Invariant Polynomials and Conjugacy Classes of Real Cartan Subalgebras

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1. Introduction. Conjugacy classes of Cartan subalgebras in real semisimple Lie algebras are of interest because of their connection with unitary representations of the corresponding Lie groups. A complete classification of the conjugacy classes was first made by Kostant [3]; explicit computations for all simple real Lie algebras are given in Suguira [7]. Our purpose here is to show an intrinsic relationship (Theorem 1) between these conjugacy classes and the connected components of the subset of regular semisimple elements in the algebra. The existence of such a relationship is suggested by the proof of conjugacy in the case of complex Lie algebras, which uses the connectedness of the subset of regular semisimple elements [6, p. III-6, Theorem 2]. In the real case, this subset is generally disconnected, with the number of its connected components far exceeding the number of conjugacy classes of Cartan subalgebras. (We give a formula for the exact number of such components in Section 4.) We shall use invariant polynomials to define a map which "collapses" the components coming from the same conjugacy class of Cartan subalgebras.

2. Notation and statement of main result. Let $\mathfrak{g}_c$ be a complex semisimple Lie algebra, and G_c any corresponding connected Lie group. G_c acts as group of automorphisms of $\mathfrak{g}_c$ through the adjoint representation, Ad; we write $g \cdot x$ for the image of $x \in \mathfrak{g}_c$ by the automorphism $Ad\ g$ for any $g \in G$. A function f on $\mathfrak{g}_c$ will be called *invariant* if $f(g \cdot x) = f(x)$ for any $x \in \mathfrak{g}_c$, $g \in G_c$. Let J be the ring of all invariant polynomials on $\mathfrak{g}_c$. A "classical" result of Chevalley states that there exist homogeneous polynomials $u_1, u_2, \cdots u_\ell$ in J, which are algebraically independent and which generate J. These generators are not unique, but any other such set would also consist of ℓ polynomials, where ℓ is the rank of $\mathfrak{g}_c$. (See Chevalley [1] and Helgason [2, pp. 429–434].)

$x \in \mathfrak{g}_c$ is called *semisimple* if $ad\ x$ is a diagonalizable endomorphism, where $ad\ x$ is the transformation of $\mathfrak{g}_c$ defined by $(ad\ x)y = [x, y]$ for all $y \in \mathfrak{g}_c$. Let $\mathfrak{g}_c'$ be a reductive subalgebra (real or complex) and let $\mathfrak{S} \subseteq \mathfrak{g}_c'$ be any com-

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mutative subset. We write $\mathfrak{g}'^{\mathfrak{S}} = \{y \in \mathfrak{g}_c \mid [x, y] = 0\}$, *the centralizer of* $\mathfrak{S}$ *in* $\mathfrak{g}'$. A semisimple $x \in \mathfrak{g}_c$ is called *regular-in-*$\mathfrak{g}'$ (or just *regular* when $\mathfrak{g}' = \mathfrak{g}_c$) if $\mathfrak{g}'^x$ is a Cartan subalgebra of $\mathfrak{g}'$. Finally, two Cartan subalgebras of $\mathfrak{g}'$ are called *conjugate* if one can be transformed into the other by an element of the adjoint group of $\mathfrak{g}'$.

We may now state our main result.

Theorem 1. *Let* $u\colon \mathfrak{g}_c \to \mathbf{C}^l$ *be defined by* $u(x) = (u_1(x), u_2(x) \cdots u_l(x))$ *and let* $\mathfrak{g}_c^* \subseteq \mathfrak{g}_c$ *be the subset of all regular semisimple elements. For any real form,* $\mathfrak{g}$, *of* $\mathfrak{g}_c$, *the map* u *sets up a one-to-one correspondence between the connected components of* $u(\mathfrak{g} \cap \mathfrak{g}_c^*)$ *and the conjugacy classes of Cartan subalgebras of* $\mathfrak{g}$.

Remark. The map u was studied by Kostant in [3].

Example. Let $\mathfrak{g}_c = \mathfrak{sl}(2, \mathbf{C})$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$, the 2×2 matrices of trace 0 over the complex and real fields, respectively. The determinant function, det, generates the ring of invariant polynomials on $\mathfrak{g}_c$. $\mathfrak{g}$ has two conjugacy classes of Cartan subalgebras, represented by the subalgebras

$$\mathfrak{h}_1 = \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbf{R}\right\} \quad \text{and} \quad \mathfrak{h}_2 = \left\{\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbf{R}\right\},$$

respectively. Any non-zero element of $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is regular. $\det(\mathfrak{g}^*) = \det(\mathfrak{h}_1 \cap \mathfrak{g}^*) \cup \det(\mathfrak{h}_2 \cap \mathfrak{g}^*)$, since det is invariant. Therefore, $\det(\mathfrak{g}^*) = \mathbf{R} - \{0\}$, which is the union of the connected components $\mathbf{R}^- = \det(\mathfrak{h}_1 - \{0\})$ and $\mathbf{R}^+ = \det(\mathfrak{h}_2 - \{0\})$.

3. Proof of main result. For notational convenience we write $\mathfrak{s}^* = \mathfrak{g}^* \cap \mathfrak{s}$ for any subalgebra $\mathfrak{s}$ of $\mathfrak{g}_c$. Now let $\mathfrak{h}_1, \mathfrak{h}_2, \cdots \mathfrak{h}_n$ be a set of representatives for the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$. Since u is invariant, we have $u(\mathfrak{g}^*) = u(\mathfrak{h}_1^*) \cup u(\mathfrak{h}_2^*) \cdots \cup u(\mathfrak{h}_n^*)$. We shall prove the following.

(i) $u(\mathfrak{h}_i^*) \cap u(\mathfrak{h}_j^*) = \phi$ if $i \neq j$,

(ii) $u(\mathfrak{h}_j^*)$ is closed in $u(\mathfrak{g}^*)$ for all j, and

(iii) For any j, $1 \leqq j \leqq n$, there exists a connected set $C_j \subset \mathfrak{h}_j^*$ such that $u(C_j) = u(\mathfrak{h}_j^*)$.

Statements (i) and (ii) together prove that $u(\mathfrak{h}_j^*)$ and $u(\mathfrak{h}_i^*)$ must be contained in different components of $u(\mathfrak{g}^*)$ unless $i = j$; statement (iii) shows that $u(\mathfrak{h}_j^*)$ is connected for any j, $1 \leqq j \leqq n$.

Before proving the above statements we must introduce more notation. Let $G \subset G_c$ be the connected subgroup corresponding to $\mathfrak{g}$. $x, y \in \mathfrak{g}$ are called *G-conjugate* (resp. *G_c-conjugate*) if there exists $g \in G_c$ (resp. $g' \in G$) such that $g \cdot x = y$ (resp. $g' \cdot x = y$). If $T \subset G_c$ is any subgroup and $\mathfrak{p} \subset \mathfrak{g}_c$ is any subset, we write $T \cdot \mathfrak{p} = \{a \cdot v\colon a \in T, v \in \mathfrak{p}\}$.

Lemma 1. *For any* $x, y \in \mathfrak{g}_c$, $u(x) = u(y)$ *iff* x *and* y *are* G_c*-conjugate.*

Proof. See Kostant [3, Proposition 10] or Helgason [2, p. 433, Lemma 6.3].

To prove (i), suppose $t \in u(\mathfrak{h}_i^*) \cap u(\mathfrak{h}_j^*)$ for some i, j. Then there exist $x \in \mathfrak{h}_i^*$, $y \in \mathfrak{h}_j^*$ such that $u(x) = u(y) = t$. By Lemma 1, this implies that x and y are G_c-conjugate. Therefore, by the author's result [5, Corollary 2.3], there exists $g \in G$ such that $g \cdot x \in \mathfrak{h}_j$. It follows that $g \cdot \mathfrak{h}_i = \mathfrak{h}_j$ since x and y are regular. This proves $i = j$, which proves (i).

To prove (ii), let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{h}_c = \mathfrak{h} + (-1)^{1/2}\,\mathfrak{h}$ be the complexification of $\mathfrak{h}$. If W_c is the Weyl group of $\mathfrak{g}_c$, then $W_c \cdot \mathfrak{h}$ is the subset of elements in $\mathfrak{h}_c$ conjugate to some element of $\mathfrak{h}$. Since $\mathfrak{h}$ is a closed subset of $\mathfrak{h}_c$, $W_c \cdot \mathfrak{h}$ is also closed in $\mathfrak{h}_c$. Therefore, its complement in $\mathfrak{h}_c$, which is written $\mathfrak{h}_c - W_c \cdot \mathfrak{h}$, is open in $\mathfrak{h}_c$, and so is $(\mathfrak{h}_c - W_c \cdot \mathfrak{h})^*$. We shall show that $G_c \cdot (\mathfrak{h}_c - W_c \cdot \mathfrak{h})$ is open in $\mathfrak{g}_c$. For, suppose that this is proved. Since the differentials du_i, $i = 1, 2, \cdots, \ell$, are linearly independent at every point of $\mathfrak{g}_c^*$, (see [3, Theorem 9]) it will follow that $U = u(G_c \cdot (\mathfrak{h}_c - W_c \cdot \mathfrak{h})^*) = u((\mathfrak{h}_c - W_c \cdot \mathfrak{h})^*)$ is open in $u(\mathfrak{g})$. By Lemma 1,

$$u(\mathfrak{g}^*) = u(U) \cup u(\mathfrak{h}^*) \quad \text{and} \quad u(U) \cap u(\mathfrak{h}^*) = \phi.$$

This will prove that $u(\mathfrak{h}^*)$ is closed in $u(\mathfrak{g}^*)$.

The proof of (ii) will be completed by the following lemma, which shows that $G_c \cdot (\mathfrak{h}_c - W_c \cdot \mathfrak{h})^*$ is open in $\mathfrak{g}_c$.

Lemma 2. Let $U \subset \mathfrak{h}_c^*$ (resp. $V \subset \mathfrak{h}^*$) be open in $\mathfrak{h}_c^*$ (resp. $\mathfrak{h}^*$). Then $G_c \cdot U$ (resp. $G \cdot V$) is an open subset of $\mathfrak{g}_c$ (resp. $\mathfrak{g}$).

Proof. We shall prove this only for $\mathfrak{g}$; the proof is the same for $\mathfrak{g}_c$. Let H be the connected subgroup of G corresponding to $\mathfrak{h}$. We define a map $f\colon G \times \mathfrak{h} \to \mathfrak{g}$ by $f(g, x) = g \cdot x$ for $g \in G$, $x \in \mathfrak{h}$. If x is regular, it is easy to show that the corresponding map of tangent spaces

$$\mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}$$

is surjective at x. By the implicit function theorem, it follows that $G \cdot V$ contains a neighborhood of x, which proves the lemma.

Before proving (iii) we must review some known facts of Cartan subalgebras. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *standard* (with respect to $\mathfrak{k}$) if $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p}$.

Lemma 3. Every Cartan subalgebra of $\mathfrak{g}$ is conjugate to a standard Cartan subalgebra.

Proof. See Suguira [1, p. 383, Theorem 2].

By Lemma 3, we may assume that $\mathfrak{h}_1, \mathfrak{h}_2, \cdots \mathfrak{h}_n$ are standard. We write $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{p}$ for any standard Cartan subalgebra $\mathfrak{h}$. If $x \in \mathfrak{h}$, we have $x = x_k + x_p$ with $x_k \in \mathfrak{h}_k$ and $x_p \in \mathfrak{h}_p$. $x \in \mathfrak{h}^*$ is called *generic* if

$$\mathfrak{g}^{x_k} = \mathfrak{g}^{\mathfrak{h}_k} \quad \text{and} \quad \mathfrak{g}^{x_p} = \mathfrak{g}^{\mathfrak{h}_p}.$$

This means that x_k and x_p have the smallest possible centralizers with respect to $\mathfrak{h}_k$ and $\mathfrak{h}_p$. An easy proof of the existence of generic elements is given in [5].

If $\mathfrak{S} \subset \mathfrak{g}_c$ is any commutative subset of semisimple elements, we write $W_c^{\mathfrak{S}}$ for the Weyl group of the reductive algebra $\mathfrak{g}_c^{\mathfrak{S}}$. Any $w \in W_c^{\mathfrak{S}}$ is the restriction, to a Cartan subalgebra of some $g \in G$, such that $Ad\ g$ leaves $\mathfrak{S}$ pointwise fixed.

We may now prove statement (iii). By Lemma 1 it suffices to show the following.

Proposition. If $\mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{g}$, *there exists a connected set* $C \subset \mathfrak{h}^*$ *such that every element of* $\mathfrak{h}^*$ *is* G_c*-conjugate to some element of* C; i.e., $G_c \cdot \mathfrak{h}^* = G_c \cdot C$.

Proof. The difficulty here arises from the fact that conjugation in $\mathfrak{h}$ is not directly given by the action of a Weyl group on $\mathfrak{h}$. However, there are Weyl groups which act on $\mathfrak{h}_k$ and $\mathfrak{h}_p$ separately; $\mathfrak{h}_k$ and $\mathfrak{h}_p$ have natural structures as Euclidean spaces, the metric being given by the Killing form, which is negative definite on $\mathfrak{h}_k$ and positive definite on $\mathfrak{h}_p$. We shall need the following well-known fact about Weyl chambres.

Lemma 4. Let $\mathfrak{h}^0$ *be a Euclidean space and* W *a Weyl group acting on* $\mathfrak{h}^0$. *Then the Weyl chambres in* $\mathfrak{h}^0$ *are convex sets.*

Proof. Let Π be a set of simple positive roots on $\mathfrak{h}^0$. It suffices to prove the lemma for the dominant chambre $D = \{x \in \mathfrak{h}^0 : \alpha(x) > 0 \text{ for all } \alpha \in \Pi\}$. Let $x, y \in D$ be arbitrary. Then for any $\alpha \in \Pi$, and any t, $0 \leqq t \leqq 1$, we have

$$\alpha(tx + (1 - t)y) > 0.$$

Hence $tx + (1 - t)y \in D$, which proves that D is convex. Q.E.D.

We shall now define the set C implicity. Let $x = x_k + x_p \in \mathfrak{h}^*$ be any generic element. Then C is defined as the connected component of $\mathfrak{h}^*$ containing x.

We must show that if $y = y_k + y_p \in \mathfrak{h}^*$, then y is G_c-conjugate to some $y' \in C$. It suffices to prove the following.

(a) There exists $w_1 \in W_c^{\mathfrak{h}_p}$ such that the element x_t defined by

$$x_t = x_p + (1 - t)x_k + tw_1 \cdot y_k$$

is regular for all t, $0 \leqq t \leqq 1$.

(b) There exists $w_2 \in W_c^{w_1 \cdot y_k}$ with $w_2 \cdot y_p \in \mathfrak{h}_p$ such that the element y_t defined by

$$y_t = w_1 \cdot y_k + (1 - t)x_p + tw_2 \cdot y_p$$

is regular for all t, $0 \leqq t \leqq 1$. For if (a) and (b) hold, let $y' = w_1 w_2 \cdot y$. By (a), the elements x_t form a line in $\mathfrak{h}^*$ connecting x and $w_1 \cdot y_k + x_p$, and by (b), the elements y_t form a line in $\mathfrak{h}^*$ connecting $w_1 \cdot y_k + x_p$ and $w_1 \cdot y_k + w_2 \cdot y_p = w_1 w_2 \cdot y = y'$. Hence $y' \in C$.

Lemma 5. If $z = z_k + z_p \in \mathfrak{h}$, *then*

$$\mathfrak{g}^{x} = (\mathfrak{g}^{x_k})^{x_p} = (\mathfrak{g}^{x_p})^{x_k}.$$

Proof. This follows easily by considering $\mathfrak{g}_c^{x}$. (See [5] for details.)

To prove (a), first note that x_k is regular-in-$\mathfrak{g}^{\mathfrak{h}_p}$ since

$$(\mathfrak{g}^{\mathfrak{h}_p})^{x_k} = (\mathfrak{g}^{x_k})^{\mathfrak{h}_p} = (\mathfrak{g}^{\mathfrak{h}_p})^{\mathfrak{h}_k} = \mathfrak{g}^{\mathfrak{h}} = \mathfrak{h}.$$

Furthermore, y_k is also regular-in-$\mathfrak{g}^{\mathfrak{h}_p}$ since

$$(\mathfrak{g}^{\mathfrak{h}_p})^{y_k} = (\mathfrak{g}^{y_p})^{y_k} = \mathfrak{h},$$

because $y = y_k + y_p$ is regular. Therefore, there exists $w_1 \in W_c^{\mathfrak{h}_p}$ such that $\mathfrak{g}^{w_1 \cdot y_k + y_p} \in C_1 \subset \mathfrak{h}_k$, where C_1 is the Weyl chambre of $\mathfrak{h}_k$ containing x, under the action of the Weyl group $W_c^{\mathfrak{h}_p}$. By Lemma 4, the element z_t defined by

$$z_t = (1 - t)x_k + ty_k$$

is in C_1 for $0 \leqq t \leqq 1$, which proves that z_t is regular-in-$\mathfrak{g}^{\mathfrak{h}_p}$. Therefore, $x_t = z_t + x_p$ is regular, since

$$\mathfrak{g}^{x_t} = (\mathfrak{g}^{x_p})^{z_t} = (\mathfrak{g}^{\mathfrak{h}_p})^{z_t} = \mathfrak{h},$$

the first equality coming from Lemma 5. This proves (a).

To prove (b), note first that y_p is regular-in-$\mathfrak{g}^{w_1 \cdot y_k}$, since

$$(\mathfrak{g}^{w_1 \cdot y_k})^{y_p} = (\mathfrak{g}^{w_1 \cdot y_k}) = \mathfrak{g}^{w_1 \cdot y} = \mathfrak{h}$$

because y, and hence $w_1 \cdot y$, are regular. Furthermore, x_p is regular-in-$\mathfrak{g}^{w_1 \cdot y_k}$ since

$$(\mathfrak{g}^{w_1 \cdot y_k})^{x_p} = (\mathfrak{g}^{\mathfrak{h}_p})^{w_1 \cdot y_k} \subseteq (\mathfrak{g}^{y_p})^{w_1 \cdot y_k} = \mathfrak{h}.$$

Let $W_1^{w_1 \cdot y_k}$ be the little Weyl group [2, p. 244] associated to the reductive algebra $\mathfrak{g}^{w_1 \cdot y_k}$. Since $x_p \in \mathfrak{h}_p$ and x_p is regular-in-$\mathfrak{g}^{w_1 \cdot y_k}$, $\mathfrak{h}_p$ contains a maximal vector subspace of $\mathfrak{g}^{w_1 \cdot y_k}$. Hence $W^{w_1 \cdot y_k}$ leaves $\mathfrak{h}_p$ invariant. Then there exists $w_2 \in W_1^{w_1 \cdot y_k}$ such that $w_2 \cdot y \in C_2 \subset \mathfrak{h}_p$, where C_2 is the Weyl chambre of $\mathfrak{h}_p$ containing x_p. The fact that y_t is regular follows by the same argument as in the proof of (a), which proves (b).

This completes the proof of the Proposition; Theorem 1 is now proved.

4. Connected components of the set of regular semisimple elements. We shall now combine the above with the author's previous result on semisimple orbits in [5, Theorem 2.6], to express $\mathfrak{g}^*$ as the union of its connected components. Suppose $\mathfrak{h}_k + \mathfrak{h}_p = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $W_0^{\mathfrak{h}_p}$ be the Cartan group of the reductive algebra $\mathfrak{k}^{\mathfrak{h}_p}$. Then let $W^{\mathfrak{h}_p} = \{w \in W_0^{\mathfrak{h}_p}: w = g \mid \mathfrak{h} \text{ for some } g \in K\}$, where $g \mid \mathfrak{h}$ is the restriction of g to $\mathfrak{h}$.

Theorem 2. Let $\mathfrak{h}_1, \mathfrak{h}_2, \cdots \mathfrak{h}_n$ *be representatives of the different conjugacy classes of Cartan subalgebras of* $\mathfrak{g}$. *Then* $\mathfrak{g}^*$ *is the union of* $m = \sum_{i=1}^{n} m_i$ *connected components, where* m_i *is the order of the coset space* $W_c^{\mathfrak{h}_{ip}}/W^{\mathfrak{h}_{ip}}$.

Proof. $\mathfrak{g}^*$ is the disjoint union of the sets $G \cdot \mathfrak{h}_j$, $j = 1, 2, \cdots n$. By Lemma 2, $G \cdot \mathfrak{h}^*$ is open in $\mathfrak{g}$ for any Cartan subalgebra $\mathfrak{h}$. Therefore, it suffices to show that $G \cdot \mathfrak{h}^*$ is the union of m components, where m is the order of the coset space $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$.

By the Proposition, there exists a connected set $C \subset \mathfrak{h}^*$ such that $G_c \cdot \mathfrak{h}^* = G_c \cdot C$. Since C is a component of the open set $\mathfrak{h}^*$, C is also open in $\mathfrak{h}$. Now let $w_1, w_2, \cdots, w_{m_0}$ be a set of representatives in G_c of the coset space $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$. By Lemma 2 the sets $G \cdot (w_i \cdot C)$ are all open. Therefore, it suffices to prove

$$\mathfrak{g}^* = \bigcup_{i=1}^{m_0} G \cdot (w_i \cdot C) \quad \text{and} \quad G \cdot (w_i \cdot C) \cap G \cdot (w_j \cdot C) = \phi \quad \text{if} \quad i \neq j.$$

However, this follows immediately from the author's theorem on orbits [5, Theorem 2.6], which states that $G_c \cdot x \cap \mathfrak{g}$ is the disjoint union of the G-orbits $\{G \cdot (w_i \cdot x)\}$, $i = 1, 2, \cdots m_0$, for any $x \in \mathfrak{g}^*$.

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