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Holomorphically nondegenerate algebraic hypersurfaces and their mappings

Linda Preiss Rothschild

1. Introduction and main results

Poincaré [P] in 1907 proved that if a germ of a biholomorphism in $\mathbb{C}^2$ maps a piece of a sphere into another sphere, then the mapping must be rational. This result was later generalized to $\mathbb{C}^N$ for $N > 2$ by Tanaka [T]. Webster [W1] gave a sweeping generalization in 1977, when he proved that any local biholomorphism mapping a Levi nondegerate algebraic hypersurface into another is necessarily algebraic.

Throughout this paper, by an algebraic hypersurface in $\mathbb{C}^N$, we shall mean a smooth hypersurface defined by the vanishing of a real valued polynomial. A holomorphic function is called algebraic if it is a root of a polynomial with holomorphic polynomial coefficients. A mapping is algebraic if its components are algebraic functions. In recent joint work [BR2] with Salah Baouendi, we obtained necessary and sufficient conditions for algebraic hypersurfaces to have the property that all germs of biholomorphisms mapping one such hypersurface into another are algebraic maps.

A real analytic hypersurface $M \subset \mathbb{C}^N$ will be called holomorphically degenerate at a point p_0 if there is a germ of a nontrivial holomorphic vector field, $\sum_{j=1}^N a_j(Z)\frac{\partial}{\partial Z_j}$, tangent to M in a neighborhood of p_0 , with $a_j(Z)$ holomorphic near p_0. A hypersurface M is called holomorphically nondegenerate at p_0 if it is not holomorphically degenerate at p_0. The main result in [BR2] is the following.

THEOREM. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$, $N > 1$. If M is connected and holomorphically nondegenerate at some point in M, and H is a germ at a point $p_0 \in M$ of a biholomorphism of $\mathbb{C}^N$ mapping M into M', then H is algebraic. Conversely, if M is algebraic and holomorphically degenerate at some point, then at any $p_0 \in M$ there exists a germ of a biholomorphism of $\mathbb{C}^N$ mapping M into itself and fixing p_0, which is not algebraic.*

We note that in $\mathbb{C}$ any real analytic hypersurface, i.e. curve, is holomorphically nondegenerate. However, the conclusion of the first part of the theorem does not hold. Indeed, the mapping $Z \mapsto e^Z$ maps the real line into itself, but is not algebraic.

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The notion of holomorphic nondegeneracy was first introduced by Stanton [S1], who later proved that holomorphic nondegeneracy is also equivalent to the finite dimensionality of the Lie algebra of infinitesimal holomorphic automorphisms of hypersurfaces [S2]. It can be shown that if M is a connected real analytic hypersurface, then M is holomorphically degenerate at a given $p_0 \in M$ if and only if it is holomorphically degenerate at every point in M. Hence if M is holomorphically nondegenerate at some point, I shall simply refer to it as a holomorphically nondegenerate hypersurface.

The theorem above implies Webster's result on algebraicity of mappings, since a connected hypersurface which is Levi nondegenerate at some point is holomorphically nondegenerate at that point. In $\mathbb{C}^2$, if M is holomorphically nondegenerate, then it is also Levi nondegenerate at most points. However, in higher dimensions one can have a holomorphically nondegenerate hypersurface which is Levi degenerate at every point. An example is the hypersurface in $\mathbb{C}^3$ given by $(\Re Z_1)^2 + (\Re Z_2)^2 - (\Re Z_3)^2 = 0$.

It should be mentioned here that the author, in joint work with S. Baouendi and P. Ebenfelt [BER1], has obtained some criteria for the algebraicity of maps between generic manifolds in $\mathbb{C}^N$ of higher codimension than one. Although the definition of holomorphic nondegeneracy extends immediately to such manifolds, it is easy to see by examples that holomorphic nondegeneracy is no longer sufficient to guarantee the algebraicity of biholomorphisms between such manifolds. For instance, the generic manifold M of codimension 2 in $\mathbb{C}^3$ given by $\Im Z_2 = |Z_1|^2, \Im Z_3 = 0$ is holomorphically nondegenerate. However the self-mapping $Z \mapsto (Z_1, Z_2, Z_3 e^{Z_3})$ is a local biholomorphism fixing the origin and mapping M into itself, but is not algebraic. Although for hypersurfaces in $\mathbb{C}^N$, $N > 1$, holomorphic nondegeneracy implies finite type (in the sense of Bloom-Graham [BG]) at most points, this is no longer the case for generic manifolds of higher codimension, as shown by the example above. It is proved in [BER1] that if a generic manifold is both holomorphically nondegenerate and of finite type at most points, then a similar conclusion of algebraicity of biholomorphisms holds. However, the methods of the proof for higher codimension are substantially more complicated and involve an inductive argument which is not needed in the case of a hypersurface. The reader can find the details in [BER1].

For the remainder of the paper, I will restrict myself to the case of a hypersurface and give some indication of the proof of the Theorem above as well as additional remarks and corollaries.

2. Necessity of holomorphic nondegeneracy

Let $p_0 \in M$ and assume that X is a nontrivial germ at p_0 of a holomorphic vector field tangent to M. To any such X, there is a holomorphic 1-parameter group of local biholomorphisms in $\mathbb{C}^N$ sending M into M defined by the complex flow of X i.e.

$$\dot{\phi}(t, Z) = X(\phi(t, Z)), \quad \phi(0, Z) = Z. \tag{2.1}$$

Then $\phi(t, Z)$ is holomorphic for $t \in \mathbb{C}, |t| < \epsilon$, and $Z \in V$, where V is an open neighborhood of p_0 in $\mathbb{C}^N$. For fixed t, the map $Z \mapsto \phi(t, Z)$ is a local biholomorphism preserving M, and if $X(p_0) = 0$, then $\phi(t, p_0) \equiv p_0$.

The second part of the Theorem will be a consequence of of the following lemma.

LEMMA 1. *Let M be a real algebraic hypersurface in $\mathbb{C}^N$, $p_0 \in M$, and X a germ at p_0 of a nontrivial holomorphic vector field tangent to M. Then there exists f, a germ at p_0 of a holomorphic function, and an arbitrarily small t such that if $\psi(t,Z)$ is the flow of $Y = fX$, the mapping $Z \mapsto \psi(t,Z)$ is a nonalgebraic local biholomorphism mapping M into itself and fixing p_0.*

PROOF. After multiplying X by an algebraic holomorphic function vanishing at p_0 if necessary, we may assume that $X(p_0) = 0$. If ϕ is the flow of X given by (2.1), then by standard arguments using the local group property, we have

$$\sum_{j=1}^{N} a_j(Z)\frac{\partial \phi_k}{\partial Z_j}(t,Z) = a_k(\phi(t,Z)), \quad k = 1,\dots,N. \tag{2.2}$$

If for some arbitrarily small t the map $Z \mapsto \phi(t,Z)$ is not algebraic, we are done. Otherwise, we assume $a_1 \not\equiv 0$, and let $f(Z) = e^{Z_1}$ and $Y = e^{Z_1}X$. We denote by $\psi(t,Z)$ the holomorphic flow of Y. By (2.2) for the vector field Y instead of X, and taking $k = 1$, we have

$$\sum_{j=1}^{N} e^{Z_1}a_j(Z)\frac{\partial \psi_1}{\partial Z_j}(t,Z) = e^{\psi_1(t,Z)}a_1(\psi(t,Z)). \tag{2.3}$$

If $Z \mapsto \psi(t,Z)$ is algebraic for some fixed t, then since all the coefficients a_k are algebraic, it would follow from (2.3) that the function $Z \mapsto e^{Z_1-\psi_1(t,Z)}$ is also algebraic. Note that $Z \mapsto Z_1 - \psi_1(t,Z)$ is algebraic and not constant (since $a_1 \not\equiv 0$). However, if $A(Z)$ is any nonconstant algebraic holomorphic function, then the function $Z \mapsto e^{A(Z)}$ cannot be algebraic. The Lemma is proved by contradiction. $\square$

3. Holomorphic nondegeneracy and the Levi type

Let M be a real analytic hypersurface in $\mathbb{C}^N$ and $p_0 \in M$. Let $\rho(Z,\bar{Z})$ be a real analytic defining function of M near p_0. Let $L_1,\dots,L_n$, $n = N-1$, given by $L_j = \sum_{k=1}^{N} a_{jk}(Z,\bar{Z})\partial/\partial\bar{Z}_k$, be a basis of the CR vector fields on M near p_0 with the a_{jk} real analytic. We introduce the following vector-valued functions. For a multi-index α, let V_α be the real analytic function defined near 0 in $\mathbb{C}^N$ by

$$V_\alpha(Z,\bar{Z}) = L^\alpha \rho_Z(Z,\bar{Z}), \tag{3.1}$$

where ρ_Z denotes the gradient of ρ with respect to Z and $L^\alpha = L_1^{\alpha_1}\cdots L_n^{\alpha_n}$. We have the following lemma whose proof could be essentially found in [BHR].

LEMMA 2. *Let M be a connected real analytic hypersurface in $\mathbb{C}^N$, $p_0 \in M$, and $V_\alpha(Z,\bar{Z}) = L^\alpha\rho_Z(Z,\bar{Z})$ given by (3.1). Then the following conditions are equivalent:*

(i) *M is holomorphically nondegenerate.*
(ii) *$\{V_\alpha(Z,\bar{Z}), \alpha \in \mathbb{Z}_+^n\}$ span $\mathbb{C}^N$ generically in a neighborhood of p_0 in M.*
(iii) *There exists an integer k, with $1 \le k \le n$, so that $\{V_\alpha(Z,\bar{Z}), |\alpha| \le k\}$ span $\mathbb{C}^N$ generically in a neighborhood of p_0 in M.*

It is not difficult to check that conditions (ii) and (iii) above are independent of the choice of the defining function of ρ, the coordinates Z, and the CR vector fields $L_1,\dots,L_n$.

We say that the hypersurface M is *k-holomorphically nondegenerate* at $Z \in M$ if $\{V_\alpha(Z,\bar{Z}), |\alpha| \le k\}$ span $\mathbb{C}^N$, with k minimal. In particular, it is easy to see that M is 1-holomorphically nondegenerate at Z if and only if the Levi form of M is nondegenerate at Z. Note that if M is connected and holomorphically nondegenerate then there exists $\ell = \ell(M)$, $1 \le \ell(M) \le N-1$, such that M is ℓ-holomorphically nondegenerate at every point in an open dense subset of M. We call $\ell(M)$ the *Levi type* of M. The Levi type of M is 1 if and only if M is generically Levi nondegenerate.

For a real analytic hypersurface $M \subset \mathbb{C}^N$ and $p_0 \in M$, we can find (see [CM], [BJT]) holomorphic coordinates (z,w), (called normal coordinates), $z \in \mathbb{C}^n$, $w \in \mathbb{C}$ vanishing at p_0 such that near p_0, M is given by

$$w = Q(z,\bar{z},\bar{w}); \tag{3.2}$$

here $Q(z,\chi,\tau)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n+1}$ and satisfies

$$Q(z,0,\tau) \equiv Q(0,\chi,\tau) \equiv \tau.$$

If M is given by (3.2), or equivalently by $\bar{w} = \bar{Q}(\bar{z},z,w)$, and $Z=(z,w)$, then

$$V_\alpha(Z,\bar{Z}) = -\bar{Q}_{\bar{z}^\alpha,Z}(\bar{z},z,w). \tag{3.3}$$

4. Sufficiency of holomorphic nondegeneracy

We associate to M the complex hypersurface $\mathcal{M}$ in $\mathbb{C}^{2N}$ locally defined near $(p_0,\bar{p}_0)$ by

$$\mathcal{M} = \{(Z,\zeta) : \rho(Z,\zeta) = 0\}, \tag{4.1}$$

where $\rho(Z,\bar{Z})$ is the defining function for M near p_0 as above.

If M, M', p_0 and H are as in the assumptions of the Theorem, then by Lemma 2, by slightly moving p_0, we may assume that M is ℓ-holomorphically nondegenerate at p_0, where ℓ is the Levi type of M. We choose normal coordinates (z,w) for M, vanishing at p_0, and normal coordinates (z',w') for M' vanishing at $H(p_0)$ and write the mapping $H=(f,g)$ with $z'=f(z,w)$ and $w'=g(z,w)$. We assume that M is given by (3.2) and M' is given by $w' = Q'(z',\bar{z}',\bar{w}')$. Since $H(M) \subset M'$, we have for $(z,w) \in M$ in a neighborhood of 0,

$$\bar{g}(\bar{z},\bar{w}) = \bar{Q}'(\bar{f}(\bar{z},\bar{w}), f(z,w), g(z,w)).$$

Since the manifold $\mathcal{M}$ defined by (4.1) is given by $\tau = \bar{Q}(\chi,z,w)$ for $(z,w,\chi,\tau) \in \mathbb{C}^{2N}$, and a similar equation for the corresponding manifold $\mathcal{M}'$, it follows from the above that we have for $(z,w,\chi,\tau) \in \mathcal{M}$

$$\bar{g}(\chi,\tau) = \bar{Q}'(\bar{f}(\chi,\tau), f(z,w), g(z,w)). \tag{4.2}$$

Consider the following holomorphic vector fields which are tangent to $\mathcal{M}$

$$\mathcal{L}_j = \frac{\partial}{\partial \chi_j} + \bar{Q}_{\chi_j}(\chi,z,w)\frac{\partial}{\partial \tau}, \quad j=1,\dots n. \tag{4.3}$$

Note that the $\mathcal{L}_j$ commute with each other, and the functions Q and Q' are algebraic. By Lemma 2 and equation (3.3), we have

$$\text{span}\ \{\bar{Q}_{\bar{z}^\alpha,Z}(0,0,0) : |\alpha| \le \ell\} = \mathbb{C}^N.$$

Observing that the matrix $(\mathcal{L}_j\bar{f}_k)(0)$ is invertible and applying repeatedly the $\mathcal{L}_j$ to (4.2) we obtain the following by the use of the implicit function theorem.

LEMMA 3. *For (z, w, χ, τ) in a neighborhood of 0 in $\mathcal{M}$ the following hold:*

$$f_j(z,w) = \Psi_j(\mathcal{L}^\gamma \bar{f}_p(\chi,\tau), \mathcal{L}^\beta \bar{g}(\chi,\tau)), \qquad j = 1,\ldots,n, \tag{4.4}$$

with $|\gamma|, |\beta| \le \ell, 1 \le p \le n$, and the Ψ_j holomorphic functions of their arguments.

It follows immediately from (4.4) that $z \mapsto f(z,0)$ is algebraic. Indeed, it suffices to take $\chi = 0$ and $\tau = w = 0$. More generally, a similar, but more complicated argument implies that for every integer q the mapping $z \mapsto \frac{\partial^q}{\partial w^q} H(z,0)$ is holomorphic algebraic in a neighborhood of 0 in $\mathbb{C}^n$.

To complete the proof of the sufficiency in the Theorem, we use (4.4) in which we take $\tau = 0$ and substitute $Q(z,\chi,0)$ for w to obtain

$$f_j(z, Q(z,\chi,0)) = \Psi_j(\mathcal{L}^\gamma \bar{f}_p(\chi,0), \mathcal{L}^\beta \bar{g}(\chi,0)), \qquad j = 1,\ldots,n, \tag{4.5}$$

which holds as an identity in $(z,\chi) \in \mathbb{C}^{2n}$ near 0. Note that after this substitution the coefficients of the vector fields $\mathcal{L}_j$ are then algebraic holomorphic in (z,χ). Since M is ℓ-holomorphically nondegenerate at 0, and the coordinates are taken to be normal, we conclude that the vector function $Q_\chi(z,\chi,0)$ does not vanish identically. Hence we may assume there is (z^0,χ^0) such that $Q_{\chi_1}(z^0,\chi^0,0) \neq 0$. Note that (z^0,χ^0) can be chosen arbitrarily close to 0 in $\mathbb{C}^{2n}$. Put $w^0 = Q(z^0,\chi^0,0)$. By the implicit function theorem , we can find an algebraic holomorphic function $\theta(z,w)$ defined near (z^0,w^0) and satisfying $\theta(z^0,w^0) = \chi_1^0$, such that the following identity holds for (z,w) near (z^0,w^0) in $\mathbb{C}^{n+1}$:

$$Q(z,\theta(z,w),\chi_2^0,\ldots,\chi_n^0,0) \equiv w. \tag{4.6}$$

We now take $\chi = (\theta(z,w),\chi_2^0,\ldots,\chi_n^0)$ in (4.5). After making this substitution, we consider that (z,w) are independent variables near (z^0,w^0). Then the functions

$$(z,w) \mapsto \Psi_j(\mathcal{L}^\gamma \bar{f}_p, \mathcal{L}^\beta \bar{g}) \tag{4.7}$$

are seen to be algebraic holomorphic. Hence the components $f_j(z,w)$ of H are algebraic. To prove that $g(z,w)$ is algebraic we again use the conjugate version of (4.2) with the same substitution as above. The already proved algebraicity of the f_j gives that of g. The proof of the sufficiency of holomorphic nondegeneracy follows.

§5. Remarks and consequences

Using the Theorem and a result in [BR1]. we can state the following.

COROLLARY. *Let M and M' be as in the first part of the Theorem and H a holomorphic mapping defined in a neighborhood of M in $\mathbb{C}^N$ with $H(M) \subset M'$. Then H is algebraic if either the Jacobian determinant of H does not vanish identically or M' does not contain any nontrivial complex variety.*

The explicit constructions of the mappings given in the proof of the Theorem make it possible to give a bound on the number of parameters which determine all mappings from a given hypersurface to another. In particular, at most points, these mappings are determined by finitely many parameters. In recent joint work of the author with S. Baouendi and P. Ebenfelt [BER2], these ideas have been expanded and carried out in the case of CR manifolds of higher codimension. Complete statements can be found in this work. For higher codimension, the proofs use the constructions of [BER1].

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Department of Mathematics-0112, University of California, San Diego, La Jolla, CA 92093-0112.

E-mail address: `lrothschild@ucsd.edu`