is given by (2.1), then

$$j(z, \bar{z}, s)\sim\sum_{\alpha, k}a_{\alpha k}\, z^{\alpha} w^{k}|_{w=s+i\varphi(z,\bar{z},s)}. \tag{2.7}$$

Proof. It suffices to prove the Proposition when the coordinates of the embedding are $(z, \bar{z}, s)$ as in the second part of the statement. We first put $y=0 (z=x+iy)$ and expand $j(x, x, s)$ and its Taylor series around 0:

$$j(x, x, s)\sim\sum_{\substack{|\alpha|\geq 0\\ k\geq 0}} b_{\alpha k}\, x^{\alpha} s^{k}.$$

Since $\varphi(0)=\nabla\varphi(0)=0$, we may inductively find $a_{\alpha k}$ so that

$$\sum_{\substack{|\alpha|\geq 0\\ k\geq 0}} b_{\alpha k}\, x^{\alpha} s^{k}\sim\sum_{\substack{|\alpha|\geq 0\\ k\geq 0}} a_{\alpha k}\, x^{\alpha}(s+i\varphi(x, x, s))^{k}.$$

Indeed,

$$x^{\alpha}(s+i\varphi(x, x, s))^{k}=x^{\alpha}s^{k}+O\Big(x^{\alpha}\Big(\sum_{j\geq 0}|s|^{j}|x|^{2k-j}+|s|^{j}|x|\,|s|)^{k-j}+|s|^{j}|s|^{2k-j}\Big)\Big).$$

Now let $\tilde{J}(z, w)$ be the formal power series

$$\tilde{J}(z, w)\sim\sum_{\substack{|\alpha|\geq 0\\ k\geq 0}} a_{\alpha k}\, z^{\alpha} w^{k} \tag{2.8}$$

and let $\tilde{j}(z, \bar{z}, s)$ be any smooth function with Taylor series

$$\tilde{j}(z, \bar{z}, s)\sim\tilde{J}(z, w)|_{w=s+i\varphi(z,\bar{z},s)}.$$

We claim that $\tilde{j}(z, \bar{z}, s) \sim j(z, \bar{z}, s)$, which will prove the Proposition. For this, we observe that

$$L_k \tilde{j}(z, \bar{z}, s) \sim 0, \quad 1 \leqq k \leqq n$$

and that

$$\tilde{j}(x, x, s) \sim j(x, x, s). \tag{2.5}$$

Hence the proof of Proposition (2.4) is completed by the following standard uniqueness result.

(2.9) **Lemma.** *If $h(z, \bar{z}, s)$ is a smooth function with*

$$L_k h(z, \bar{z}, s) \sim 0, \; 1 \leqq k \leqq n, \quad \text{and} \quad h(x, x, s) \sim 0, \tag{2.10}$$

then

$$h(z, \bar{z}, s) \sim 0, \tag{2.11}$$

i.e. *the Taylor series of a formal* CR *function is determined by its restriction to $y=0$.*

Proof. We write

$$2L_k = \frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} + \lambda_k(z, \bar{z}, s) \frac{\partial}{\partial s} \tag{2.12}$$

and expand h as a Taylor series in y:

$$h(z, \bar{z}, s) \sim \sum_{|\alpha| \geqq 0} h_\alpha(x, s)\, y^\alpha \tag{2.13}$$

so that $h_0(x, s) \sim h(x, x, s) \sim 0$. Then (2.11) will be proved if we show that for each α, $|\alpha| \geqq 1$, h_α is determined by the h_β, $|\beta| = |\alpha| - 1$.

By applying L_k to the right hand side of (2.13) we obtain for $\alpha_k \geqq 1$

$$\begin{aligned} i\alpha_k h_{(\alpha_1, \dots, \alpha_k, \dots, \alpha_n)}(x, s) &+ \frac{\partial}{\partial x_k} h_{(\alpha_1, \dots, \alpha_k - 1, \dots, \alpha_n)}(x, s) \\ &+ \lambda_k(x, x, s) \frac{\partial}{\partial s} h_{(\alpha_1, \dots, \alpha_k - 1, \dots, \alpha_n)}(x, s) \\ &\sim 0 \quad k = 1, \dots, n, \end{aligned} \tag{2.14}$$

which completes the proof.

The following observations are important in proving the analyticity of H.

(2.15) **Lemma.** *For any multi-index α, we have*

$$L^\alpha|_0 = \frac{\partial^\alpha}{\partial \bar{z}^\alpha}, \tag{2.16}$$

with $L^\alpha = L_1^{\alpha_1} \dots L_n^{\alpha_n}$, and the L_j defined by (2.3).

Proof. Since by (2.3) we have

(2.17) $$L^\alpha=\left(\frac{\partial}{\partial\bar z_1}-i\frac{\varphi_{\bar z_1}}{1+i\varphi_s}\frac{\partial}{\partial s}\right)^{\alpha_1}\left(\frac{\partial}{\partial\bar z_2}-i\frac{\varphi_{\bar z_2}}{1+i\varphi_s}\frac{\partial}{\partial s}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial\bar z_n}-i\frac{\varphi_{\bar z_n}}{1+i\varphi_s}\frac{\partial}{\partial s}\right)^{\alpha_n},$$

the desired conclusion (2.16) is equivalent to the vanishing at 0 of the coefficient of $\frac{\partial}{\partial s}$ in the expansion of (2.17). This follows immediately from the assumption (2.2).

(2.18) **Lemma.** *If* $j(z,\bar z,s)$ *is a* CR *function and* $J(z,w)$ *is the formal power series given by Proposition* (2.4) *then for any multi-index* α,

(2.19) $$L^\alpha\bar j(0)=\left(\frac{\partial}{\partial\bar z}\right)^\alpha\bar J(0,0).$$

Proof. By definition

(2.20) $$\left(\frac{\partial}{\partial\bar z}\right)^\alpha\bar J(0,0)=\alpha!\,\bar a_{\alpha 0},$$

where

$$\bar J(\bar z,\bar w)\sim\sum_{\substack{|\beta|\geq 0\\ k\geq 0}}\bar a_{\beta k}\,\bar z^\beta\,\bar w^k.$$

Since $\left(\frac{\partial}{\partial\bar z}\right)^\beta(s+i\varphi)(0)=0$ for all β, by (2.2), using (2.7) of Proposition (2.4) and Lemma (2.15) we also have

$$L^\alpha\bar j(0)=\alpha!\,\bar a_{\alpha 0},$$

which completes the proof of Lemma (2.18).

§ 3. CR mappings of finite multiplicity

Let H be a CR mapping $H\colon M\to M'$, where M and M' are embedded hypersurfaces in $\mathbb{C}^{n+1}$ and $H(0)=0$. We write $H=(h_1,\ldots,h_{n+1})$ where the h_j are smooth CR functions defined on M. Suppose the coordinates Z in $\mathbb{C}^{n+1}$ have been chosen so that $z=(Z_1,\ldots,Z_n)$, $w=Z_{n+1}$ with M defined by (2.1). Similarly, we assume that M' is given by

(3.1) $$\operatorname{Im} w'=\psi(z',\bar z',\operatorname{Re} w'),$$

with $z'\in\mathbb{C}^n$, $w'\in\mathbb{C}$, $s'=\operatorname{Re} w'$, and ψ satisfies $\psi(0,\zeta',s')\equiv\psi(z',0,s')\equiv 0$ (See [4]). Then we write $H=(f_1,\ldots,f_n,g)$ and let $F_1,\ldots,F_n,G$ be the associated power series given by Proposition (2.4). By the choice of the coordinates we have the identity

$$\frac{g-\bar{g}}{2i}=\psi\left(f,\bar{f},\frac{g+\bar{g}}{2}\right) \tag{3.2}$$

on M.

(3.3) **Proposition.** *Let H be a* CR *map from M to M' given by $H=(f,g)=(f_1,\ldots,f_n,g)$ with local coordinates $(z,\bar{z},s)$ on M, and let $G(z,w)$ be the power series associated to g by Proposition* (2.4). *Then $\frac{\partial G}{\partial w}(0)$ is real and the following are equivalent.*

(i) $$H'(\mathbb{C}T_0M)\not\subset\mathscr{V}_0'\oplus\bar{\mathscr{V}}_0',$$

(ii) $$\frac{\partial g}{\partial s}(0)\neq 0,$$

and

(iii) $$\frac{\partial G}{\partial w}(0)\neq 0.$$

Proof. Since $g(z,\bar{z},s)\sim G(z,w)|_{w=s+i\varphi(z,\bar{z},s)}$, and $\nabla\varphi(0)=0$, it follows that

$$\frac{\partial g}{\partial s}(0)=\frac{\partial G}{\partial w}(0).$$

On the other hand since $\frac{\partial}{\partial s}(\operatorname{Im} g)(0)=0$, by (3.2) and the fact that $\nabla\psi(0)=0$, we have $\frac{\partial g}{\partial s}(0)=\frac{\partial}{\partial s}(\operatorname{Re} g)(0)$. This shows that $\frac{\partial G}{\partial w}(0)$ is real and the equivalence of (ii) and (iii).

In order to show that (i) and (ii) are equivalent, note first that (i) holds if and only if $H'\left(\left.\frac{\partial}{\partial s}\right|_0\right)\notin\mathscr{V}_0'\oplus\bar{\mathscr{V}}_0'$. Indeed, since H is a CR map $H'(L_j|_0)$ and $H'(\bar{L}_j|_0)$ are in $\mathscr{V}_0'\oplus\bar{\mathscr{V}}_0'$. Now compute the coefficient of $\frac{\partial}{\partial s'}$ in $H'\left(\left.\frac{\partial}{\partial s}\right|_0\right)$, where $(z',\bar{z}',s')$ are local coordinates on M' with $z'=f(z,\bar{z},s)$ and $s'=\operatorname{Re} g(z,\bar{z},s)$ and $s'=\operatorname{Re} g(z,\bar{z},s)$. By the chain rule we have

$$H'\left(\left.\frac{\partial}{\partial s}\right|_0\right)=\sum\frac{\partial f_j}{\partial s}(0)\frac{\partial}{\partial z_j'}+\sum\frac{\partial\bar{f}_j}{\partial s}(0)\frac{\partial}{\partial\bar{z}_j'}+\frac{\partial}{\partial s}(\operatorname{Re} g)(0)\frac{\partial}{\partial s'},$$

and the equivalence of (i) and (ii) follows.

(3.4) **Lemma.** *Let $H\colon M\to M'$, $H=(j_1,\ldots,j_{n+1})$, be a* CR *map, and M be given by $\rho(Z,\bar{Z})=0$, $Z\in\mathbb{C}^{n+1}$, and $d\rho(0)\neq 0$. If $Z=(z_1,\ldots,z_n,w)$ and $\rho((z,w),0)=\alpha(z,w)\,w$, $\alpha(0)\neq 0$, then*

$$\dim_{\mathbb{C}}\mathcal{O}[\![z]\!]/(J_1(z,0),\ldots,J_{n+1}(z,0)) \tag{3.5}$$

is independent of the choice of the holomorphic coordinates. Here $(J_1(z,0), \ldots, J_{n+1}(z,0))$ *denotes the ideal generated by* $J_1(z,0), \ldots, J_{n+1}(z,0)$, *and dimension is taken in the sense of complex vector spaces. In addition if* $H=(f_1, \ldots, f_n, g)$ *where* (f, g) *satisfies* (3.2) *then the number given by* (3.5) *equals*

(3.6) $$\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/(F_1(z,0), \ldots, F_n(z,0)).$$

Proof. The proof of the lemma is straightforward using (3.8) proved below.

We shall say that a smooth CR mapping H from M to M' is of *finite multiplicity* at 0 if the number given in (3.5) is finite.

(3.7) **Lemma.** *If* $H=(f_1, \ldots, f_n, g)$ *is a* CR *map as above, then*

(3.8) $$G(z,w)=wG_1(z,w)$$

where G_1 *is another formal power series. In addition, if* $\frac{\partial G}{\partial w}(0)\neq 0$, *then* $G_1(0)$ *is real and nonzero.*

Proof. By applying L^α to (3.2) we conclude

$$L^\alpha \bar{g}(0)=0$$

for all α. Therefore, by Lemma (2.18), $\partial_{\bar{z}}^\alpha G(0)=0$ for all α, which yields (3.8). The rest is immediate since $\frac{\partial G}{\partial w}(0)$ is real by Proposition (3.3).

(3.9) **Proposition.** *Let* M *and* M' *be defined by* (2.1) *and* (3.1) *respectively,* $H: M \to M'$ *a* CR *map. Then there exists a formal power series* $\alpha_1(z, \zeta)$, *such that*

(3.10) $$\alpha_1(z,\zeta)\varphi(z,\zeta,0) \sim \psi(F(z,0), \bar{F}(\zeta,0), 0).$$

Furthermore, if $\frac{\partial G}{\partial w}(0)\neq 0$ *then* $\alpha_1(0)\neq 0$.

Proof. We substitute in (3.2) the formal power series $G(z,w)$ and $F(z,w)$ defined in Proposition (2.4), evaluated at $w=s+i\varphi(z,\bar{z},s)$. After putting $s=0$ we obtain from (3.2)

(3.11) $$G(z, i\varphi(z,\bar{z},0)) - \bar{G}(\bar{z}, -i\varphi(z,\bar{z},0)) \sim 2i\psi\left(F(z, i\varphi(z,\bar{z},0)), \bar{F}(\bar{z}, -i\varphi(z,\bar{z},0)), \frac{G(z,i\varphi)+\bar{G}(\bar{z},-i\varphi)}{2}\right).$$

By Lemma (3.7) we have

(3.12) $$\frac{G(z,u)-\bar{G}(\bar{z},-u)}{2i} \sim \alpha u$$

where $\alpha(0)\neq 0$ if $\frac{\partial G}{\partial w}(0)\neq 0$. Since (3.11) and (3.12) hold in formal power series in z and $\bar{z}$ we may replace $\bar{z}$ by an independent variable ζ, to obtain

$$\alpha\varphi(z,\zeta,0)\sim\psi\left(F(z,i\varphi),\bar{F}(\zeta,-i\varphi),\frac{G(z,i\varphi)+\bar{G}(\bar{z},-i\varphi)}{2}\right). \tag{3.13}$$

On the other hand, we see that

$$\psi\left(F(z,u),\bar{F}(\zeta,-u),\frac{G(z,u)+\bar{G}(\zeta,-u)}{2}\right)\sim\psi_0(F(z,0),\bar{F}(\zeta,0)) + u\beta(z,\zeta,u), \tag{3.14}$$

where $\beta(0)=0$ and $\psi_0(z,\zeta)=\psi(z,\zeta,0)$. Now substitute $u=i\varphi(z,\zeta,0)$ in (3.14). Combining this with (3.13) we obtain

$$\alpha\varphi(z,\zeta,0)\sim\psi_0(F(z,0),\bar{F}(\zeta,0))+\varphi(z,\zeta,0)\,\beta. \tag{3.15}$$

By combining the term on the left in (3.15) with the second term on the right we obtain (3.10).

Proof of Theorem 2. Since the proof of Theorem 1 does not make use of Theorem 2, we may assume, by Theorem 1, that H is real analytic. Both sides of (3.10) are now convergent power series and (3.10) becomes an equality

$$\alpha_1(z,\zeta)\,\varphi(z,\zeta,0)=\psi(F(z,0),\bar{F}(\zeta,0),0). \tag{3.16}$$

By assumptions we know that $F_1(z,0),\ldots,F_n(z,0)$ have no common zeros other than $z=0$, since otherwise the dimension of $\mathcal{O}[\![z]\!]/(F_1(z,0),\ldots,F_n(z,0))$ would not be finite (see for example [16]) contradicting the finite multiplicity property of H.

We prove the theorem by contradiction. Assume there exists $z_0\neq 0$, such that $\varphi(z_0,\zeta,0)=0$ for all $\zeta\in\mathbb{C}^n$, ζ small. It follows from (3.16) that

$$\psi(z_0',\bar{F}(\zeta,0),0)=0 \tag{3.17}$$

with $z_0'=F(z_0,0)\neq 0$.

Since the mapping $\zeta\mapsto\bar{F}(\zeta,0)$ is open (see [16]) we conclude that

$$\psi(z_0',\zeta',0)=0$$

for all ζ' small, contradicting the essential finiteness of M' (see [4]). The proof of Theorem 2 is complete.

For mappings of finite multiplicity the following result will be essential in the proof of Theorem 1.

(3.18) **Proposition.** *Let H: $M \to M'$ be a CR map with $H=(f_1, \dots, f_n, g)$ with notation as above. If H is of finite multiplicity at 0 then there exists a multi-index α such that*

$$L^\alpha(\det(L_j \bar{f}_k))(0) \neq 0$$

where $L^\alpha = L_1^{\alpha_1} \dots L_n^{\alpha_n}$ and the L_j are defined in (2.13).

Proof. By Lemma (2.18) we are reduced to the following.

(3.19) **Lemma.** *If H is of finite multiplicity, then*

$$\det\left(\frac{\partial}{\partial \bar{z}_j} \bar{F}_k(\bar{z}, 0)\right) \not\equiv 0. \tag{3.20}$$

Proof. Using the fact that the number given by (3.6) is finite, this is a special case of a known result which can be found, for instance, in [14].

§4. Proof of Theorem 3

Since by Proposition (3.3) any of the conditions (i), (ii), or (iii) implies $\frac{\partial G}{\partial w}(0)$ is nonzero, by Lemma (3.4) it suffices to prove that the number given by (3.6) is finite, in order to prove that H is of finite multiplicity. By Proposition (3.9), (3.10) holds with $\alpha_1(0) \neq 0$.

Now write $\varphi(z, \zeta, 0) = \sum a_\alpha(z) \zeta^\alpha$, a convergent power series. By (3.10) and the fact that $\psi(0, \zeta, 0) \equiv 0$, we obtain that the ideal generated by all the $\alpha_\alpha(z)$ in $\mathcal{O}[\![z]\!]$ is contained in that generated by the $F_j(z, 0)$, $1 \leqq j \leqq n$, i.e.

$$(a_\alpha(z)) \subset (F_1(z, 0), \dots, F_n(z, 0)). \tag{4.1}$$

By the assumption that $\varphi(z, \zeta, 0) = 0$ for all ζ implies $z=0$, it follows that the $a_\alpha(z)$ have no common zero other than $z=0$. Hence, by the Nullstellensatz,

$$\dim \mathcal{O}[\![z]\!]/(a_\alpha(z)) < \infty. \tag{4.2}$$

By the inclusion (4.1) it follows that (3.6) is finite also. This proves that H is of finite multiplicity.

In order to prove that M' is essentially finite, we shall essentially reduce the question to the case where F is holomorphic by the following.

(4.3) **Lemma.** *Let $(a_\alpha(z))$ be an ideal in $\mathcal{O}[\![z]\!]$ and p a positive integer such that $z_i^p \in (a_\alpha(z))$ for all i, $1 \leqq i \leqq n$. Suppose that $b_\alpha(z) \in \mathcal{O}[\![z]\!]$ satisfy $a_\alpha(z) - b_\alpha(z) \in (z_i^{p+1})_{i=1,\dots,n}$ for all α. Then $(a_\alpha(z)) = (b_\alpha(z))$.*

Proof. Let E be the $\mathcal{O}[\![z]\!]$ module generated by the $a_\alpha(z)$. Then $b_\alpha(z) \in E$ for all α, since $b_\alpha(z) - a_\alpha(z) \in ME$, where M is the module generated by the z_i, $i = 1, \dots, n$. Since $\{b_\alpha(z)\}$ generate E modulo ME, by Nakayama's Lemma, (see e.g. [16]), $b_\alpha(z)$ generate E.

We may now return to the proof of Theorem 3. We begin with the formal identity (3.10). Let $F^k(z)$ be the polynomial mapping obtained from the formal power series $F(z, 0)$ by dropping all monomials containing a factor z_j^{k+1} for some j. We define α_1^k in a similar manner truncating in both z and ζ. We define the analytic function $\varphi^{(k)}(z, \zeta)$ by

$$\alpha_1^k(z, \zeta)\, \varphi^{(k)}(z, \zeta) = \psi(F^k(z), \bar{F}^k(\zeta), 0). \tag{4.4}$$

We write $\varphi^{(k)}(z, \zeta) = \sum a_\alpha^{(k)}(z) \zeta^\alpha$. Let N be sufficiently large so that $(a_\alpha(z))$ is generated by $\{a_\alpha(z), |\alpha| \leqq N\}$. Let p be chosen so that $z_i^p \in (a_\alpha(z))$, $i = 1, \ldots, n$.

Now we choose k sufficiently large to satisfy the following conditions.

(i) $\dim \mathcal{O}[\![z]\!]/(F^k(z)) < \infty$, i.e. F^k defines a finite holomorphic mapping in the terminology of [16]. This holds for k sufficiently large by Lemma (4.3) since H is of finite multiplicity.

(ii) $k \geqq N$,

(iii) $k \geqq p + 1$.

We first claim that $(a_\alpha^{(k)}(z), |\alpha| \leqq N)$ is the same as $(a_\alpha(z), |\alpha| \leqq N)$, which is $(a_\alpha(z))$. Indeed $a_\alpha^{(k)} - a_\alpha \in M^k$ for $|\alpha| \leqq N$, by (ii), where M is defined as in Lemma (4.3). Hence by (iii) and Lemma (4.3) the claim follows.

Hence

$$\dim \mathcal{O}[\![z]\!]/(a_\alpha^{(k)}(z), |\alpha| \leqq N) < \infty$$

which implies that $\{a_\alpha^{(k)}, |\alpha| \leqq N\}$ have no common zeros. To complete the proof that M' is essentially finite we shall show that for every $z_0' \in \mathbb{C}^n \setminus \{0\}$ sufficiently small, there exists an arbitrarily small ζ_0' such that $\psi(z_0', \zeta_0', 0) \neq 0$. Indeed, since $F^k(z)$ is a finite holomorphic mapping there exists z_0 such that $F^k(z_0) = z_0'$. Since $\varphi^{(k)}(z_0, \zeta)$ is not identically zero as a function of ζ, by the above, there exists an arbitrarily small ζ_0 such that $\varphi^{(k)}(z_0, \zeta_0) \neq 0$. It suffices then to take $\zeta_0' = F^{(k)}(\zeta_0)$ and to apply the identity (4.4).

It remains to show (1.4). Since the proof of Theorem 1 requires only that H is of finite multiplicity and M' is essentially finite, we may assume that H is real analytic and therefore that (3.10) holds as an identity of convergent power series with $\alpha_1 \neq 0$. We shall use the following result from commutative algebra.

(4.5) **Lemma.** *Let $F\colon \mathbb{C}^n \to \mathbb{C}^n$ be a germ of a finite holomorphic mapping (in the terminology of [16]) with $F(0) = 0$. Let $\mathscr{I}$ be an ideal of $\mathcal{O}[\![z]\!]$ and denote by $\mathscr{I} \circ F$ the ideal whose elements are obtained by composing the elements of $\mathscr{I}$ by F. We assume furthermore that $\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/\mathscr{I} < \infty$. Then $\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/\mathscr{I} \circ F < \infty$ also, and we have the equality*

$$\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/\mathscr{I} \cdot \dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/(F(z)) = \dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/\mathscr{I} \circ F. \tag{4.6}$$

Here $(F(z))$ denotes the ideal generated by $F_1(z), \ldots, F_n(z)$ in $\mathcal{O}[\![z]\!]$.

Proof. The proof is based on standard arguments in commutative algebra. Let $B = \mathcal{O}[\![z]\!]$ and $A = \mathcal{O}[\![F_1, \ldots, F_n]\!]$. Since $\dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/(F)$ is finite, it follows from Zariski-Samuel [22], Vol. 2, p. 259, Corollary 2, and Nagata [20], Theorem

(25.16), that B is a free A-module of rank $\dim_{\mathbb{C}} B/(F)$. The desired conclusion (4.6) follows by observing that

$$B/\mathcal{I}\circ F = B\otimes_A A/\mathcal{I}\circ F,$$

and the fact that $\dim_{\mathbb{C}} A/\mathcal{I}\circ F = \dim_{\mathbb{C}} \mathcal{O}[\![z]\!]/\mathcal{I}$.

We also need the following.

(4.7) **Lemma.** *Let $a(x, y)$ be a convergent power series in $\mathbb{C}^n \times \mathbb{C}^p$, $x=(x_1, \ldots, x_n)$, $y=(y_1, \ldots, y_p)$, with $a(x, 0)=a(0, y)=0$. We write $a(x, y)=\sum a_\alpha(x) y^\alpha$ and let $\mathcal{I}=(a_\alpha(x))$ be the ideal generated by the coefficients of y^α in $\mathcal{O}[\![x]\!]$, and let $\mathcal{I}_\varepsilon$ be the ideal in $\mathcal{O}[\![x]\!]$ generated by $\{(a(x, y), y\in\mathbb{C}^p, |y|<\varepsilon\}$. Then there exists $\varepsilon_0>0$ such that*

$$\mathcal{I} = \mathcal{I}_\varepsilon$$

for all ε, $0<\varepsilon\leqq\varepsilon_0$.

Proof. This result is also standard, and can be proved, for instance, by using the Krull Intersection Theorem (see e.g. [22], Vol. 1, Theorem 12') in the form $\bigcap_n (\mathcal{I}+\mathcal{J}^n)=I$ for any ideal $\mathcal{I}$ of $\mathcal{O}[\![z]\!]$ and $\mathcal{J}$ the maximal ideal.

We may now prove (1.4). We begin with the identity (3.10), which gives equality of the ideals in $\mathcal{O}[\![z]\!]$ generated by $\{\varphi(z, \zeta, 0): |\zeta|<\varepsilon\}$ and $\{\psi(F(z, 0), \bar{F}(\zeta, 0), 0): |\zeta|<\varepsilon\}$. The latter is the same as the ideal generated by $\{\psi(F(z, 0), \zeta', 0): |\zeta'|<\varepsilon\}$, provided $\varepsilon>0$ is sufficiently small (see Lemma (4.7)). The desired equality (1.4) follows from the definition of the essential type given in § 1 and from Lemmas (4.5) and (4.7).

§ 5. Technical consequences of the finite multiplicity of H

This section contains technical lemmas to be used in the proof of Theorem 1. We will assume here that H: $M\to M'$, $H(0)=0$, is a CR map, that H is of finite multiplicity at 0, and that M' is essentially finite.

We begin with (3.2). By the implicit function theorem we can solve in $\bar{g}$ to obtain

$$\bar{g}=Q(f, \bar{f}, g), \tag{5.1}$$

where $Q(z, \zeta, w)$ is holomorphic and satisfies

$$Q(z, 0, w)\equiv Q(0, \zeta, w)\equiv w. \tag{5.2}$$

By Theorem 3, since M' is assumed essentially finite and H is of finite multiplicity, M is essentially finite also. It is therefore of finite type, so that every CR function extends holomorphically to at least one side of M [6]. In particular, we may assume that $f_j(z, \bar{z}, s)$ $1\leqq j\leqq n$, and $g(z, \bar{z}, s)$ extend holomorphically to the upper half plane in s for fixed z, $|z|$ small (see [4]).

We prove here the following,

(5.3) **Lemma.** *There exists $r>0$ such that for every $z_0 \in \mathbb{C}^n$ fixed, $|z_0|<r$, and every multi-index α, there exist functions $a(s+it)$, $b(s+it)$ holomorphic in the domain*

$$R=\{s+it: |s|<r, -r<t<0\}, \tag{5.4}$$

smooth in $\bar{R}$, such that $b \not\equiv 0$ and

$$Q_{\zeta^\alpha}(f, \bar{f}, g)(z_0, \bar{z}_0, s)=\frac{a(s)}{b(s)}, \quad |s|<r. \tag{5.5}$$

Proof. For $\alpha=0$ (5.5) follows from (5.1) and the above remarks on holomorphic extendability of CR functions by taking $a(s)=\bar{g}(z_0, \bar{z}_0, s)$, $b(s)\equiv 1$. Now suppose $|\alpha|=1$. From (5.1) and the fact that $L_i f_i = L_i g = 0$ we obtain,

$$L_i \bar{g}=\sum_{j=1}^{n} Q_{\zeta_j}(f, \bar{f}, g)(L_i \bar{f}_j), \quad i=1, 2, \ldots, n.$$

Let $D(z, s)=\det(L_i \bar{f}_j(z, \bar{z}, s))$. By Cramer's rule we write for $j=1, 2, \ldots, n$

$$D(z, s)\, Q_{\zeta_j}(f, \bar{f}, g)=P_j(L_i \bar{f}_j, L_i \bar{g}), \tag{5.6}$$

where P_j is a polynomial in $L_i \bar{f}_j$ and $L_i \bar{g}$, $1 \leqq i \leqq n$. Suppose first that $D(z_0, s) \not\equiv 0$ as a function of s. Then (5.5) is proved by dividing (5.6) by D, since the coefficients of the L_i are real analytic in s. If, on the other hand $D(z_0, s)\equiv 0$, then by applying L_{j_1} to (5.5) we have

$$(L_{j_1} D)\, Q_{\zeta_j}+D(L_{\zeta_1} Q_{\zeta_j})=L_{\vartheta_1} P(L_i \bar{f}_j, L_i \bar{g}). \tag{5.7}$$

Since $D(z_0, s)\equiv 0$, if $L_{j_1} D(z_0, s) \not\equiv 0$, we may divide by it in (5.7) to obtain (5.5) for Q_{ζ_j}. If $L_j D(z_0, s)\equiv 0$ for all j_1, $1 \leqq j_1 \leqq n$, then we may apply another L, say L_{j_2}, to (5.7) and prove (5.5) provided $L_{j_2} L_{j_1} D(z_0, s) \not\equiv 0$. Since by Proposition (3.18) there exists a multi-index α for which $L^\alpha D(0) \neq 0$, by choosing the proper string $L_{j_p} L_{j_{p-1}} \ldots L_{j_1}$ the above process will terminate in proving (5.5) for any Q_{ζ_j}.

Now we prove (5.5) when $|\alpha|=2$. Dividing (5.6) by $D(z, s)$, and applying L_p, $1 \leqq p \leqq n$, and then Cramer's rule again we obtain for $1 \leqq j, k \leqq n$

$$D^3(z, s)\, Q_{\zeta_j \zeta_k}(f, \bar{f}, g)=P_{jk}(L^\gamma \bar{f}, L^\gamma \bar{g}), \tag{5.8}$$

where P_{jk} is a polynomial in $L^\gamma \bar{f}$, $L^\gamma \bar{g}$, $|\gamma| \leqq 2$. We proceed as in the case $|\alpha|=1$. If $D(z_0, s) \not\equiv 0$ we divide (5.8) by D^3 to reach the desired conclusion (5.5) for $Q_{\zeta_j \zeta_k}$. If not we apply the proper string of L's to (5.8) to reach the same conclusion, using again Proposition (3.18), since there must be a multi-index α' for which $L^{\alpha'} D^3(0) \neq 0$.

It is now clear that by repeating the same argument we obtain for any multi-index α

$$D^{2|\alpha|-1}(z, s)\, Q_{\zeta^\alpha}(f, \bar{f}, g)=P_\alpha(L^\gamma \bar{f}, L^\gamma \bar{g}), \tag{5.9}$$

where P_α is a polynomial in $L^\gamma \bar{f}$, $L^\gamma \bar{g}$, $|\gamma| \leq |\alpha|$. We reach the conclusion (5.5) for Q_{ζ^α} by using (5.9) and Proposition (3.18), as in the case $|\alpha| = 1, 2$. The proof of Lemma (5.3) is complete.

§ 6. Polynomial identities for the coordinate functions of H

Under the same assumptions as in § 5, we prove here that the coordinate functions $f_1, \ldots, f_n$, where $H = (f_1, \ldots, f_n, g)$ as in § 5, satisfy polynomial identities. More precisely we have

(6.1) **Lemma.** *There exists a finite set S of multi-indices such that for each j, $1 \leq j \leq n$, there exists an integer N_j and functions $a_k^j(u_p^\gamma, v^\beta)$, $1 \leq p \leq n$, $\beta, \gamma \in S$, holomorphic near $u_{p,0}^\gamma = L^\gamma \bar{f}_p(0)$ and $v_0^\beta = 0$ such that*

(6.2)
$$f_j^{N_j} + \sum_{k=0}^{N_j - 1} a_k^j(L^\gamma \bar{f}_p, L^\beta \bar{g}) f_j^k = 0,$$

in a neighborhood of $z = 0$, $s = 0$.

Proof. We have by (5.1)

(6.3)
$$\bar{g} = Q(f, \bar{f}, g) = R(f, \bar{f}) + g P(f, \bar{f}, g),$$

where R and P are holomorphic. We write

(6.4)
$$R(f, \bar{f}) = \sum_\alpha a_\alpha(f) \bar{f}^\alpha.$$

Since M' is essentially finite it follows easily that for all ζ_0 sufficiently small there exists a multi-index α_0 such that

(6.5)
$$a_{\alpha_0}(\zeta_0) \neq 0.$$

We claim that for every ζ_0 sufficiently small there exists a multi-index β_0 such that

(6.6)
$$L^{\beta_0} R(\zeta_0, \bar{f})(0) = \sum_\alpha a_\alpha(\zeta_0) L^{\beta_0} \bar{f}^\alpha(0) \neq 0.$$

Indeed, since by Lemma (2.18)

(6.7)
$$L^\beta \bar{f}^\alpha(0) = \left(\frac{\partial}{\partial \bar{z}}\right)^\beta \bar{F}^\alpha(0)$$

for all multi-indices α, β, it suffices to show

(6.8)
$$\sum_\alpha a_\alpha(\zeta_0) \bar{F}^\alpha(z, 0) \not\equiv 0$$

as a formal power series in z. Reasoning by contradiction, assuming (6.8) does not hold and using (6.5), we obtain a nontrivial relation

$$U(F_1(z,0)), \ldots, F_n(z,0)) \equiv 0, \tag{6.9}$$

with U a holomorphic function in n variables.

Using the Weierstrass Preparation Theorem, and after a linear change of variables, we can see that (6.9) implies that $F_1(z,0)$ is a zero divisor in $\mathcal{O}[[z]]/(F_2(z,0), \ldots, F_n(z,0))$, contradicting the assumption that the number given by (3.6) is finite (see e.g. [16]). This establishes (6.6).

As β varies over all multi-indices the convergent power series in ζ

$$\sum_\alpha a_\alpha(\zeta) L^\beta \bar{f}^\alpha(0) \tag{6.10}$$

(which are finite sums over α) have no common zeroes, by (6.6). By the Noetherian Theorem there exist finitely many multi-indices, $\beta_1, \ldots, \beta_r$ such that the holomorphic functions

$$\sum_\alpha a_\alpha(\zeta) L^{\beta_j} \bar{f}^\alpha(0), \qquad 1 \leqq j \leqq r, \tag{6.11}$$

have no common zeroes in ζ. By the Nullstellensatz, for N large we can write for $p = 1, \ldots, n$,

$$\zeta_p^N = \sum_{j=1}^{r} b_{jp}(\zeta) \sum_\alpha a_\alpha(\zeta) L^{\beta_j} \bar{f}^\alpha(0), \tag{6.12}$$

with $b_{jp}(\zeta)$ convergent power series. Substituting f in (6.12) we have

$$f_p^N = \sum_{j,\alpha} b_{jp}(f)\, a_\alpha(f)\, L^{\beta_j} \bar{f}^\alpha(0). \tag{6.13}$$

On the other hand by applying L^{β_j} to (6.3) and using (6.4) we have

$$L^{\beta_j} \bar{g} = \sum_\alpha a_\alpha(f) L^{\beta_j} \bar{f}^\alpha + \bar{Q}(\bar{f}, f, \bar{g}) L^{\beta_j} P(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g})). \tag{6.14}$$

By multiplying (6.14) by $b_{jp}(f)$ and summing over j we obtain

$$\begin{aligned} &\sum_{\alpha,j} b_{jp}(f)\, a_\alpha(f)\, L^{\beta_j} \bar{f}^\alpha \\ &\quad = \sum_j b_{jp}(f) \left[L^{\beta_j} \bar{g} - \bar{Q}(\bar{f}, f, \bar{g}) L^{\beta_j} P(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g})) \right]. \end{aligned} \tag{6.15}$$

Using (6.13) and (6.15) we have

$$\begin{aligned} f_p^N &= \sum_{j,\alpha} b_{jp}(f)\, a_\alpha(f) (L^{\beta_j} \bar{f}^\alpha(0) - L^{\beta_j} \bar{f}^\alpha) \\ &\quad + \sum_j b_{jp}(f) \left[L^{\beta_j} \bar{g} - \bar{Q}(\bar{f}, f, \bar{g}) L^{\beta_j} P(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g})) \right]. \end{aligned} \tag{6.16}$$

Let $u_p^\gamma = L^\gamma \bar{f}_p$ and $v^\gamma = L^\gamma \bar{g}$, considered as independent variables, for $1 \leqq p \leqq n$, and $0 \leqq |\gamma| \leqq \max_j |\beta_j|$. We let $u_{p,o}^\gamma = L^\gamma \bar{f}_p(0)$ and $v_0^\gamma = 0$ (since $L^\gamma \bar{g}(0) = 0$). Since $\tilde{Q}(0, z, 0) \equiv 0$, we may rewrite (6.16) in the form

$$f_p^N + K_p(f, u, v) = 0, \qquad 1 \leqq p < n, \tag{6.17}$$

where $K_p(Z, u, v)$ is holomorphic, $Z \in \mathbb{C}^n$, $u = (u_p^\gamma)$, $v = (v^\gamma)$ as above and

$$K_p(Z, u_0, v_0) \equiv 0, \tag{6.18}$$

with $u_0 = (u_{p,0}^\gamma)$, $v_0 = (v_0^\gamma)$.

We apply the Weierstrass Preparation Theorem to (6.17) with $p = 1$ to obtain

$$f_1^N + \sum_{j=0}^{N-1} c_j^1(f_2, \ldots, f_n, u, v) f_1^j = 0 \tag{6.19}$$

where the c_j^1 are holomorphic and satisfy

$$c_j^1(0, \ldots, 0, u_0, v_0) = 0.$$

Let $\rho_j(f_2, \ldots, f_n, u, v)$ be the roots in f_1 $(1 \leqq j \leqq N)$ of the polynomial (6.19). Then using (6.17) with $p = 2$, and replacing f_1 by ρ_j, $j = 1, \ldots, N$, we obtain

$$\begin{aligned} &\prod_{j=1}^{N} [f_2^N + K_2(\rho_j(f_2, \ldots, f_n, u, v), f_2, \ldots, f_n, u, v))] \\ &= H_2(f_2, \ldots, f_n, u, v) = 0, \end{aligned} \tag{6.20}$$

where H_2 is holomorphic (since it is a symmetric function of the roots), and

$$H_2(Z_2, \ldots, Z_n, u_0, v_0) = 0$$

(by (6.18)). Applying the Weierstrass Preparation Theorem with respect to f_2 at $(0, u_0, v_0)$ to (6.20) we obtain

$$f_2^{N^2} + \sum_{j=0}^{N^2-1} a_j^2(f_3, \ldots, f_n, u, v) f_2^j = 0. \tag{6.21}$$

We continue this argument to obtain

$$f_k^{N^k} + \sum_{j=0}^{N^k-1} a_j^k(f_{k-1}, \ldots, f_n, u, v) f_k^j = 0, \tag{6.22}$$

$1 \leqq k \leqq n$. Beginning with (6.22) for $k = n$, substituting the roots into (6.22) with $k = n - 1$ and taking the product over all roots, we successively eliminate the f_j from the coefficients of (6.22) and obtain the desired conclusion.

§ 7. Polynomial identities for Q_ζ.

As in § 6 we shall prove that for each α, $Q_{\zeta^\alpha}(f, \bar{f}, g)$ satisfies a polynomial relation with coefficients as in Lemma (6.1). This result is the following.

(7.1) **Lemma.** *There exists an integer N such that for each multi-index α, $Q_{\zeta^\alpha}(f, \bar{f}, g)$ is a root of a polynomial of the form*

$$X^N + \sum_{j=0}^{N-1} b_j^\alpha(L^\gamma \bar{f}_p, L^\beta \bar{g}) X^j = 0 \tag{7.2}$$

with γ, $\beta \in S$ defined in Lemma (6.1), *and b_j^α are holomorphic and satisfying*

$$|b_j^\alpha(L^\gamma \bar{f}, L^\gamma \bar{g})(z, \bar{z}, s+it)| \leqq (C^{|\alpha|+1} |\alpha|!)^{N-j} \tag{7.3}$$

for $|z|<r$, $|s|<r$ and $-r \leqq t \leqq 0$.

Proof. Taking the complex conjugate of (5.1) we have

$$g = \bar{Q}(\bar{f}, f, \bar{g})$$

therefore

$$Q_{\zeta^\alpha}(f, \bar{f}, g) = Q_{\zeta^\alpha}(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g})) = R_\alpha(f, \bar{f}, \bar{g}). \tag{7.4}$$

Substituting $f_1, \ldots, f_n$ in $R_\alpha(f, \bar{f}, \bar{g})$ by the roots of the polynomials in Lemma (6.1) we obtain that $X = Q_{\zeta^\alpha}(f, \bar{f}, g)$ satisfies the equation

$$\prod_\beta (X - R_\alpha(\rho_\beta, \bar{f}, \bar{g})) = 0, \tag{7.5}$$

where $\rho_\beta = (\rho_1^{\beta_1}, \ldots, \rho_n^{\beta_n})$ and $\rho_j^{\beta_j}$ is the β_jth root of the polynomial (6.2). Then (7.2) follows by expanding the left hand side of (7.5) and observing that the coefficients are symmetric functions of the roots of (6.2).

In order to prove estimates (7.3) we observe that

$$|Q_{\zeta^\alpha}(z, \zeta, w)| \leqq C^{|\alpha|+1} |\alpha|!$$

for $|z|$, $|\zeta|$, $|w| \leqq r$ and that the roots of (6.2) are well defined for $|z|<r$, $s+it$, $|s| \leqq r$, $-r \leqq t \leqq 0$. Since the coefficients of the polynomial defined by the left hand side of (7.5) are the basic symmetric functions of its roots, (7.3) follows.

§ 8. Proof of Theorem 1 and Corollary 2. Remark

By the first part of Theorem 3, it suffices to prove the conclusion of Theorem 1 under the assumptions that H is of finite multiplicity at 0 and M' is essentially finite at 0, (which implies that M is essentially finite, by Theorem 2). We follow the general approach of [4]. Starting with (5.1) and taking the Taylor expansion we obtain

$$(8.1)\qquad Q(f, \lambda, g)(z, \bar{z}, s) = \sum_{\alpha=0}^{\infty} \frac{(\lambda - \bar{f})^{\alpha}}{\alpha!} Q_{\zeta^{\alpha}}(f, \bar{f}, g)(z, \bar{z}, s),$$

valid for $(z, \bar{z}, s)$ in a neighborhood of 0 in $\mathbb{R}^{2n+1}$ and any small complex number λ. We use Lemma (8.15) of [3], combined with Lemmas (5.3) and (7.1) to conclude that the right hand side of (8.1) extends holomorphically as a function of s to the domain R defined by (5.4) uniformly for $|z| \leqq r$. On the other hand, the left hand side of (8.1) extends holomorphically as a function of s to the domain $-R$ since f and g have this property.

Therefore, we conclude from (8.1) that the function

$$s \mapsto Q(f, \lambda, g)(z, \bar{z}, s)$$

is real analytic in s, $|s| < r$, uniformly in z, λ, $|z| < r$, $|\lambda| < r$. The desired real analyticity of g follows by taking $\lambda = 0$ and using (5.2). As in [3] applying the Weierstrass Preparation Theorem to (8.1) with respect to f and Lemma (6.1) of [4] we obtain the real analyticity of f, which completes the proof of Theorem 1.

Proof of Corollary 2. Here we take $M = \partial D$, the boundary of D, and $M' = \partial D'$. Since $F \in C^{\infty}(\bar{D})$ by [9] and [12], the mapping $H = F|_M$ is a smooth CR mapping from M into M'. We claim that assumptions (ii) of Theorem 1 are satisfied at every point $p_0 \in M$. Indeed since M is a real analytic compact manifold it is essentially finite at each point (see [4]). On the other hand using an argument due to Fornaess [15] it is easy to show as in [3] that the pseudoconvexity of D and D' implies that (1.3) (or its equivalent form (ii) of Proposition (3.3)) holds at each point of M. Corollary 2 is then an immediate consequence of Theorem 1.

Remark. In $\mathbb{C}^2$ it is shown in [3] that if M and M' are essentially finite at p_0 and p'_0 respectively (i.e. of finite type), then any CR map $H = (f, g)$, $H(p_0) = p'_0$, extends holomorphically at p_0 unless both f and g are flat at p_0. This is false in higher dimension as shown by the following example. Let $M = \{(z_1, z_2, w) \in \mathbb{C}^3;\ \operatorname{Im} w = |z_1|^2 + |z_2|^2\}$ and $M' = \{(z_1, z_2, w) \in \mathbb{C}^3;\ \operatorname{Im} w = |z_1|^2 - |z_2|^2\}$. Then M and M' are both essentially finite. Let f be a smooth CR function on M which does not extend holomorphically at 0, $f(0) = 0$. Then the mapping $H\colon M \to M'$ given by $H = (f_1, f_2, g)$ with $f_1 = f_2 = f$ and $g = 0$ does not extend holomorphically. Note that here H is not of finite multiplicity at 0.

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