Holomorphic Mappings of Real Analytic Hypersurfaces

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1. Introduction. The Schwarz reflection principle in one complex variable can be stated as follows.

Let M and M' be two real analytic curves in $\mathbf{C}$ *and* $\mathscr{H}$ *a holomorphic function defined on one side of M, extending continuously through M, and mapping M into M'. Then* $\mathscr{H}$ *has a holomorphic extension across M.*

We address here the question of extending this classical theorem to higher complex dimensions.

To make this precise, we first introduce some notation and basic definitions. By a real analytic (germ of a) hypersurface in $\mathbf{C}^{n+1}$, $n \geq 1$, we mean the set M of zeroes of a real analytic function $\rho(Z, \overline{Z})$, $\rho(0) = 0$, with nonvanishing differential $d\rho$. If Ω is a sufficiently small open neighborhood of 0 in $\mathbf{C}^{n+1}$, we denote by Ω^+ the set $\Omega^+ = \{Z \in \Omega : \rho(Z, \overline{Z}) > 0\}$. We consider a mapping $\mathscr{H}$, holomorphic in Ω^+, smooth in $\overline{\Omega^+}$, valued in $\mathbf{C}^{n+1}$ and satisfying $\mathscr{H}(M) \subset M'$, where M' is another real analytic hypersurface in $\mathbf{C}^{n+1}$ containing the origin. We shall always assume $\mathscr{H}(0) = 0$. We shall say that $\mathscr{H}$ *extends holomorphically at* 0 if there exists a smaller neighborhood of 0 in $\mathbf{C}^{n+1}$ to which $\mathscr{H}$ has a holomorphic extension. We discuss conditions on M, M', and $\mathscr{H}$ which guarantee that $\mathscr{H}$ extends holomorphically at 0.

The following simple example shows that the conclusion of the Schwarz reflection principle does not hold for "flat" hypersurfaces in $\mathbf{C}^{n+1}$.

EXAMPLE 1. Let M be defined by $\rho(Z, \overline{Z}) = \operatorname{Im} Z_{n+1}$ and put $M' = M$. Let $\mathscr{H}$ be the mapping defined by

$$\mathscr{H}_1(Z) = Z_1 + h(Z_{n+1}), \qquad \mathscr{H}_j(Z) = Z_j, \quad j = 2, \ldots, n+1,$$

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where h is a holomorphic function in the upper half plane in $\mathbf{C}$, smooth up to the boundary, but which does not extend holomorphically across 0. If we take h to satisfy $h(0) = h'(0) = 0$, the mapping $\mathscr{H}$ is a local diffeomorphism near 0 in $\overline{\Omega^+}$, for a sufficiently small Ω, such that $\mathscr{H}(M) \subset M'$, and for which $\mathscr{H}$ does not extend holomorphically at 0.

The reflection principle can also fail to hold even if M is not flat, if the mapping $\mathscr{H}$ is in some sense too degenerate, as shown by the following example.

EXAMPLE 2. Let M be the hypersurface in $\mathbf{C}^2$ given by

$$\rho(Z, \overline{Z}) = \operatorname{Im} Z_2 - (\operatorname{Re} Z_2)|Z_1|^2,$$

and take $M' = M$. Take $\mathscr{H}$ to be the mapping given by $\mathscr{H}_1(Z) = f(Z_2)$, and $\mathscr{H}_2(Z) = 0$, where f is holomorphic in the complex plane with a cut in the negative imaginary axis, and does not extend holomorphically in a neighborhood of 0. It is easily checked that $\mathscr{H}$ is holomorphic above M, maps M into M' and does not extend holomorphically to any neighborhood of 0 in $\mathbf{C}^2$.

In light of the above examples, we make the following definitions.

DEFINITION. If $\mathscr{H}: \Omega^+ \to \mathbf{C}^{n+1}$ is a holomorphic mapping with $\mathscr{H} \in C^\infty(\overline{\Omega^+})$, we shall say that $\mathscr{H}$ is not totally degenerate at 0 if its Jacobian determinant $J(\mathscr{H})(Z) = \det(\partial \mathscr{H}_j / \partial Z_k)(Z)$ has nonvanishing Taylor series at 0. We shall say that a real analytic hypersurface M has the reflection property at 0 if any holomorphic mapping defined on one side of M and not totally degenerate at 0, mapping M into another real analytic hypersurface M' of $\mathbf{C}^{n+1}$, extends holomorphically at 0.

In the 1970s, H. Lewy [21] and S. Pinčuk [22] gave the first positive results for reflection in higher dimensions. In fact, their work shows that the reflection principle holds when M and M' are strictly pseudoconvex and $\mathscr{H}$ is a local diffeomorphism. In the past decade these results have been considerably sharpened by weakening the conditions both on the hypersurfaces as well as on the mappings. The reader should consult the excellent survey article by Forstnerič [18] and its extensive references. We mention here in particular the papers of Webster [24], Diederich and Webster [15], Bedford and Bell [9], Baouendi, Jacobowitz, and Treves [7], Baouendi, Rothschild, and Bell [3], Diederich and Fornaess [14], and the authors [4, 5, 6].

For the case of $\mathbf{C}^2$, i.e. $n = 1$, a necessary and sufficient condition for a hypersurface to have the reflection property is now known. In §2 we shall give an account of these results, and in §3 discuss what is known for higher dimensions. §3 deals with globally-defined proper holomorphic mappings between bounded domains in $\mathbf{C}^{n+1}$. In §5 we give some indications of proofs for some of the results cited.

2. General reflection principle in $\mathbb{C}^2$. We shall say that M is *flat* if after a holomorphic change of coordinates in $\mathbb{C}^2$, M is given by $\operatorname{Im} Z_2 = 0$. Example 1 of §1 shows that a flat hypersurface does not have the reflection property. The following characterization of the reflection property shows that the converse also holds.

THEOREM 1 [6]. *A real analytic hypersurface M in $\mathbb{C}^2$ has the reflection property at 0 if and only if M is not flat.*

We may choose holomorphic coordinates $(z, w) \in \mathbb{C}^2$, such that M is given by

$$\operatorname{Im} w = \phi(z, \bar{z}, s), \quad \text{with } s = \operatorname{Re} w, \tag{2.1}$$

where ϕ is real analytic and satisfies $\phi(0) = 0$ and $d\phi(0) = 0$. We may also require that $\phi(z, 0, s) = 0$. With this condition, the (z, w) are called *normal coordinates* for M and w is called a *transversal coordinate.* If $\mathscr{H}$ is a holomorphic map in Ω^+, smooth up to M, mapping M into M', we write $\mathscr{H} = (f, g)$ if $z' = f(z, w)$ and $w' = g(z, w)$, where (z', w') are normal coordinates for M'. Then g is called a *transversal component* of $\mathscr{H}$. We shall denote by $F(z, w)$ and $G(z, w)$ the formal power series at the origin of f and g respectively.

If M is given by (2.1) then the induced Cauchy-Riemann operator on M is given by the vector field

$$L = \frac{\partial}{\partial \bar{z}} - i\frac{\phi_{\bar{z}}(z, \bar{z}, s)}{1 + i\phi_s(z, \bar{z}, s)}\frac{\partial}{\partial s},$$

where M is parametrized by (z, s). Recall that M is of (finite) type m at 0 if m is the smallest integer for which there exists a commutator of L and $\overline{L}$ of length m which is not in the linear span of L and $\overline{L}$ at 0 (see Kohn [20]).

Two distinct cases are studied in the proof of Theorem 1: M of finite type at 0 and M of infinite type. We shall consider first the case where M is of finite type. For this case, Theorem 1 can be strengthened considerably.

THEOREM 2 [3]. *Suppose that M is of finite type m and $\mathscr{H} = (f, g)$ as above, mapping M into M'. If the formal power series F and G do not both vanish identically, then the following hold:*

(i) $\frac{\partial G}{\partial w}(0) \neq 0$.

(ii) *There exists an integer $k \geq 1$ such that $\overline{L}^k f(0) \neq 0$.*

(iii) *M' is of finite type m', with $m = km'$, where k is the minimal integer satisfying* (ii).

(iv) *$\mathscr{H}$ extends holomorphically at 0 as a (finite) k-to-one map in a neighborhood of 0.*

It is unknown whether a nonzero map $\mathscr{H}$ as in Theorem 2 can have both formal power series F and G vanishing identically. (However, if M and

M' are replaced by totally real smooth manifolds, then it is proved in [1] that such a mapping cannot exist.)

For the case where M is of infinite type, there are a number of integers associated with M and the mapping $\mathscr{H}$. If M is given by (2.1) and (z, w) are normal coordinates, we write

$$\phi(z, \bar{z}, s) = \sum_{j=j_0+1}^{\infty} \phi_j(z, \bar{z}) s^j, \tag{2.2}$$

where $j_0 = j_0(M)$ is the minimal integer ≥ -1 with $\phi_{j_0+1}(z, \bar{z}) \neq 0$. Note that j_0 is finite if and only if M is not flat, and $j_0 \geq 0$ if and only if M is of infinite type. For the mapping $\mathscr{H} = (f, g)$ we write the power series F and G in the following form:

$$F(z, w) = \sum_{j=0}^{\infty} F_j(z) w^j, \qquad G(z, w) = \sum_{j=0}^{\infty} G_j(z) w^j. \tag{2.3}$$

We define p and l to be minimal, $0 \leq p \leq l \leq \infty$, for which

$$F_p(z) \not\equiv 0, \qquad F_l(z) \not\equiv F_l(0). \tag{2.4}$$

If $l < \infty$ we define $k_1 \geq 1$ so that $F_{l, z^{k_1}}(0) \neq 0$. Finally, we define k_0 to be minimal so that $G_{k_0} \not\equiv 0$. It is shown in [6] that the integers j_0, l, k_0, and k_1 are biholomorphic invariants, but p may change under a biholomorphism.

Unlike the finite type case, when M is of infinite type, M' may be either of finite type or of infinite type, as shown by the following example.

Example. Let $\mathscr{H} = (\mathscr{F}, \mathscr{G})$ be the holomorphic map in $\mathbf{C}^2$ given by $\mathscr{F}(z, w) = zw$ and $\mathscr{G}(z, w) = w$. If M and M' are given by

$$M\colon w - \overline{w} = 2i|z|^2|w|^2, \qquad M'\colon w' - \overline{w'} = 2i|z'|^2,$$

then $\mathscr{H}$ maps M into M', with M and M' of infinite and finite types respectively.

The following geometric properties of the hypersurfaces M and M' and the mapping $\mathscr{H}$ are essential for the proof of Theorem 1, and also are of independent interest.

Theorem 3. *Let $\mathscr{H} = (f, g)$ be a holomorphic map in Ω^+, mapping M into M'. If $G \not\equiv 0$, then for any choice of normal coordinates the following holds for every $j \geq 1$:*

$$\overline{L}^j f(z, \bar{z}, s) = s^l f_j(z, \bar{z}, s),$$

for some smooth functions f_j on M with $f_{k_1}(0) \neq 0$, and $f_1(0) = \cdots = f_{k_1-1}(0) = 0$. In addition, if M' is of infinite type then $j_0(M) \geq l - p$, and if M' is of infinite type, then $G_{k_0}(z)$ is a real nonzero constant and $j_0(M) \geq 2l$. Here the integers j_0, l, k_0, k_1 are the ones introduced above.

3. Reflection principle in higher dimensions. In the case of higher complex dimension, extendability of mappings from a hypersurface M to M' has been proved only under the assumption that M and M' are at least of finite type. The following example shows that even in the case in which $\mathscr{H}$ is a diffeomorphism, the conclusion can fail for M and M' of finite type.

EXAMPLE 3.1. Let $M = M'$ be the hypersurface in $\mathbf{C}^3$ given by $\operatorname{Im} Z_3 = |Z_1|^2$. Let $\mathscr{H}$ be the mapping defined by

$$\mathscr{H}_1(Z) = Z_1, \qquad \mathscr{H}_2(Z) = Z_2 + h(Z_3), \qquad \mathscr{H}_3(Z) = Z_3,$$

where h is a holomorphic function in the upper half plane in $\mathbf{C}$, smooth up to the boundary, but which does not extend holomorphically across 0. If we take h to satisfy $h(0) = h'(0) = 0$ the mapping $\mathscr{H}$ is a local diffeomorphism near 0 in $\operatorname{Im} Z_3 > |Z_1|^2$, such that $\mathscr{H}(M) \subset M'$, and for which $\mathscr{H}$ does not extend holomorphically at 0. Note that M is of finite type, in contrast to the $\mathbf{C}^2$ case.

In the above example the defining equation $\rho = \operatorname{Im} Z_3 - |Z_1|^2 = 0$ is independent of Z_2. An algebraic condition on ρ which excludes this type of deficiency was introduced by Baouendi, Jacobowitz, and Treves [7] for studying the case of a diffeomorphism.

As in §2 we may introduce holomorphic coordinates $(z, w) \in \mathbf{C}^{n+1}$, $z = (z_1, \dots, z_n)$ and w in $\mathbf{C}$, such that M is given by

$$\operatorname{Im} w = \phi(z, \bar{z}, s), \qquad \text{with } s = \operatorname{Re} w, \tag{3.2}$$

where ϕ is real analytic and satisfies $\phi(z, 0, s) = 0$. As before, we shall call (z, w) normal coordinates for M and w a transversal coordinate. Using notation similar to that of §2 we write $\mathscr{H} = (f, g)$ with $f = (f_1, \dots, f_n)$, and also use the notation $F(z, w)$ and $G(z, w)$ for the corresponding power series. We write $\phi(z, \zeta, 0) = \sum_\alpha a_\alpha(z)\zeta^\alpha$. As in [7], we shall say that M is *essentially finite* at 0 if the ideal $(a_\alpha(z))$ in the ring of formal power series $\mathbf{C}[[z]]$ generated by all the $a_\alpha(z)$ is of finite codimension, i.e. $\dim_{\mathbf{C}} \mathbf{C}[[z]]/(a_\alpha(z)) = m$ is finite. We shall call m the *essential type* of M at 0. (See also D'Angelo [2] and [4].)

Even if M and M' are both essentially finite, a stronger nondegeneracy condition than that of Theorem 2 in the case of $\mathbf{C}^2$ must be imposed in higher dimensions, as is shown by the following example.

EXAMPLE 3.3. Let M and M' be the hypersurfaces in $\mathbf{C}^3$ given by

$$M\colon \operatorname{Im} w = |z_1|^2 + |z_2|^2, \qquad M'\colon \operatorname{Im} w' = |z_1'|^2 - |z_2'|^2,$$

and $\mathscr{H}$ given by $\mathscr{H} = (f, f, 0)$, where f is holomorphic in $\operatorname{Im} w > |z_1|^2 + |z_2|^2$, smooth up to M and which does not extend holomorphically at 0. Then $\mathscr{H}$ maps M into M', both of which are essentially finite, but $\mathscr{H}$ does not extend holomorphically at 0.

We shall write $F(z, w) = F = (F_1, \dots, F_n)$. We shall say that the mapping $\mathscr{H} = (f, g)$ is *not totally degenerate* if

$$\det\left(\frac{\partial F_j}{\partial z_k}\right)(z, 0) \not\equiv 0, \tag{3.4}$$

where the left-hand side of (3.4) is regarded as a formal power series.

Another condition on F is the following. We say that $\mathscr{H}$ is of *finite multiplicity* if the ideal $(F(z, 0))$ in the ring of formal power series $\mathbf{C}[[z]]$ generated by all the $F_j(z, 0)$ is of finite codimension, i.e. $\dim_{\mathbf{C}} \mathbf{C}[[z]]/(F(z, 0)) = d$ is finite. We shall call d the *multiplicity* of $\mathscr{H}$ at 0. It is known from techniques of commutative algebra, that finite multiplicity implies not totally degenerate (see, e.g., [16]).

The following extendability result holds.

THEOREM 4 [5]. *Let $\mathscr{H}$ be holomorphic in Ω^+, smooth in $\overline{\Omega^+}$, and satisfying $\mathscr{H}(M) \subset M'$, where M and M' are real analytic hypersurfaces in $\mathbf{C}^{n+1}$ containing the origin. Let g be a transversal component of $\mathscr{H}$. Then $\mathscr{H}$ extends holomorphically to a neighborhood of 0 in $\mathbf{C}^{n+1}$ if any of the following conditions holds.*

(i) *M is essentially finite and g is not flat at 0.*
(ii) *M' is essentially finite and $\mathscr{H}$ is of finite multiplicity at 0.*
(iii) *M' is essentially finite and $\mathscr{H}$ is not totally degenerate at 0.*

Note that a special case of Theorem 4 was also obtained independently by Diederich and Fornaess [14].

By the remark preceding Theorem 4, condition (iii) of the Theorem implies condition (ii). Also, (i) is a consequence of (iii) by the following result relating these conditions.

THEOREM 5 [5]. *Let $\mathscr{H}$ as above mapping M into M'. Let w be a transversal coordinate for M and g any transversal coordinate for $\mathscr{H}$. If M is essentially finite, then the following hold.*

(i) *If g is flat (i.e., $G \equiv 0$), then either $\mathscr{H}$ is not of finite multiplicity or M' is not essentially finite.*
(ii) *If $G \not\equiv 0$, then $\frac{\partial G}{\partial w}(0) \neq 0$, $\mathscr{H}$ is of finite multiplicity, and M' is essentially finite.*

The essential types of M and M' and the multiplicity of the mapping $\mathscr{H}$ are related by the formula

$$\text{ess type } M = (\text{mult}\,\mathscr{H}) \text{ ess type } M' \tag{3.5}$$

if all three integers are finite. Note, by Theorem 5, that this is the case if either (i) or (ii) of Theorem 4 holds. However, (iii) of Theorem 4 does not imply this, as is shown by the following example.

EXAMPLE 3.6. Let M and M' be the hypersurfaces in $\mathbf{C}^3$ given by

$$M\colon \operatorname{Im} w = |z_1|^2 - |z_1 z_2|^2 \quad \text{and} \quad M'\colon \operatorname{Im} w = |z_1|^2 - |z_2|^2,$$

and $\mathscr{H} = (f_1, f_2, g)$ with $f_1(z, w) = z_1$, $f_2(z, w) = z_1 z_2$, and $g = w$. Here M' is essentially finite, M is of finite type (but not essentially finite), $\mathscr{H}$ is not totally degenerate, but not of finite multiplicity.

The extension of $\mathscr{H}$, when the three numbers in (3.5) are finite, is a *finite holomorphic map* in the sense of algebraic geometry (see, e.g., [19] and [23]), and the map is d-to-one in a neighborhood of 0.

From (3.5) we may also obtain additional information for the case where $M = M'$. The following result, to the best of our knowledge, is new.

THEOREM 6. *Let $\mathscr{H}$ be as in Theorem 4 mapping M into itself. If M is essentially finite and either $\mathscr{H}$ is of finite multiplicity or g is not flat at the origin, then $\mathscr{H}$ extends to a local biholomorphism of $\mathbf{C}^{n+1}$ into itself.*

Theorem 6 is proved by first applying Theorem 4 to conclude that $\mathscr{H}$ extends holomorphically in a neighborhood of 0 in $\mathbf{C}^{n+1}$. Then Theorem 5 and formula (3.4) may be used to conclude that the multiplicity of $\mathscr{H}$ is 1 and $\frac{\partial G}{\partial w}(0) \neq 0$, which implies that $\mathscr{H}$ is a local biholomorphism.

4. Applications to holomorphic mappings of domains in $\mathbf{C}^{n+1}$. In this section we shall assume that we are given a proper holomorphic map $\mathscr{H}$, (i.e., the inverse image of a compact set is compact), taking one bounded domain with real analytic boundary into another. In this case, $\mathscr{H}$ is an open mapping and is locally finite-to-one (see, e.g., Rudin [23]). We discuss here boundary behavior and multiplicity of these mappings. For other results concerned with smoothness and multiplicity, the reader is referred to the survey article by Bedford [8].

If D and D' are strongly pseudoconvex domains in $\mathbf{C}^{n+1}$ with smooth boundary, and $\mathscr{H}$ a biholomorphic mapping from D to D' then it follows from the celebrated theorem of Fefferman [17] that $\mathscr{H}$ extends smoothly to the boundary of D. This result, first proved in the mid 70s, has been considerably simplified and strengthened since then. Fefferman's proof was based on a difficult explicit calculation of the Bergman kernel function. A much simplified proof was found by Bell and Ligocka [12] by using Kohn's solution of the $\overline{\partial}$-Neumann problem. This simplification led to a proof for some classes of weakly pseudoconvex domains, including those with real analytic boundary. Subsequently, the result was generalized (independently by Bell and Catlin [11] and Diederich and Fornaess [13]) from biholomorphic mappings to proper holomorphic ones. We state here this result in the case of real analytic boundaries.

THEOREM 6 [11, 13]. *If $\mathscr{H}$ is a proper holomorphic mapping from D to D', where D and D' are pseudoconvex domains in $\mathbf{C}^{n+1}$ with real analytic boundaries, then $\mathscr{H}$ extends smoothly to $\overline{D}$, the closure of D.*

It follows also from the results in [11] and [13] that at each point of the boundary of D any transversal component g of $\mathscr{H}$ has a nonzero differential. We can then combine Theorem 4(i) of §3 with Theorem 6 above to obtain the following result, which was proved in $\mathbf{C}^2$ by the authors together with S. Bell [3] and, for general n, independently by the authors [4] and Diederich and Fornaess [14].

THEOREM 7 [4, 14]. *Under the assumptions of Theorem* 6, $\mathscr{H}$ *extends holomorphically to a neighborhood of the closure of* D *in* $\mathbf{C}^{n+1}$.

The other global results cited here do not require pseudoconvexity. We begin with a proper holomorphic mapping $\mathscr{H}: D \to D'$ smooth up to the closure $\overline{D}$ of D, where D and D' are bounded domains in $\mathbf{C}^{n+1}$ with real analytic boundaries. We shall also need to assume that at every point of ∂D no transversal component of $\mathscr{H}$ is flat. Under these hypotheses it follows from Theorems 4 and 5 of §3 that $\mathscr{H}$ extends holomorphically to a neighborhood D_1 of $\overline{D}$ and that at every $q \in \partial D$ we have $\frac{\partial G}{\partial w}(q) \neq 0$.

For $p \in D_1$ we define $m(\mathscr{H}, p)$, the holomorphic multiplicity of $\mathscr{H}$ at p, by

$$m(\mathscr{H}, p) = \dim_{\mathbf{C}} \mathbf{C}[[Z - p]]/(\mathscr{H}_j(Z) - \mathscr{H}_j(p)). \tag{4.1}$$

Note that for $q \in \partial D$, $m(\mathscr{H}, q)$ is the same as the multiplicity of $\mathscr{H}$ introduced in §3, since $\frac{\partial G}{\partial w}(q) \neq 0$.

THEOREM 8 [5]. *Let* $\mathscr{H}: D \to D'$ *be as above. Then there exists* $\varepsilon > 0$ *such that*

$$\sup_{p \in D_\varepsilon} m(\mathscr{H}, p) \leq \sup_{q \in \partial D} m(\mathscr{H}, q), \tag{4.2}$$

where $D_\varepsilon = \{z \in D: \operatorname{dist}(z, \partial D) < \varepsilon\}$. *Also, if* $m(\mathscr{H}, q) = 1$ *for all* $q \in \partial D$ *then* $m(\mathscr{H}, p) = 1$ *for all* $p \in D$. *Finally, if* $\sup_{q \in \partial D} m(\mathscr{H}, q) = 2$ *then equality holds in* (4.2).

The following example shows that D_ε cannot be replaced by D in (4.2).

EXAMPLE. Let D and D' be contained in $\mathbf{C}^2$ and defined by

$$D = \{(z, w) \in \mathbf{C}^2: |z^3 + zw|^2 + |w|^2 < 1\},$$
$$D' = \{(z', w') \in \mathbf{C}^2: |z'|^2 + |w'|^2 < 1\}.$$

Let $\mathscr{H} = (z^3 + zw, w)$. Clearly $\mathscr{H}$ is a proper map from D into D'. We have

$$m(\mathscr{H}, (z, w)) = \begin{cases} 1 & \text{if } 3z^2 + w \neq 0, \\ 2 & \text{if } 3z^2 + w = 0,\ z \neq 0, \\ 3 & \text{if } z = w = 0. \end{cases}$$

Therefore

$$\sup_{p \in D} m(\mathscr{H}, p) = 3 > \sup_{p \in \partial D} m(\mathscr{H}, p) = 2.$$

The following example shows that the inequality in (4.2) can be strict.

EXAMPLE. Let D and D' be domains in $\mathbf{C}^3$ given by

$$D = \{z_1, z_2, w) \in \mathbf{C}^3 : |z_1^3 + wz_1^2|^2 + |z_2|^2 + |w-1|^2 < 1\},$$
$$D' = \{(z_1', z_2', w') \in \mathbf{C}^3 : |z_1'|^2 + |z_2'|^2 + |w'-1|^2 < 1\},$$

and let $\mathscr{H} = (z_1^3 + wz_1^2, z_2, w)$. Then

$$m(\mathscr{H}(z, w)) = \begin{cases} 1 & \text{if } z_1 \neq 0 \text{ and } 3z_1 + 2w \neq 0, \\ 2 & \text{if } (z_1 = 0 \text{ and } w \neq 0) \text{ or } (z_1 \neq 0 \text{ and } w = -\frac{3}{2}z_1), \\ 3 & \text{if } z_1 = w = 0. \end{cases}$$

Hence

$$\sup_{q \in \partial D} m(\mathscr{H}, q) = 3 > \sup_{p \in D} m(\mathscr{H}, p) = 2.$$

For $p \in \overline{D}$ we let $\mu(\mathscr{H}, p)$ be the number of preimages of $\mathscr{H}(p)$ in $\overline{D}$. The following is a corollary of Theorem 8.

THEOREM 9 [5]. *If $\mathscr{H} : D \to D'$ is as above, we have*

$$\sup_{q \in \partial D} \mu(\mathscr{H}, q) = \sup_{p \in D} \mu(\mathscr{H}, p). \tag{4.3}$$

For proper self-mappings the following theorem holds.

THEOREM 10 [5]. *Let $\mathscr{H} : D \to D'$ as above with $D = D'$. Then $\mathscr{H}$ extends as a biholomorphism from an open neighborhood of $\overline{D}$ onto another.*

Theorem 10 was first proved in the pseudoconvex case by Bedford and Bell [10] and in the case of $\mathbf{C}^2$ by the authors and Bell [3].

5. Methods of proof. In order to prove Theorems 1, 2, and 4 dealing with holomorphic extension of mappings, we start with normal coordinates (z, w) for M in $\mathbf{C}^{n+1}$ with M given as in (2.1). If M is parametrized by $(z, \bar{z}, s)$ then a function $h(z, \bar{z}, s)$ defined on M is called CR (for Cauchy-Riemann) if and only if $L_j h = 0$, where the L_j, $j = 1, \dots, n$, form a basis for the induced Cauchy-Riemann operators on M. We may take, for instance,

$$L_j = \frac{\partial}{\partial \bar{z}_j} - i \frac{\phi_{z_j}(z, \bar{z}, s)}{1 + i\phi_s(z, \bar{z}, s)} \frac{\partial}{\partial s}, \qquad j = 1, \dots, n.$$

Note that the restriction to M of any function holomorphic in Ω^+ is a CR function; in particular, the restrictions of the components of $\mathscr{H}$ are CR functions.

It is easily shown, using the analyticity of ϕ (see [7]), that a CR function h is the restriction to M of a holomorphic function defined in a neighborhood of 0 in $\mathbf{C}^{n+1}$ if and only if h is real analytic in s near the origin uniformly in z for $|z|$ small. Since the components of $\mathscr{H}$ are holomorphic in Ω^+, they extend holomorphically in s to the upper half plane for fixed z; therefore, it suffices to prove that for fixed z they extend holomorphically to the lower half plane in s.

We write $\mathscr{H} = (f, g)$ as in §2 and §3. By the above comments, it suffices to show that f and g extend holomorphically to the lower half plane in s for fixed z. Rewriting the equation for M' in the form $\bar{w}' = Q(z', \bar{z}', w')$, we obtain the nonlinear equation valid on M,

$$\bar{g} = Q(f, \bar{f}, g). \tag{5.1}$$

The Cauchy-Riemann operators L_j are successively applied to (5.1) to find a system of nonlinear differential equations relating the components f and g.

To illustrate the use of these nonlinear equations we shall prove the result for a simple example. We consider the case of hypersurfaces M and M' in $\mathbf{C}^2$ where M' is given by $\operatorname{Im} w' = |z'|^2$, with the assumption that $\mathscr{H}$ is a local diffeomorphism from M to M'. In this case, it is easily checked that $L\bar{f}(0) \neq 0$. We apply L to (5.1), which may be written $\bar{g} = g + 2if\bar{f}$. We obtain, after dividing by $L\bar{f}$,

$$\frac{L\bar{g}}{L\bar{f}} = 2if, \tag{5.2}$$

since $Lf = Lg = 0$. Because f and g extend holomorphically to the upper half plane in s for fixed z, their conjugates $\bar{f}$ and $\bar{g}$ extend holomorphically to the lower half plane in s. Since $L\bar{f}(0) \neq 0$, the left-hand side of (5.2) also extends holomorphically to the lower half plane, and hence so does f. This proves the desired extendability for f, and the desired extendability for g follows from (5.1).

In the general case, even in $\mathbf{C}^2$, Q in (5.1) is not linear in f and one can not expect to solve for f as in the simple case (5.2). In general, one must use the Weierstrass Preparation Theorem to find polynomial relations satisfied by f and g with coefficients extending holomorphically down in s. The case when $\mathscr{H}$ is a diffeomorphism was treated in [7]. If this condition is dropped there are additional technical difficulties. In $\mathbf{C}^2$ we distinguish between the cases where M is of finite type or M is of infinite type. If M is of finite type (and g not flat at 0) then it can be shown that $Q(z, \zeta, 0) \not\equiv 0$, i.e. M' is also of finite type. By Theorem 2(ii) there exists $k \geq 0$ such that $L^k\bar{f}(0) \neq 0$. By successively applying L to (5.1) and using the Weierstrass Preparation Theorem it can be shown that f satisfies a polynomial equation of the form

$$f^N + \sum_{j=0}^{N-1} a_j(L^\alpha \bar{f}, L^\beta \bar{g}) f^j \equiv 0, \tag{5.2}$$

where the a_j are holomorphic with respect to their arguments and $0 \leq \alpha, \beta \leq N_1$. Hence f satisfies a polynomial equation with coefficients which extend holomorphically down in s for fixed z. Unfortunately, this alone is not sufficient to conclude that f also extends holomorphically down. For this, the following one variable lemma, whose proof is elementary, is used.

LEMMA 5.3 [3]. *Let $r > 0$ and R the rectangle in the complex plane*

$$R = \{s + it \in \mathbf{C} : |s| < r, -r < t < 0\}.$$

Suppose that u and v are two functions defined in $\overline{R}$ and satisfying

(i) *$u, v \in C^\infty(\overline{R})$ and u, v holomorphic in R,*

(ii) *there exists $h(s) \in C^\infty([-r, r])$ such that $h(s)v(s) = u(s)$,*

(iii) *$h(s)$ satisfies a polynomial equation of the form*

$$(h(s))^N + \sum_{j=0}^{N-1} a_j(s)(h(s))^j \equiv 0,$$

in $[-r, r]$, where the a_j are smooth in $\overline{R}$ and holomorphic in R.

Then h extends holomorphically to R, agrees with $u(w)/v(w)$, $w = s+it$, wherever $v(w) \neq 0$, and $h \in C^\infty(R \cup (-r, r))$.

When M is of infinite type, another elementary one-variable lemma is also needed.

LEMMA 5.4 [6]. *Let R be as in Lemma (5.3), and $a(s + it)$, $b(s + it)$ be holomorphic in R, smooth in $\overline{R}$ and satisfying the following conditions.*

(i) *$a(s) = b(s)c(s)$ in $(-r, r)$, with $c \in C^\infty(-r, r)$.*

(ii) *$|b(s + it)| \geq C|s + it|^l$ for some $C > 0$ and some $l \geq 0$, $s + it \in R$.*

Then c extends holomorphically to R and is smooth in $\overline{R}$.

By using Lemma (5.4) and Theorem 3 of §2, one can prove (5.2) in the infinite type case also.

In higher dimensions, in order to show that the components of $\mathscr{H}$ satisfy polynomial relations of the form (5.2), some tools of commutative algebra are needed. In particular, the Nullstellensatz in both convergent and formal power series is a crucial ingredient in the proofs of Theorems 4 and 5.

REFERENCES

1. S. Alinhac, M. S. Baouendi, and L. P. Rothschild, *Unique continuation and regularity at the boundary for holomorphic functions*, Duke Math. J. **61** (1990), 635–653.
2. J. D'Angelo, *The notion of formal essential finiteness for smooth real hypersurfaces*, Indiana Univ. Math. J. **36** (1987), 897–903.
3. M. S. Baouendi, S. R. Bell, and L. P. Rothschild, *Mappings of three-dimensional CR manifolds and their holomorphic extension*, Duke Math. J. **56** (1988), 503–530.
4. M. S. Baouendi and L. P. Rothschild, *Germs of CR maps between real analytic hypersurfaces*, Invent. Math. **93** (1988), 481–500.
5. ——, *Geometric properties of smooth and holomorphic mappings between hypersurfaces in complex space*, J. Differential Geom. **31** (1990), 473–499.
6. ——, *A general reflection principle in $\mathbf{C}^2$*, J. Funct. Anal. (to appear).
7. M. S. Baouendi, H. Jacobowitz, and F. Treves, *On the analyticity of CR mappings*, Ann. of Math. **122** (1985), 365–400.

8. E. Bedford, *Proper holomorphic mappings*, Bull. Amer. Math. Soc. **10** (1984), 157–175.
9. E. Bedford and S. Bell, *Extension of proper holomorphic mappings past the boundary*, Manuscripta Math. **50** (1985), 1–10.
10. ——, *Proper self maps of weakly pseudoconvex domains*, Math. Ann. **261** (1982), 47–49.
11. S. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J. **49** (1982), 385–396.
12. S. Bell and E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. **57** (1980), 283–289.
13. K. Diederich and J. E. Fornaess, *Boundary regularity of proper holomorphic mappings*, Invent. Math. **67** (1982), 363–384.
14. ——, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* C^n, Math. Ann. **282** (1988), 681–700.
15. K. Diederich and S. M. Webster, *A reflection principle for degenerate real hypersurfaces*, Duke Math. J. **47** (1980), 835–843.
16. D. Eisenbud and H. Levine, *An algebraic formula for the degree of a* C^∞ *map germ*, Ann. of Math. **106** (1977), 19–44.
17. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.
18. F. Forstnerič, *Proper holomorphic mappings: A survey*, Preprint Proceedings of the Special Year in Several Complex Variables, Institute Mittag-Leffler, 1987/88 (to appear).
19. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Interscience, New York, 1978.
20. J. J. Kohn, *Boundary behavior of* $\bar{\partial}$ *on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.
21. H. Lewy, *On the boundary behavior of holomorphic mappings*, Acad Naz. Lincei, **35** (1977), 1–8.
22. S. I. Pinčuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J. **15** (1974), 909–917.
23. W. Rudin, *Function theory on the unit ball in* $\mathbf{C}^n$, Springer-Verlag, 1980.
24. S. Webster, *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc. **71** (1978), 26–28.

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