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EQUATIONS AUX DERIVEES PARTIELLES

HOLOMORPHIC MAPPINGS BETWEEN ALGEBRAIC HYPERSURFACES IN COMPLEX SPACE

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0. Introduction

We give here an account of recent work [BR4] of the authors characterizing those real algebraic hypersurfaces in \mathbb{C}^N between which all holomorphic mappings must be algebraic. Some applications of this work were given in joint work with X. Huang [BHR] to prove analyticity of sufficiently smooth CR mappings between such hypersurfaces. We outline here some of the proofs in [BR4], including a simplification of one part, as well as some other improvements.

A real hypersurface in \mathbb{C}^N is *algebraic* if it is given by the vanishing of a real valued polynomial with nonvanishing gradient. A germ of a holomorphic function is *algebraic* if it is the root of a polynomial with holomorphic polynomial coefficients. Similarly, a germ of a holomorphic map is algebraic if its components are. We need to introduce the following definition. A real analytic hypersurface M in \mathbb{C}^N is *holomorphically degenerate* at a point $p_0 \in M$ if there exists a nontrivial germ of a holomorphic vector field, with holomorphic coefficients, tangent to M in a neighborhood of p_0 . (See also Stanton [S], where this definition was introduced.) We have the following.

Proposition 0.1. *If M is a connected real analytic hypersurface, then M is holomorphically degenerate at a given $p_0 \in M$ if and only if it is holomorphically degenerate at every point in M .*

If M is connected, we shall say that M is *holomorphically nondegenerate* if it is not holomorphically degenerate at any point in M , or, equivalently, by Proposition 0.1, it is not holomorphically degenerate at a given point p_0 in M . We can now state our main result.

Theorem. *Let M and M' be two algebraic hypersurfaces in \mathbb{C}^N and $p_0 \in M$. If M is connected and holomorphically nondegenerate, and H is a germ at p_0 of a biholomorphism of \mathbb{C}^N mapping M into M' , then H is algebraic. Conversely, if M is algebraic and holomorphically degenerate at p_0 , then there exists a germ of a biholomorphism of \mathbb{C}^N at p_0 , mapping M into itself and fixing p_0 , which is not algebraic.*

The following is an easy consequence of the Theorem and of a result in [BR3].

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Corollary. *Let M and M' be as in the first part of the Theorem and H a holomorphic mapping defined in a neighborhood of M in \mathbb{C}^N with $H(M) \subset M'$. Then H is algebraic if either the Jacobian determinant of H does not vanish identically or M' does not contain any nontrivial complex variety.*

We shall say that a property holds *generically* on M if it holds in an open, dense subset of M . It should be noted that if M is generically Levi nondegenerate, then M is holomorphically nondegenerate. The converse, however, holds only in \mathbb{C}^2 . For instance, the hypersurface in \mathbb{C}^3 given by $(\Re Z_1)^2 + (\Re Z_2)^2 - (\Re Z_3)^2 = 0$ is Levi degenerate at every point (with $\Re Z \neq 0$), but holomorphically nondegenerate. In 1977 Webster [W1] proved the first part of the above theorem for the Levi nondegenerate case. Previous results were proved by Poincaré [P] and Tanaka [T] for pieces of spheres in \mathbb{C}^N . We note here that Webster's result has been extended in some cases to nondegenerate hypersurfaces of different dimensions. See e.g. Webster [W2], Forstnerič [Fo], Huang [H] and their references.

1. Normal coordinates and the Levi type of a hypersurface

Let $M \subset \mathbb{C}^N$ be a real analytic hypersurface, given by $\rho(Z, \bar{Z}) = 0$ near p_0 with $d\rho \neq 0$ and $N = n + 1$. We can find (see [CM], [BJT]) holomorphic coordinates (z, w) , (called *normal coordinates*), $z \in \mathbb{C}^n, w \in \mathbb{C}$ vanishing at p_0 such that near p_0 , M is given by

$$(1.1) \quad w = Q(z, \bar{z}, \bar{w}),$$

where $Q(z, \chi, \tau)$ is holomorphic in a neighborhood of 0 in \mathbb{C}^{2n+1} and satisfies $Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau$. We associate to M the complex hypersurface \mathcal{M} in \mathbb{C}^{2N} locally defined near (p_0, \bar{p}_0) by

$$(1.2) \quad \mathcal{M} = \{(Z, \zeta) : \rho(Z, \zeta) = 0\},$$

where $\rho(Z, \bar{Z})$ is the defining function for M near p_0 as above. We define the germ of an analytic subset $\mathcal{V}_{p_0} \subset \mathbb{C}^N$ through p_0 by

$$(1.3) \quad \mathcal{V}_{p_0} = \{Z : \rho(Z, \zeta) = 0 \text{ for all } \zeta \text{ near } \bar{p}_0 \text{ with } \rho(p_0, \zeta) = 0\}.$$

Note in fact that $\mathcal{V}_{p_0} \subset M$. Recall that M is called *essentially finite* at p_0 if $\mathcal{V}_{p_0} = \{p_0\}$. We also recall (see [BR2]) that the set of essentially finite points in each connected component of M is either empty or open and dense. In the sequel we assume $p_0 = 0$.

Let $L_1, \dots, L_n, n = N - 1$, given by $L_j = \sum_{k=1}^N a_{jk}(Z, \bar{Z}) \partial / \partial \bar{Z}_k$ be a basis of the CR vector fields on M near 0 with the a_{jk} real analytic. We need to introduce the following vector-valued functions. For a multi-index α , let V_α be the real analytic function defined near 0 in \mathbb{C}^N by

$$(1.4) \quad V_\alpha(Z, \bar{Z}) = L^\alpha \rho_Z(Z, \bar{Z}),$$

where ρ_Z denotes the gradient of ρ with respect to Z and $L^\alpha = L_1^{\alpha_1} \dots L_n^{\alpha_n}$. We have the following lemma whose proof could be essentially found in [BHR].

Lemma 1.5. *Let M be a connected real analytic hypersurface in \mathbb{C}^N . The following conditions are equivalent:*

- (i) M is holomorphically nondegenerate.
- (ii) $\{V_\alpha(Z, \bar{Z}), \alpha \in \mathbb{Z}_+^n\}$ span \mathbb{C}^N generically in a neighborhood of p_0 in M .
- (iii) There exists an integer k , with $1 \leq k \leq n$ so that $\{V_\alpha(Z, \bar{Z}), |\alpha| \leq k\}$ span \mathbb{C}^N generically in a neighborhood of p_0 in M .

We say that the hypersurface M is k -holomorphically nondegenerate at $Z \in M$ if $\{V_\alpha(Z, \bar{Z}), |\alpha| \leq k\}$ span \mathbb{C}^N , with k minimal. In particular, it is easy to see that M is 1-holomorphically nondegenerate at Z if and only if the Levi form of M is nondegenerate at Z . Note that if M is connected and holomorphically nondegenerate then there exists $\ell = \ell(M)$, $1 \leq \ell(M) \leq N - 1$, such that M is ℓ -holomorphically nondegenerate at every point in an open dense subset of M . We call $\ell(M)$ the *Levi type* of M . The Levi type of M is 1 if and only if M is generically Levi nondegenerate.

If M is given by (1.1), or equivalently by $\bar{w} = \bar{Q}(\bar{z}, z, w)$, and $Z = (z, w)$, then

$$(1.6) \quad V_\alpha(Z, \bar{Z}) = -\bar{Q}_{\bar{z}\alpha, Z}(\bar{z}, z, w).$$

2. Proof of the first part of the Theorem

Assume M, M', p_0 and H are as in the assumptions of the Theorem. By Lemma 1.5 and the comments following, by slightly moving p_0 , we may assume that M is ℓ -holomorphically nondegenerate at p_0 , with $\ell = \ell(M)$, the Levi type of M as defined in §1. We choose normal coordinates (z, w) for M , vanishing at p_0 , and normal coordinates (z', w') for M' vanishing at $H(p_0)$. We write the mapping $H = (f, g)$ with $z' = f(z, w)$ and $w' = g(z, w)$. We assume that M is given by (1.1) and M' is given by $w' = Q'(z', \bar{z}', \bar{w}')$. Since $H(M) \subset M'$, we have for $(z, w) \in M$ in a neighborhood of 0,

$$\bar{g}(\bar{z}, \bar{w}) = \bar{Q}'(\bar{f}(\bar{z}, \bar{w}), f(z, w), g(z, w)).$$

Since the manifold \mathcal{M} introduced in (1.2) is given by $\tau = \bar{Q}(\chi, z, w)$ for $(z, \chi, w, \tau) \in \mathbb{C}^{2N}$, and a similar equation for \mathcal{M}' , it follows from the above that we have for $(z, w, \chi, \tau) \in \mathcal{M}$

$$(2.1) \quad \bar{g}(\chi, \tau) = \bar{Q}'(\bar{f}(\chi, \tau), f(z, w), g(z, w)).$$

We now introduce the following holomorphic vector fields which are tangent to \mathcal{M} :

$$(2.2) \quad \mathcal{L}_j = \frac{\partial}{\partial \chi_j} + \bar{Q}_{\chi_j}(\chi, z, w) \frac{\partial}{\partial \tau}, \quad j = 1, \dots, n,$$

Note that the \mathcal{L}_j commute with each other. Since M and M' are algebraic, and the functions Q and Q' are obtained by the implicit function theorem, it follows that they are algebraic also.

Lemma 2.3. *For (z, w, χ, τ) in a neighborhood of 0 in \mathcal{M} the following holds:*

$$(2.4) \quad f_j(z, w) = \Psi_j(\mathcal{L}^\gamma \bar{f}_p(\chi, \tau), \mathcal{L}^\beta \bar{g}(\chi, \tau)), \quad j = 1, \dots, n,$$

with $|\gamma|, |\beta| \leq \ell$ and the Ψ_j holomorphic functions of their arguments.

Proof. By Lemma 1.5, identity (1.6), and the choice of p_0 , we have

$$(2.5) \quad \text{span} \{ \bar{Q}_{z^\alpha, z}(0, 0, 0) : |\alpha| \leq \ell \} = \mathbb{C}^N.$$

Since H is a biholomorphism at the origin, it follows from the choice of normal coordinates, that the matrix $(\mathcal{L}_j \bar{f}_k)(0)$ is invertible. Using this, applying repeatedly the \mathcal{L}_j to (2.1) and using (2.5), we may obtain the lemma by the use of the implicit function theorem. \square

We are now ready to prove the first part of the Theorem. We first prove the following preliminary result.

Lemma 2.6. *For every integer q the mapping $z \mapsto \frac{\partial^q}{\partial w^q} H(z, 0)$ is holomorphic algebraic in a neighborhood of 0 in \mathbb{C}^n .*

Proof. We begin with the identity (2.4). We note that for $z \in \mathbb{C}^n$ close to 0, the point $(z, w, \zeta, \tau) = (z, 0, 0, 0)$ is in \mathcal{M} , since $Q(z, 0, 0) \equiv 0$. Since the coefficients of \mathcal{L}_j given by (2.2) are algebraic holomorphic, for any holomorphic function $J(\zeta, \tau)$, the functions $(z, w) \mapsto (\mathcal{L}^\gamma J(\zeta, \tau))|_{\zeta=0, \tau=0}$ are algebraic holomorphic. Evaluating (2.4) at $(z, 0, 0, 0)$, we obtain the conclusion of the lemma for $q = 0$ since $g(z, 0) \equiv 0$.

To prove the lemma for $q > 0$, we take $\chi = 0$ and $\tau = w$ in (2.4). Differentiating the resulting identity q times with respect to w and evaluating at $w = 0$ gives the desired result for the f_j . From (2.1) we have on \mathcal{M}

$$(2.7) \quad g(z, w) = Q'(f(z, w), \bar{f}(\chi, \tau), \bar{g}(\chi, \tau)).$$

As before, we take $\chi = 0$ and $\tau = w$ in (2.7), differentiate q times in w , and evaluate at $w = 0$. The conclusion of the lemma for g then follows from that for the f_j . \square

To complete the proof of the first part of the Theorem, we use (2.4) in which we take $\tau = 0$ and substitute $Q(z, \chi, 0)$ for w to obtain

$$(2.8) \quad f_j(z, Q(z, \chi, 0)) = \Psi_j(\mathcal{L}^\gamma \bar{f}_p(\chi, 0), \mathcal{L}^\beta \bar{g}(\chi, 0)), \quad j = 1, \dots, n,$$

which holds as an identity in $(z, \chi) \in \mathbb{C}^{2n}$ near 0. Note that after this substitution the coefficients of the vector fields \mathcal{L}_j are then algebraic holomorphic in (z, χ) . Since M is ℓ -holomorphically nondegenerate at 0, and the coordinates are taken to be normal, we conclude that the vector function $Q_\chi(z, \chi, 0)$ does not vanish identically. Hence we may assume there is (z^0, χ^0) such that $Q_{\chi^0}(z^0, \chi^0, 0) \neq 0$. Note that (z^0, χ^0) can be chosen arbitrarily close to 0 in \mathbb{C}^{2n} . Put $w^0 = Q(z^0, \chi^0, 0)$. By the implicit function theorem, we can find an algebraic holomorphic function $\theta(z, w)$ defined near (z^0, w^0) and satisfying $\theta(z^0, w^0) = \chi^0$, such that the following identity holds for (z, w) near (z^0, w^0) in \mathbb{C}^{n+1} :

$$(2.9) \quad Q(z, \theta(z, w), \chi_2^0, \dots, \chi_n^0, 0) \equiv w.$$

We now take $\chi = (\theta(z, w), \chi_2^0, \dots, \chi_n^0)$ in (2.8). After making this substitution, we consider that (z, w) are independent variables near (z^0, w^0) . (Recall that τ has been set to 0 throughout this part of the proof.) After this substitution the functions

$$(2.10) \quad (z, w) \mapsto \Psi_j(\mathcal{L}^\gamma \bar{f}_p, \mathcal{L}^\beta \bar{g})$$

are seen to be algebraic holomorphic, by using Lemma 2.6. This proves that the components $f_j(z, w)$ of H are algebraic. To prove that $g(z, w)$ is algebraic we again use (2.7) with the same substitution as above. The already proved algebraicity of the f_j gives that of g . This finishes the proof of the first part of the Theorem. \square

3. Proof of the second part of the Theorem; Flow of holomorphic vector fields

We now give the proof of the second part of the Theorem. Let $p_0 \in M$ and assume that X is a nontrivial germ at p_0 of a holomorphic vector field tangent to M . To any such X , there is a holomorphic one parameter group of local biholomorphisms in \mathbb{C}^N sending M into M defined by the complex flow of X i.e.

$$(3.1) \quad \dot{\phi}(t, Z) = X(\phi(t, Z)), \quad \phi(0, Z) = Z.$$

Then $\phi(t, Z)$ is holomorphic for $t \in \mathbb{C}, |t| < \epsilon$, and $Z \in V$, where V is an open neighborhood of p_0 in \mathbb{C}^N . For fixed t , the map $Z \mapsto \phi(t, Z)$ is a local biholomorphism preserving M , and if $X(p_0) = 0$, then $\phi(t, p_0) \equiv p_0$.

The second part of the Theorem will be a consequence of (ii) of the following proposition.

Proposition 3.2. *Let M be a real algebraic hypersurface in \mathbb{C}^N , $p_0 \in M$, and X a germ at p_0 of a nontrivial holomorphic vector field tangent to M . Then the following hold.*

- (i) *The germ at 0 of the holomorphic complex curve $t \mapsto \phi(t, p_0)$, where $\phi(t, p_0)$ is the flow of X starting from p_0 given by (3.1), is contained in \mathcal{V}_{p_0} , as defined by (1.3).*
- (ii) *There exists f , a germ at p_0 of a holomorphic function and arbitrarily small t such that if $\psi(t, Z)$ is the flow of $Y = fX$, the mapping $Z \mapsto \psi(t, Z)$ is a nonalgebraic local biholomorphism mapping M into itself and fixing p_0 .*

Proof. We show that the function $t \mapsto h(t) = \rho(\phi(t, p_0), \zeta)$ vanishes identically for $\zeta \in \mathbb{C}^N$ close to \bar{p}_0 with $\rho(p_0, \zeta) = 0$. If $X = \sum_{j=1}^N a_j(Z) \frac{\partial}{\partial Z_j}$, then $\frac{dh}{dt}(t) = \sum_{j=1}^N a_j(\phi(t, p_0)) \frac{\partial \rho}{\partial Z_j}(\phi(t, p_0), \zeta)$, which must be a multiple of $h(t)$. Since $h(0) = 0$, by the uniqueness of the solution of differential equations, we conclude that $h(t) \equiv 0$, proving (i).

To prove (ii), by standard arguments using the local group property, we have

$$(3.3) \quad \sum_{j=1}^N a_j(Z) \frac{\partial \phi_k}{\partial Z_j}(t, Z) = a_k(\phi(t, Z)), \quad k = 1, \dots, N.$$

We may assume $X(p_0) = 0$. If for some arbitrarily small t the map $Z \mapsto \phi(t, Z)$ is not algebraic, we are done. Otherwise, we assume $a_1 \neq 0$, and let $f(Z) = e^{Z_1}$ and $Y = e^{Z_1}X$. We denote by $\psi(t, Z)$ the holomorphic flow of Y . By (3.3) for the vector field Y instead of X , and taking $k = 1$, we have

$$(3.4) \quad \sum_{j=1}^N e^{Z_1} a_j(Z) \frac{\partial \psi_1}{\partial Z_j}(t, Z) = e^{\psi_1(t, Z)} a_1(\psi(t, Z)).$$

If $Z \mapsto \psi(t, Z)$ is algebraic for some fixed t , then since all the coefficients a_k are algebraic, it would follow from (3.4) that the function $Z \mapsto e^{Z_1 - \psi_1(t, Z)}$ is also algebraic. Note that $Z \mapsto Z_1 - \psi_1(t, Z)$ is algebraic and not constant (since $a_1 \neq 0$). However, if $A(Z)$ is any nonconstant algebraic holomorphic function, then the function $Z \mapsto e^{A(Z)}$ cannot be algebraic. This contradiction proves (ii). \square

4. Remarks

Remark 4.1. It follows from Proposition 0.1, Proposition 3.2 (ii) and the openness of the set of essentially finite points that a connected real analytic hypersurface M is holomorphically nondegenerate if and only if M is essentially finite at some point $p_0 \in M$.

Remark 4.2. An algebraic holomorphic function $h(Z)$ is said to be of degree m if it satisfies a polynomial equation $P(Z, f(Z)) = 0$, where $P(Z, X)$ is an irreducible polynomial in $N + 1$ variables of total degree m . By the total degree of an algebraic hypersurface M we mean the total degree of its defining real polynomial. An inspection of the proof of the Theorem shows that the degrees of the components of H are bounded by a constant depending only on the dimension N and the total degrees of M and M' .

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