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Integrability and Holomorphic Extendibility for Rigid CR Structures

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We consider here surfaces M in $\mathbb{C}^{n+l}$ defined locally by

(1) $$M = \{(z, w) \in \mathbb{C}^{n+l}: z \in \mathbb{C}^n, w \in \mathbb{C}^l \text{ and } \operatorname{Im} w = \Phi(z, \bar{z})\},$$

where $\Phi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ is smooth; such a surface will be called *rigid*. We give an invariant characterization of CR (Cauchy–Riemann) structures giving rise to rigid surfaces and give sufficient conditions for holomorphic extendibility of solutions of the tangential Cauchy–Riemann equations. The results announced here represent joint work with M. S. Baouendi and F. Treves; details will appear elsewhere.

Extendibility of CR functions has been studied extensively, beginning with the classical paper of Hans Lewy [11]. Later results were obtained by R. Nirenberg [12], Hill and Taiani [9], Boggess and Polking [7], and others, using the method of analytic disks. In [3]–[5], and in [2] with Chang, Baouendi and Treves obtained new results on extendibility by using microlocal methods—in particular, the Fourier–Bros–Iagolnitzer (FBI) integral (see [13]). The FBI integral is again a major technical tool in the present work.

Let $\Omega \subset \mathbb{R}^{2n+l}$ be open. A generic (abstract) CR structure on Ω is a subbundle $\mathscr{V} \subset \mathbb{C}T(\Omega)$ such that

(2) $$\mathscr{V} \cap \bar{\mathscr{V}} = (0),$$

and

(3) $$[\mathscr{V}, \mathscr{V}] \subset \mathscr{V}.$$

Let $\mathbb{L} = C^\infty(\Omega, \mathscr{V})$, and let $L_1, L_2, \ldots, L_n$ be a basis of $\mathbb{L}$. The inclusion (3) means that $[L, L'] \in \mathbb{L}$ for any $L, L' \in \mathbb{L}$. The structure $\mathscr{V}$ (or $\mathbb{L}$) is called *locally*

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integrable at ω_0 if there exist C^∞ functions $\zeta_1,\ldots,\zeta_{n+l}$, $\zeta_i\colon U \to \mathbb{C}$, $\omega_0 \in U \subset \Omega$, with $\{d\zeta_i(\omega_0)\}$ linearly independent and

$$\mathbb{L}\zeta_i = 0, \qquad i = 1, 2, \ldots, n. \tag{4}$$

If the ζ_i exist, they define a local embedding of Ω onto a submanifold M of $\mathbb{C}^{n+l}$; M is then called a *CR manifold*.

THEOREM 1. *Let $\mathscr{V}$ be an abstract generic CR structure of dimension n. Suppose there exists an l-dimensional real subspace $\mathbb{T} \subset C^\infty(\Omega, \mathbb{C}T(M))$ such that*

$$[\mathbb{T}, \mathbb{T}] = 0, \tag{5}$$

and

$$[\mathbb{L}, \mathbb{T}] \subset \mathbb{L}. \tag{6}$$

Then $\mathscr{V}$ is integrable and there exist coordinates (z, w) on $\mathbb{C}^{n+l}$ such that

$$\zeta_j = z_j, \quad j = 1, 2, \ldots, n, \qquad \text{and} \qquad \zeta_{n+j} = \operatorname{Re} w_j + i\Phi_j(z, \bar{z}), \tag{7}$$

where Φ_j is a real-valued smooth function on $\mathbb{R}^{2n}$ for $j = 1, \ldots, n$.

Conversely, if $\mathscr{V}$ is a generic integrable CR structure for which the ζ_i can be taken to be in the form (7), *then $\mathbb{T}$, satisfying* (4) *and* (5), *exists.*

A complex-valued function h on a CR manifold M is called *CR* if $\zeta^*(\mathbb{L})h = 0$ in a neighborhood of $\zeta(\omega_0)$, where $\zeta^*(L)$ is the push forward of $L \in \mathbb{L}$ from Ω to M. Now suppose M is rigid. By a *wedge* in $\mathbb{C}^{n+l}$ we mean an open set of the form

$$W = \{(z, w)\colon \operatorname{Im} w = \Phi(z, \bar{z}) + v, v \in V_\delta, z \in U\}, \tag{8}$$

where $V_\delta = \{v \in \Gamma\colon |v| < \delta\}$, where $\Gamma \subset \mathbb{R}^l - \{0\}$ is a strictly conic set and U is a neighborhood of 0 in $\mathbb{C}^n$. If H is holomorphic in the wedge W and h is a CR function on M, we write $h = bH$ if h is (locally) the boundary value of H.

The following theorem generalizes a result of Andreotti–Hill [1].

THEOREM 2. *Let M be rigid, and let h be a CR function on M. Then we may write $h = h_1 + h_2 + \cdots + h_k$ such that, for each h_i, there is a wedge W_i and a holomorphic function H_i defined in W_i such that $h_i = bH_i$.*

In order to guarantee extendibility for CR functions, it is necessary to add the following hypothesis. M is of *finite type* if there is a number m such that the set of all commutators of $\mathbb{L} \oplus \bar{\mathbb{L}}$ of length $\leqslant m$ spans the tangent space to Ω. The following generalizes Theorem I.1 of [5] for hypersurfaces in the rigid case.

THEOREM 3. *If M is a rigid CR manifold of finite type, then for any CR function h on M there is a wedge W of the form* (8) *in $\mathbb{C}^{n+l}$ and a holomorphic function H on W such that $h = bH$ near the origin.*

We now recall the definition of the hypoanalytic wave-front set [2] for CR functions on M. If h is a CR function, let $\tilde{h} = h \circ \zeta$, $\tilde{h}\colon \Omega \to \mathbb{C}$, with $\mathbb{L}\tilde{h} = 0$ near ω_0. We let $\tilde{h}_0(x, s) = \tilde{h}(x, 0, s)$ and $\zeta_0(x, s) = \xi(x, 0, s)$. For $u(x, s)$ compactly supported let

$$F^K(u; v, \eta) = \int e^{-i\eta\cdot\zeta_0(x,s) - K\langle\eta\rangle[v - \zeta_0(x,s)]^2} u(x, s)\, d\zeta_0(x, s),$$

where $v, \eta \in \mathbb{C}^{n+l}$, with $|\mathrm{Im}\,\eta| < |\mathrm{Re}\,\eta|$, $\langle\eta\rangle = (\eta_1^2 + \cdots + \eta_{n+l}^2)^{1/2}$, and $[v]^2 = v_1^2 + \cdots + v_{n+l}^2$. Then h is *hypoanalytic* at $(0;\, 0, \eta_0)$, $\eta_0 \in \mathbb{R}^l$, if there exists $\varphi \in C_0^\infty(\mathbb{R}^{n+l})$, $R > 0$, $K > 0$, and an open cone $\mathscr{C}$ in $C^{n+l} \setminus \{0\}$ containing η_0 such that

$$(9) \qquad |F^K(\varphi h_0, v, \eta)| \leq Ce^{-|\eta|/R}$$

for all $\eta \in \mathscr{C}$ and all $v \in V^{\mathbf{C}}$, a neighborhood of 0 in $\mathbb{C}^m$. Then or any $\eta \in \mathbb{R}^l$,

$$(0;\, 0, \eta) \notin \mathrm{WF}_{\mathrm{ha}}\, h \quad \text{iff} \quad h \text{ is hypoanalytic at } (0;\, 0, \eta).$$

The connection of $\mathrm{WF}_{\mathrm{ha}}$ with holomorphic extendibility is the following. Let $\Gamma \subset \mathbb{R}^l - \{0\}$ be a strictly convex cone with Γ^0 its polar; i.e., $\Gamma^0 = \{v \in \mathbb{R}^l\colon v \cdot \xi > 0 \text{ for all } \xi \in \Gamma\}$. Then if h is a CR function, $\mathrm{WF}_{\mathrm{ha}} \subset \mathrm{h}\{(0;\, 0, \eta)\colon \eta \in \Gamma\}$ if and only if $h = bH$, where H is holomorphic in a wedge W of the form (8) with Γ replaced by Γ^0. Our final result gives a sufficient condition for implying that $(0;\, 0, \xi) \notin \mathrm{WF}_{\mathrm{ha}}\, h$.

In [5] Baouendi and Treves introduced the notion of sector property in order to give a sufficient condition for extendibility for CR functions on a hypersurface. A real-valued polynomial $p_m(c, \bar{c})$, $c \in \mathbb{C}$, homogeneous of degree m satisfies the *sector property* if there exists $q_m(c, \bar{c})$, harmonic and also of degree m, such that

$$(10) \qquad p_m(c, \bar{c}) + q_m(c, \bar{c}) < 0 \text{ on a sector of angle } > \pi/m \text{ in the plane.}$$

THEOREM 4. *Let M be a rigid CR manifold defined by* (1) *and* $\gamma\colon \mathbb{C} \to \mathbb{C}^n$ *a* (*possibly singular*) *holomorphic curve. For $\xi \in \mathbb{R}^l - \{0\}$ let $\Phi(\gamma(c)) \cdot \xi = p_m(c, \bar{c}) + O(m + 1)$, where $O(m + 1)$ is of degree $\geq m + 1$. Then if $p_m(c, \bar{c})$ satisfies the sector property*, $(0;\, (0, \xi)) \notin \mathrm{WF}_{\mathrm{ha}}$ *h for any CR function h on M.*

REMARK. E. Bedford [6] has recently defined a weaker property, called the *amalgamated sector property*, which implies extendibility on some classes of hypersurfaces. It is possible that Theorem 4 might extend to this case also.

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