Invariant Polynomials and Conjugacy Classes of Real Cartan Subalgebras

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1. Introduction. Conjugacy classes of Cartan subalgebras in real semisimple Lie algebras are of interest because of their connection with unitary representations of the corresponding Lie groups. A complete classification of the conjugacy classes was first made by Kostant [3]; explicit computations for all simple real Lie algebras are given in Sugiuira [7]. Our purpose here is to show an intrinsic relationship (Theorem 1) between these conjugacy classes and the connected components of the subset of regular semisimple elements in the algebra. The existence of such a relationship is suggested by the proof of conjugacy in the case of complex Lie algebras, which uses the connectedness of the subset of regular semisimple elements [6, p. III-6, Theorem 2]. In the real case, this subset is generally disconnected, with the number of its connected components far exceeding the number of conjugacy classes of Cartan subalgebras. (We give a formula for the exact number of such components in Section 4.) We shall use invariant polynomials to define a map which “collapses” the components coming from the same conjugacy class of Cartan subalgebras.

2. Notation and statement of main result. Let \( g_e \) be a complex semisimple Lie algebra, and \( G_e \) any corresponding connected Lie group. \( G_e \) acts as group of automorphisms of \( g_e \) through the adjoint representation, \( \text{Ad} \); we write \( g \cdot x \) for the image of \( x \in g_e \) by the automorphism \( \text{Ad} \ g \) for any \( g \in G_e \). A function \( f \) on \( g_e \) will be called invariant if \( f(g \cdot x) = f(x) \) for any \( x \in g_e, \ g \in G_e \). Let \( J \) be the ring of all invariant polynomials on \( g_e \). A “classical” result of Chevalley states that there exist homogeneous polynomials \( u_1, u_2, \ldots, u_t \) in \( J \), which are algebraically independent and which generate \( J \). These generators are not unique, but any other such set would also consist of \( t \) polynomials, where \( t \) is the rank of \( g_e \). (See Chevalley [1] and Helgason [2, pp. 429–434].)

\( x \in g_e \) is called semisimple if \( \text{ad} \ x \) is a diagonalizable endomorphism, where \( \text{ad} \ x \) is the transformation of \( g_e \) defined by \( (\text{ad} \ x)y = [x, y] \) for all \( y \in g_e \). Let \( g' \) be a reductive subalgebra (real or complex) and let \( \mathcal{S} \subseteq g' \) be any com-

mutative subset. We write \( g' = \{ y \in g_\alpha \mid [x, y] = 0 \} \), the centralizer of \( g \) in \( g' \). A semisimple \( x \in g_\alpha \) is called regular-in-\( g' \) (or just regular when \( g' = g_\alpha \)) if \( g' = g' \iota \) is a Cartan subalgebra of \( g' \). Finally, two Cartan subalgebras of \( g' \) are called conjugate if one can be transformed into the other by an element of the adjoint group of \( g' \).

We may now state our main result.

**Theorem 1.** Let \( u : g_\alpha \to C' \) be defined by \( u(x) = (u_1(x), u_2(x), \ldots, u_r(x)) \) and let \( g'_* \subseteq g_\alpha \) be the subset of all regular semisimple elements. For any real form, \( g_\alpha \), of \( g_\alpha \), the map \( u \) sets up a one-to-one correspondence between the connected components of \( u(g_\alpha \cap g'_*) \) and the conjugacy classes of Cartan subalgebras of \( g_\alpha \).

**Remark.** The map \( u \) was studied by Kostant in [3].

**Example.** Let \( g_\alpha = \mathfrak{sl}(2, \mathbb{C}) \) and \( g = \mathfrak{sl}(2, \mathbb{R}) \), the \( 2 \times 2 \) matrices of trace 0 over the complex and real fields, respectively. The determinant function, \( \det \), generates the ring of invariant polynomials on \( g_\alpha \). \( g_\alpha \) has two conjugacy classes of Cartan subalgebras, represented by the subalgebras

\[
\mathfrak{h}_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} : a \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{h}_2 = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\},
\]

respectively. Any non-zero element of \( \mathfrak{h}_1 \) or \( \mathfrak{h}_2 \) is regular. \( \det (g_*^\ast) = \det (\mathfrak{h}_1 \cap g_*^\ast) \cup \det (\mathfrak{h}_2 \cap g_*^\ast) \), since \( \det \) is invariant. Therefore, \( \det (g_*^\ast) = \mathbb{R} - \{0\} \), which is the union of the connected components \( \mathbb{R}^- = \det (\mathfrak{h}_1 - \{0\}) \) and \( \mathbb{R}^+ = \det (\mathfrak{h}_2 - \{0\}) \).

3. **Proof of main result.** For notational convenience we write \( g_*^\ast = g_*^\ast \cap g \) for any subalgebra \( g \) of \( g_\alpha \). Now let \( \mathfrak{h}_1, \mathfrak{h}_2, \cdots, \mathfrak{h}_n \) be a set of representatives for the conjugacy classes of Cartan subalgebras of \( g_\alpha \). Since \( u \) is invariant, we have \( u(g_*^\ast) = u(\mathfrak{h}_1^*) \cup u(\mathfrak{h}_2^*) \cdots \cup u(\mathfrak{h}_n^*) \). We shall prove the following.

(i) \( u(\mathfrak{h}_i^*) \cap u(\mathfrak{h}_j^*) = \emptyset \) if \( i \neq j \),

(ii) \( u(\mathfrak{h}_i^*) \) is closed in \( u(g_*^\ast) \) for all \( j \), and

(iii) For any \( j, 1 \leq j \leq n \), there exists a connected set \( C_j \subseteq \mathfrak{h}_j^* \) such that \( u(C_j) = u(\mathfrak{h}_j^*) \).

Statements (i) and (ii) together prove that \( u(\mathfrak{h}_i^*) \) and \( u(\mathfrak{h}_j^*) \) must be contained in different components of \( u(g_*^\ast) \) unless \( i = j \); statement (iii) shows that \( u(\mathfrak{h}_i^*) \) is connected for any \( j, 1 \leq j \leq n \).

Before proving the above statements we must introduce more notation. Let \( G \subseteq G_\alpha \) be the connected subgroup corresponding to \( g_\alpha \). \( x, y \in g \) are called \( G \)-conjugate (resp. \( G_\alpha \)-conjugate) if there exists \( g \in G_\alpha \) (resp. \( g' \in G \)) such that \( g \cdot x = y \) (resp. \( g' \cdot x = y \)). If \( T \subseteq G_\alpha \) is any subgroup and \( \mathfrak{p} \subseteq g_\alpha \) is any subset, we write \( T \cdot \mathfrak{p} = \{ a \cdot v : a \in T, v \in \mathfrak{p} \} \).

**Lemma 1.** For any \( x, y \in g_\alpha \), \( u(x) = u(y) \) iff \( x \) and \( y \) are \( G_\alpha \)-conjugate.

To prove (i), suppose \( t \in u(\mathfrak{h}_i^*) \cap u(\mathfrak{h}_j^*) \) for some \( i, j \). Then there exist \( x \in \mathfrak{h}_i^* \), \( y \in \mathfrak{h}_j^* \) such that \( u(x) = u(y) = t \). By Lemma 1, this implies that \( x \) and \( y \) are \( G_\epsilon \)-conjugate. Therefore, by the author's result [5, Corollary 2.3], there exists \( g \in G \) such that \( g \cdot x \in \mathfrak{h}_i \). It follows that \( g \cdot \mathfrak{h}_i = \mathfrak{h}_i \), since \( x \) and \( y \) are regular. This proves \( i = j \), which proves (i).

To prove (ii), let \( \mathfrak{h} \) be any Cartan subalgebra of \( g \), and let \( \mathfrak{h}_e = \mathfrak{h} + (-1)^{i/2} \mathfrak{h} \) be the complexification of \( \mathfrak{h} \). If \( W_\epsilon \) is the Weyl group of \( g_\epsilon \), then \( W_\epsilon \cdot \mathfrak{h} \) is the subset of elements in \( \mathfrak{h}_e \) conjugate to some element of \( \mathfrak{h} \). Since \( \mathfrak{h} \) is a closed subset of \( \mathfrak{h}_e \), \( W_\epsilon \cdot \mathfrak{h} \) is also closed in \( \mathfrak{h}_e \). Therefore, its complement in \( \mathfrak{h}_e \), which is written \( \mathfrak{h}_e - W_\epsilon \cdot \mathfrak{h} \), is open in \( \mathfrak{h}_e \), and so is \( (\mathfrak{h}_e - W_\epsilon \cdot \mathfrak{h})^* \). We shall show that \( G_\epsilon \cdot (\mathfrak{h}_e - W_\epsilon \cdot \mathfrak{h})^* \) is open in \( g_\epsilon \). For, suppose that this is proved. Since the differentials \( d\mu_i \), \( i = 1, 2, \ldots, \ell \), are linearly independent at every point of \( g_\epsilon^* \), (see [3, Theorem 9]) it will follow that \( U = u(G_\epsilon \cdot (\mathfrak{h}_e - W_\epsilon \cdot \mathfrak{h})^*) = u((\mathfrak{h}_e - W_\epsilon \cdot \mathfrak{h})^*) \) is open in \( u(g) \).

By Lemma 1,

\[
u(g^*) = u(U) \cup u(\mathfrak{h}^*) \quad \text{and} \quad u(U) \cap u(\mathfrak{h}^*) = \phi.\]

This will prove that \( u(\mathfrak{h}^*) \) is closed in \( u(g^*) \).

The proof of (ii) will be completed by the following lemma, which shows that \( G_\epsilon \cdot (\mathfrak{h}_e - W_\epsilon \cdot \mathfrak{h})^* \) is open in \( g_\epsilon \).

Lemma 2. Let \( U \subset \mathfrak{h}^*_e \) (resp. \( V \subset \mathfrak{h}^* \)) be open in \( \mathfrak{h}^*_e \) (resp. \( \mathfrak{h}^* \)). Then \( G_\epsilon \cdot U \) (resp. \( G \cdot V \)) is an open subset of \( g_\epsilon \) (resp. \( g \)).

Proof. We shall prove this only for \( g \); the proof is the same for \( g_\epsilon \). Let \( H \) be the connected subgroup of \( G \) corresponding to \( \mathfrak{h} \). We define a map \( f: G \times \mathfrak{h} \to g \) by \( f(g, x) = g \cdot x \) for \( g \in G \), \( x \in \mathfrak{h} \). If \( x \) is regular, it is easy to show that the corresponding map of tangent spaces

\[
g \times \mathfrak{h} \to g\]

is surjective at \( x \). By the implicit function theorem, it follows that \( G \cdot V \) contains a neighborhood of \( x \), which proves the lemma.

Before proving (iii) we must review some known facts of Cartan subalgebras. Let \( g = \mathfrak{t} + \mathfrak{p} \) be a Cartan decomposition of \( g \), where \( \mathfrak{t} \) is a maximal compact subalgebra of \( g \). A Cartan subalgebra \( \mathfrak{h} \subset g \) is called standard (with respect to \( \mathfrak{t} \)) if \( \mathfrak{h} = \mathfrak{h} \cap \mathfrak{t} + \mathfrak{h} \cap \mathfrak{p} \).

Lemma 3. Every Cartan subalgebra of \( g \) is conjugate to a standard Cartan subalgebra.

Proof. See Suguira [1, p. 383, Theorem 2].

By Lemma 3, we may assume that \( \mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_n \) are standard. We write \( \mathfrak{h}_e = \mathfrak{h} \cap \mathfrak{t} \) and \( \mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{p} \) for any standard Cartan subalgebra \( \mathfrak{h} \). If \( x \in \mathfrak{h}_e \), we have \( x = x_e + x_p \) with \( x_e \in \mathfrak{h}_e \) and \( x_p \in \mathfrak{h}_p \). \( x \in \mathfrak{h}^* \) is called generic if

\[
g^{x*} = g^{x*} \quad \text{and} \quad g^{x*} = g^{x*}.
\]
This means that $x_\xi$ and $x_\eta$ have the smallest possible centralizers with respect to $\mathfrak{h}_\xi$ and $\mathfrak{h}_\eta$. An easy proof of the existence of generic elements is given in [5].

If $\mathfrak{g} \subset \mathfrak{g}_e$ is any commutative subset of semisimple elements, we write $\mathcal{W}_e^\mathfrak{g}$ for the Weyl group of the reductive algebra $\mathfrak{g}_e^\mathfrak{g}$. Any $w \in \mathcal{W}_e^\mathfrak{g}$ is the restriction, to a Cartan subalgebra of some $g \in G$, such that $Ad g$ leaves $\mathfrak{g}$ pointwise fixed.

We may now prove statement (iii). By Lemma 1 it suffices to show the following.

**Proposition.** If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, there exists a connected set $C \subset \mathfrak{h}^*$ such that every element of $\mathfrak{h}^*$ is $G_e$-conjugate to some element of $C$; i.e., $G_e \cdot \mathfrak{h}^* = G_e \cdot C$.

**Proof.** The difficulty here arises from the fact that conjugation in $\mathfrak{h}$ is not directly given by the action of a Weyl group on $\mathfrak{h}$. However, there are Weyl groups which act on $\mathfrak{h}_\xi$ and $\mathfrak{h}_\eta$ separately; $\mathfrak{h}_\xi$ and $\mathfrak{h}_\eta$ have natural structures as Euclidean spaces, the metric being given by the Killing form, which is negative definite on $\mathfrak{h}_\xi$ and positive definite on $\mathfrak{h}_\eta$. We shall need the following well-known fact about Weyl chambers.

**Lemma 4.** Let $\mathfrak{h}^0$ be a Euclidean space and $W$ a Weyl group acting on $\mathfrak{h}^0$. Then the Weyl chambers in $\mathfrak{h}^0$ are convex sets.

**Proof.** Let $\Pi$ be a set of simple positive roots on $\mathfrak{h}^0$. It suffices to prove the lemma for the dominant chamber $D = \{x \in \mathfrak{h}^0 : \alpha(x) > 0 \text{ for all } \alpha \in \Pi\}$. Let $x, y \in D$ be arbitrary. Then for any $\alpha \in \Pi$, and any $t, 0 \leq t \leq 1$, we have

$$\alpha(tx + (1 - t)y) > 0.$$  

Hence $tx + (1 - t)y \in D$, which proves that $D$ is convex. Q.E.D.

We shall now define the set $C$ implicitly. Let $x = x_\xi + x_\eta \in \mathfrak{h}^*$ be any generic element. Then $C$ is defined as the connected component of $\mathfrak{h}^*$ containing $x$.

We must show that if $y = y_\xi + y_\eta \in \mathfrak{h}^*$, then $y$ is $G_e$-conjugate to some $y' \in C$. It suffices to prove the following.

(a) There exists $w_1 \in \mathcal{W}_e^\mathfrak{h}_e$ such that the element $x_\xi$ defined by

$$x_\xi = x_\eta + (1 - t)x_\xi + tw_1 \cdot y_\eta$$

is regular for all $t, 0 \leq t \leq 1$.

(b) There exists $w_2 \in \mathcal{W}_e^\mathfrak{h}_e$ with $w_2 \cdot y_\eta \in \mathfrak{h}_e$ such that the element $y_\eta$ defined by

$$y_\eta = w_1 \cdot y_\eta + (1 - t)x_\eta + tw_2 \cdot y_\eta$$

is regular for all $t, 0 \leq t \leq 1$. For if (a) and (b) hold, let $y' = w_1 w_2 \cdot y$. By (a), the elements $x_\xi$ form a line in $\mathfrak{h}_e^*$ connecting $x$ and $w_1 \cdot y_\eta + x_\eta$, and by (b), the elements $y_\eta$ form a line in $\mathfrak{h}_e^*$ connecting $w_1 \cdot y_\eta + x_\eta$ and $w_1 \cdot y_\eta + w_2 \cdot y_\eta = w_1 w_2 \cdot y = y'$. Hence $y' \in C$.

**Lemma 5.** If $z = x_\xi + x_\eta \in \mathfrak{h}$, then
\[ g^* = (g^{\ast \ast})^{\ast 
abla} = (g^{\ast \ast})^{\ast}. \]

**Proof.** This follows easily by considering \( g^*_x \). (See [5] for details.)

To prove (a), first note that \( x_\lambda \) is regular-in-\( g^{b'} \) since
\[ (g^{b'})^{\ast \ast} = (g^{\ast \ast})^{b'} = (g^{b'})^{\ast \ast} = g^b = h. \]

Furthermore, \( y_\lambda \) is also regular-in-\( g^{b'} \) since
\[ (g^{b'})^{\ast \ast} = (g^{\ast \ast})^{b'} = h, \]
because \( y = y_\lambda + y_\rho \) is regular. Therefore, there exists \( w_1 \in W_1^{b'} \) such that \( g^{w_1 \cdot w_1} \in C_1 \subseteq h, \) where \( C_1 \) is the Weyl chamber of \( h \), containing \( x_\lambda \). Under the action of the Weyl group \( W_1^{b'} \). By Lemma 4, the element \( z_t \) defined by
\[ z_t = (1 - t) x_\lambda + t y_\lambda \]
is in \( C_1 \) for 0 \( \leq t \leq 1 \), which proves that \( z_t \) is regular-in-\( g^{b'} \). Therefore, \( z_t = z_\lambda + z_\rho \) is regular, since
\[ g^{z_t} = (g^{z_\lambda})^{z_\rho} = (g^{b'})^{z_\rho} = h, \]
the first equality coming from Lemma 5. This proves (a).

To prove (b), note first that \( y_\rho \) is regular-in-\( g^{w_1 w_1} \), since
\[ (g^{w_1 w_1})^{y_\rho} = (g^{w_1 w_1}) = g^{w_1 w_1} = h \]
because \( y_\rho \) and hence \( w_1 \cdot y_\rho \) are regular. Furthermore, \( x_\rho \) is regular-in-\( g^{w_1 w_1} \) since
\[ (g^{w_1 w_1})^{x_\rho} = (g^{b'})^{w_1 w_1} \subseteq (g^{b'})^{w_1 w_1} = h. \]

Let \( W_1^{w_1 w_1} \) be the little Weyl group [2, p. 244] associated to the reductive algebra \( g^{w_1 w_1} \). Since \( x_\rho \in h \) and \( x_\rho \) is regular-in-\( g^{w_1 w_1} \), \( h \) contains a maximal vector subspace of \( g^{w_1 w_1} \). Hence \( W_1^{w_1 w_1} \) leaves \( h \) invariant. Then there exists \( w_2 \in W_1^{w_1 w_1} \) such that \( w_2 \cdot y \in C_2 \subseteq h \), where \( C_2 \) is the Weyl chamber of \( h \), containing \( x_\rho \). The fact that \( y_\rho \) is regular follows by the same argument as in the proof of (a), which proves (b).

This completes the proof of the Proposition; Theorem 1 is now proved.

4. **Connected components of the set of regular semisimple elements.** We shall now combine the above with the author's previous result on semisimple orbits in [5, Theorem 2.6], to express \( g^* \) as the union of its connected components. Suppose \( h_{\lambda} + h_{\rho} = h \) is a Cartan subalgebra of \( g \). Let \( W_{b'}^{b'} \) be the Cartan group of the reductive algebra \( f^{b'} \). Then let \( W^{b'} = \{ w \in W_{b'}^{b'} : w = g \mid h \} \) for some \( g \in K \), where \( g \mid h \) is the restriction of \( g \) to \( h \).

**Theorem 2.** Let \( h_{\lambda} \), \( h_{\mu} \), \( \cdots \), \( h_{\nu} \) be representatives of the different conjugacy classes of Cartan subalgebras of \( g \). Then \( g^* \) is the union of \( m = \sum_{i=1}^m m_i \) connected components, where \( m_i \) is the order of the coset space \( W_{b'}^{b'}/W_{b'}^{b'}. \)
Proof. $g^*$ is the disjoint union of the sets $G \cdot \mathfrak{h}_j$, $j = 1, 2, \ldots, n$. By Lemma 2, $G \cdot \mathfrak{h}^*$ is open in $g$ for any Cartan subalgebra $\mathfrak{h}$. Therefore, it suffices to show that $G \cdot \mathfrak{h}^*$ is the union of $m$ components, where $m$ is the order of the coset space $W_e^{br}/W^b$.

By the Proposition, there exists a connected set $C \subset \mathfrak{h}^*$ such that $G_e \cdot \mathfrak{h}^* = G_e \cdot C$. Since $C$ is a component of the open set $\mathfrak{h}^*$, $C$ is also open in $\mathfrak{h}$. Now let $w_1, w_2, \ldots, w_m$ be a set of representatives in $G_e$ of the coset space $W_e^{br}/W^b$.

By Lemma 2 the sets $G \cdot (w_i \cdot C)$ are all open. Therefore, it suffices to prove

$$g^* = \bigcup_{i=1}^{m} G \cdot (w_i \cdot C) \quad \text{and} \quad G \cdot (w_i \cdot C) \cap G \cdot (w_j \cdot C) = \emptyset \quad \text{if} \quad i \neq j.$$

However, this follows immediately from the author's theorem on orbits [5, Theorem 2.6], which states that $G_e \cdot x \cap g$ is the disjoint union of the $G$-orbits $\{G \cdot (w_i \cdot x)\}$, $i = 1, 2, \ldots, m_0$, for any $x \in g^*$.

References


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