INVERSION OF ANALYTIC MATRICES AND LOCAL SOLVABILITY 
OF SOME INVARIANT DIFFERENTIAL OPERATORS 
ON NILPOTENT LIE GROUPS

Linda Preiss Rothschild and David S. Tartakoff

Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706

Department of Mathematics
University of Illinois
Chicago, Illinois 60680

1. INTRODUCTION

We shall prove a necessary and sufficient condition for the local solvability of some left invariant operators on a class of 2-step nilpotent groups including the Heisenberg groups. Our method involves the use of the Plancherel formula and the interpretation of the inverse of an analytic matrix as a matrix of distributions. Here we will say that a partial differential operator \( L \) is \textit{locally solvable} at \( x_0 \) provided there is a neighborhood \( U_{x_0} \) of \( x_0 \) such that

\[
(1.1) \quad L \sigma = f
\]

has a solution \( \sigma \in C^\infty(U_{x_0}) \) for every \( f \in C^\infty_0(U_{x_0}) \).

Copyright © 1981 by Marcel Dekker, Inc.
Let \( g \) be a 2-step nilpotent Lie algebra such that \( g = g_1 + g_2 \) with \( [g_1, g_1] = g_2 \) and \( g_2 \) contained in the center of \( g \). Following [17] we shall say that \( g \) is of type \( H \) if for every non-zero linear functional \( \eta \) on \( g_2 \) and basis \( Y_1, \ldots, Y_{p_1} \) of \( g_1 \),

\[
\det_B = \det \eta ( [Y_i, Y_j] )_{1 \leq i, j \leq p_1} \neq 0.
\]

Let \( G \) be the simply connected Lie group corresponding to \( g \) and \( \mathfrak{u}(g) \) the universal enveloping algebra, which we identify with the set of all left invariant differential operators on \( G \).

The dilations \( \delta_s \) defined by \( \delta_s|_{g_1} = s \cdot I \) and \( \delta_s|_{g_2} = s^2 \cdot I, \ s > 0, \) form a family of automorphisms which extend to a family of dilations, again denoted \( \delta_s \), on \( \mathfrak{u}(g) \).

An element \( L \in \mathfrak{u}(g) \) is homogeneous of degree \( d \) if \( \delta_s(L) = s^d L \). Any such \( L \) can be written in the form

\[
L = \sum_{|\alpha| = d} c_\alpha Y_{I_1}^{I_2} \cdots Y_{I_{|\alpha|}}^{I_{|\alpha|}}
\]

where \( Y_{I_1}^{I_2} \cdots Y_{I_{|\alpha|}}^{I_{|\alpha|}} \) can denote any \( \alpha \) and the \( c_\alpha \) are constants. We shall assume that \( L \) is "elliptic in the generating directions", i.e., that

\[
\sum_{|\alpha| = d} c_\alpha \xi^{\alpha} \neq 0, \quad 0 \neq \xi \in \mathbb{R}^{p_1}.
\]

Let \( \hat{G} \) be the set of all irreducible unitary representations of \( G \). If \( \pi \in \hat{G} \) we again denote by \( \pi \) the representation
of \( \mathcal{U}(g) \) on the space of \( C^\infty \) vectors. For \( \lambda \in g^* \), the linear dual of \( g \), let \( \pi_\lambda \) be the corresponding representation. We may now state our main results.

**Theorem I.** Let \( L \in \mathcal{U}(g) \), \( g \) of type \( H \), be homogeneous and satisfy (1.4). Then the following are equivalent:

1) \( L \) is locally solvable.

2) There is no open set \( U \subset g^* \) such that \( \pi_\lambda(L^t) \) has a non-trivial kernel (in \( L^2 \)) for all \( \lambda \in U \).

3) \( \ker L^t \cap L^2(G) = \{0\} \).

**Theorem II.** If \( L \in \mathcal{U}(g) \), \( g \) of type \( H \) with \( p_2 > 1 \), is homogeneous of degree 2 and satisfies (1.4), then \( \ker L \) and \( \ker L^t \) are trivial and thus \( L \) is locally solvable.

**Remark 1.** The implications 1) implies 3) and 3) implies 2) of Theorem I are contained in Corwin-Rothschild [3].

**Remark 2.** Theorem I was known for the Heisenberg group [6].

**Remark 3.** Theorem 2 improves the result of Lévy-Bruhl [13], who showed that 1) and 2) are equivalent when \( \deg(L) = 2 \).

Results for other 2-step groups with \( \deg(L) = 2 \) had previously been obtained by the first author [22].

The first example of an unsolvable linear differential operator, given by H. Lewy [14], may also be interpreted as
a left invariant operator on the three dimensional Heisenberg algebra, homogeneous of degree 1. A representation-theoretic condition for hypoellipticity and local solvability for a related class of operators on the Heisenberg group was given by Folland-Stein [5]. Rockland [20] then proved that a homogeneous left invariant operator \( L \) on the Heisenberg group is hypoelliptic if and only if \( \pi(L) \) has trivial kernel for every irreducible, nontrivial unitary representation \( \pi \) of \( G \), and conjectured the validity of the statement for all nilpotent Lie groups with dilations. The general case was later proved by Helffer and Nourrigat [8]. Rockland also conjectured that the existence of a right inverse for \( \pi(L) \) would give local solvability. This was proved by the first author [21] and Lion [15], independently, and generalized by Corwin [2].

For operators on the Heisenberg group which are elliptic in the generating directions, a very detailed analysis was made by Geller [6], who obtained as a side result the necessary and sufficient conditions of Theorem 1. (See also Greiner, Kohn and Stein [7].)

The most general results for local solvability on Lie groups have been obtained for bi-invariant operators, not necessarily homogeneous. The first result was proved by Raïs [17] (see also [24]), who used the method of division by analytic functions as in Atiyah [1] and the Plancherel formula to prove that bi-invariant differential operators on nilpotent Lie groups are
locally solvable. The general result for any Lie group was later obtained by Duflo [4], who again relied on the resolution on singularities in [1].

The present work is inspired by that of Rais, in that we make essential use of the division by analytic functions, and by that of Levy-Bruhl [13], whose work on local solvability on groups of type \( H \) emphasized the simplicity of dealing with solvability in that case. In addition, our results extend some previously obtained by the first author [22] for second order operators.

2. Harmonic analysis on \( G \).

We shall use some calculations given explicitly by Mérivier in [16]. Recall that the Kirillov theory [12] identifies, up to unitary equivalence, \( \hat{G} \) with the orbits of \( g^* \), the linear dual of \( g \), under the action of the co-adjoint representation of \( G \). If \( g \) is of type \( H \), the infinite dimensional representations of \( G \) may be parameterized by \( g^*_2 \setminus \{0\} \). We introduce polar coordinates \( \eta = (\rho, \omega) \) in \( g^*_2 \setminus \{0\} \). Then there exists a local basis \( \{X_1^\omega\} \) of \( g_1 \) with

\[
(2.1) \quad \eta[R_1,S_j] = \rho \delta_{ij}, \quad \eta[R_i,R_j] = \eta[S_i,S_j] = 0,
\]

where \( R_i = X_i^\omega \) if \( i \leq p_1/2 \) and \( S_j = X_j^\omega + (p_1/2) \) if \( j \leq p_1/2 \) and

\[
(2.2) \quad X_i^\omega = \sum Y_{ij}(\omega) Y_j
\]
with \( \gamma_{ij}(\omega) \) analytic in \( \omega \). Indeed, this is a simple application of the Gram-Schmidt orthogonalization process. Then we may write locally (and use a partition of unity in §3)

\[
L = \sum_{|\alpha|=d} b_{\alpha}(\omega) X_{\omega}^{\alpha}
\]

where the \( b_{\alpha}(\omega) \) are analytic.

Now for \( \eta = (\rho, \omega) \) we may define \( \pi(\rho, \omega) \in \hat{G} \) as follows. Let \( T_1, \cdots, T_{p_2} \) be a basis of \( g_2 \) (independent of \( \omega \)), and define global coordinates on \( G \) (for fixed \( \omega \)) by

\[
(r, s, t) \leftrightarrow \exp(\rho^{-1/2} r \cdot R + \rho^{-1/2} s \cdot S + \rho^{-1} t \cdot T)
\]

for \( r, s \) in \( R_{1/2} \) and \( t \) in \( R_2 \), with \( r \cdot R = \sum r_j R_j \), etc. Then put

\[
\pi(\rho, \omega)(r, s, t)f(u) = e^{i(t \cdot \eta + s \cdot r/2 + s \cdot u)}f(u + r).
\]

It follows that

\[
\pi(\rho, \omega)(R_j) = \rho^{1/2} \partial/\partial u_j, \quad \pi(\rho, \omega)(S_j) = i \rho^{1/2} u_j,
\]

and

\[
\pi(\rho, \omega)(T_j) = i \eta_j.
\]

We shall also need the Plancherel formula on \( G \). For this, we define, for \( \varphi \in C_0^\infty(G) \),

\[
\pi(\rho, \omega)(\varphi) = \int_G \varphi(g) \pi(\rho, \omega)(g^{-1}) \, dg,
\]
where \( dg \) is Haar measure (= Lebesgue measure) on \( G \). Then the Plancherel formula for \( G \), in polar coordinates, is (see, e.g. [16])

\[
(2.6) \quad \varphi(0) = c_0 \int_0^\infty \int_{S^{p_2-1}} \text{tr} \left( \pi(\rho, \omega)(\varphi) \right) \left( \det B_{\eta(\rho, \omega)} \right)^{1/2} \rho^{p_2-1} \, \omega \, d\rho \, d\omega,
\]

where \( S^{p_2-1} \) denotes the unit sphere in \( \mathbb{R}^{p_2} \), \( c_0 \) is a constant, and \( \text{tr} \) denotes trace. An easy calculation, given in [16], shows that the distribution kernel \( K(u, v) \) of the operator \( \pi(\rho, \omega)(\varphi) \) is given by

\[
(2.7) \quad K(u, v) = (\det B_{\eta(\rho, \omega)})^{-1/2} (\varphi \ast \exp)^{\Lambda_{2, 3}}(u-v, \frac{u+v}{2}, \eta(\rho, \omega))
\]

where \( \exp \) denotes the exponential map \( \exp: g \rightarrow G \) and \( \Lambda_{2, 3} \) is the partial Fourier Transform in \( s \) and \( t \). Finally, we shall need the general fact that for any \( L \in \mathfrak{u}(g) \), \( \varphi \in C_0^\infty(G) \)

\[
(2.8) \quad \pi(\rho, \omega)(L \varphi) = \pi(\rho, \omega)(L) \pi(\rho, \omega)(\varphi),
\]

which follows easily from (2.5).

The Laplacian in \((t_1, \cdots, t_{p_2})\) will be written \( \Delta_t = -\sum_{j=1}^{p_2} T_j^2 \), and \( \pi(\rho, \omega)(\Delta_t) = \rho^2 \).

For technical reasons involved in the proof of Theorem 1, we shall need to consider a more complicated operator.
Lemma 2.1. Suppose that $L$ satisfies (ii) of Theorem 1. Then the same is true of

$$L' = LL^* M \quad (\sum Y_j^2)^N$$

for any positive integer $N$. Further, if $L'$ satisfies (i) of the theorem, so does $L$. Thus it suffices to prove that (ii) implies (i) for the operator $L'$.

Proof: Suppose $\pi((L')^*) f = 0$, $f \in L^2$. Then $\pi(\sum Y_j^2)^N \pi(LL^*) f = 0$, and we claim that also $\pi(LL^*) f = 0$. If so, then $\pi(L^*) f = 0$ also since the kernel of $\pi(LL^*)$ is in the Schwartz space $S$ [8], so that (ii) holds also for $L'$. To prove that $\pi(LL^*) f = 0$, we show that

$$\int f(u) \pi(LL^*) h(u) du = 0$$

for all $h$ in $S$. Since any such $h$ can be written as $\pi(\sum Y_j^2)^N h'$ with $h'$ in $S$ [13], the result follows. Finally, if $L' = LL''$ is locally solvable, it is clear that $L$ is also.

3. Main outlines of the proof.

In view of Lemma 2.1, we may replace $L$ by $LL^*$ and from now on we shall assume that $L$ is self-adjoint and that $L' = L(\sum Y_j^2)^N$.

The first key step in the proof of Theorem I is the following Proposition, whose proof is given in section 4.
Proposition 3.1. Fix \( \omega_0 \). Then there exists a neighborhood, \( \mathcal{U}_{\omega_0} \) of \( \omega_0 \) in \( S^1 \) and \( \varepsilon > 0 \) and for each \( \omega \) in \( \mathcal{U}_{\omega_0} \) an \( L^2 \) projection \( P_\omega \) with range in \( \text{Dom}(\pi_{(\rho, \omega)}(L)) \) such that

1) The spectrum of \( \pi_{(1, \omega)}(L)|\text{Im}(I-P_\omega) \subset (\varepsilon, \infty) \),

2) The rank of \( P_\omega \) is finite and constant in \( \mathcal{U}_{\omega_0} \),

3) \( V_\omega = \text{Image} \, P_\omega \) varies analytically with \( \omega \in \mathcal{U}_{\omega_0} \); that is, there exists a basis \( \{e_{i}^{\omega}\} \) of \( V_\omega \) with each \( e_{i}^{\omega} \) strongly differentiable of all orders and \( \omega \to (e_{i}^{\omega}, w) \) analytic for all \( w \) in \( L^2(R_{1/2}^1) \), and

4) \( \pi_{(1, \omega)}(L) : V_\omega \to V_\omega \) is given by an analytic matrix \( (A_{ij}(\omega)) \).

In those regions in \( \omega \)-space where \( 0 \) is not in the spectrum of \( \pi_{(1, \omega)}(L) \), \( P_\omega \) is \( 0 \) and we may invert \( \pi_{(1, \omega)}(L) \) boundedly in view of part 1) of the Proposition. Near values of \( \omega \) where \( P_\omega \) is not trivial, we may still invert \( \pi_{(1, \omega)}(L) \) boundedly on \( \ker P_\omega \). Since \( \ker (I-P_\omega) \) is finite dimensional, we are reduced to inverting an analytic matrix on a finite dimensional space. We do this by using the method of Lojasiewicz for inverting analytic functions as in [16]. By means of the Plancherel formula, these pieces will sum to give the desired solution.

Next, we observe that it is sufficient to show the local solvability of the problem.
(3.2) \[ L'\sigma = Zf, \quad f \in C_0^\infty(G) \]

where \( Z \) is a fixed, constant coefficient operator, since such \( Z \) are known to be locally solvable (cf. [10]). To solve (3.2), we shall construct a global distribution solution to the problem

(3.3) \[ L'\sigma = Z\delta, \]

\( \delta \) the Dirac distribution. We shall put \( Z = \Delta_{\frac{k}{t}}^{p_2-1} \) where \( k \) is an integer which will be chosen later.

By the compactness of \( S^{p_2-1} \), we may choose a cover of finitely many open sets of the form \( U_{\omega_0} \), whose existence is asserted by Proposition 3.1 and choose a partition of unity, \( \{\psi_j(\omega)\} \), subordinate to \( \{U_{\omega_j}\} \).

By using (i), the inverse of \( M'_{\omega} = \pi(1,\omega)(L)|\text{Im}(I-P_{\omega}) \) is bounded, and so we may define the linear functional \( \sigma_{j,2} \) on \( C_0^\infty(G) \) by the formula

\[
\sigma_{j,2}(\varphi) = c_0 \int_0^\infty \int_{S^{p_2-1}} \rho^{2k-d-N} \frac{d}{2} \text{tr}( (I-P_{\omega}) \overline{M}_{\omega}^{-1}) \\
\cdot (\pi(1,\omega)(\sum Y_j^2)^{-N})_{\pi(\rho,\omega)}(\varphi) \psi_j(\omega) (\det B_{\eta(\rho,\omega)})^{\frac{1}{2}} \rho^{p_2-1} d\omega \, d\rho.
\]

(3.4)

Then, since \( L^t = \overline{L} \), for \( \chi \in C_0^\infty(G) \),

\[
\sigma_{j,2}(L^t\chi) = c_0 \int_0^\infty \int_{S^{p_2-1}} \rho^{2k} \text{tr}( (I-P_{\omega})\overline{M}_{\omega}^{-1}) \pi(1,\omega)(\sum Y_j^2)^{-N}.
\]
\[(3.5)\quad \pi_1(1,\omega)(L)\pi_2(\rho,\omega)(x)\psi_j(\omega)\rho^{p_2-1} \left(\det B_{\eta(\rho,\omega)}\right)^{\frac{1}{2}} \, d\omega \, d\rho
\]

\[
= c_0 \int_0^\infty \int_{P_2^{-1}} \rho^{2k} \text{tr}((1-P_\omega)\pi(\rho,\omega)(x))\psi_j(\omega)\left(\det B_{\eta(\rho,\omega)}\right)^{\frac{1}{2}} \rho^{p_2-1} \, d\omega \, d\rho.
\]

We will show in the section 6 that each \(\sigma_{j,2}\) is actually a distribution.

Now we must invert \(M_\omega''' = \pi_1(1,\omega)(L)|\text{Im } P_\omega\). According to the proposition, it is the matrix \(A(\omega)\) which we must invert.

By hypothesis \(|A(\omega)| = \det A(\omega) \neq 0\). When \(|A(\omega)| \neq 0\) we have

\[
A(\omega)^{-1} = |A(\omega)|^{-1} B(\omega)
\]

where \(B(\omega)\) is the cofactor matrix of \(A(\omega)\), and hence is analytic even at points where \(|A(\omega)| = 0\). The result of Lojasiewicz on division by analytic functions (discussed in the next section) allows us locally to construct a distribution \(\tilde{A}\) on a neighborhood \(V_{\omega_0}\) of \(\omega_0\) such that \(\tilde{A}(A(\omega)x(\omega)) = \int x(\omega) \, d\omega\) for all \(x \in C^\infty(V_{\omega_0})\).

We then define linear functionals \(\sigma_{j,1}^L\) on \(C^\infty_0(G)\) by

\[
\sigma_{j,1}^L(\phi) = c_0 \int_0^\infty \rho^{2k-d/2} \tilde{A}((\pi(\rho,\omega)(\phi)e^\omega, \pi(\rho,\omega)((\sum_j x_j^2)^{-N})
\]

\[
\cdot \bar{B}(\omega)e^\omega \psi_j(\omega)) \left(\det B_{\eta(\rho,\omega)}\right)^{\frac{1}{2}} \rho^{p_2} \, d\rho
\]

where the inner product (in \(\omega\)) is in \(L^2(R^{p_1/2})\) and this definition is justified by the following
Proposition 3.2. For $k$ sufficiently large, the linear functional defined by (3.6) exists; i.e., for $\varphi \in C_0^\infty(G)$ the function

$$I_\ell(\rho, \omega) = (\pi(\rho, \omega)(\varphi)e^\omega, \pi(\rho, \omega) (\sum_j Y_j^2 - N B(\omega) \psi_j(\omega)e^\omega)_L^2 \rho^{-k}$$

belongs to $C^\infty(S^{p_2-1})$ for each $\rho > 0$ and \[\rho^{2k} A(I_\ell(\rho, \omega)) \in L^1(0, \infty).\]

In section 5 we shall prove Proposition 3.2 and show that the linear functional given by (3.6) defines a distribution on $G$, i.e., that for any compact $K$ in $G$ there exist $C_K$ and $N_K$ such that for all $\varphi \in C_0^\infty(K)$,

$$|\int_0^\infty \rho^{2k-(d/2)+p_2-1} A(I_\ell(\rho, \omega)) d\rho| \leq C_K \sup_{|\alpha| \leq N_K} |D^\alpha \varphi|.$$

Finally, we claim that when $\varphi = L^t x$, $x \in C_0^\infty(G)$,

$$\sigma_{j,1}(L^t x) = c_0 \int_0^\infty \int_{S^{p_2-1}} \rho^{2k} (\pi(\rho, \omega)(x)e^\omega, e^\omega)_L^2 \psi_j(\omega) \rho^{2k} (\pi(\rho, \omega)(x)e^\omega, e^\omega)_L^2 \psi_j(\omega) \cdot (\det B_{\eta(\rho, \omega)})^{1/2} \rho^{p_2-1} d\omega d\rho.$$

Hence by summing over $\ell$ we obtain

$$\sum_{\ell} \sigma_{j,1}(L^t x) = c_0 \int_0^\infty \int_{S^{p_2-1}} \rho^{2k} \operatorname{tr}(\pi(\rho, \omega)(x)\psi_j(\omega)) \rho^{p_2-1}.$$
(3.10) \[ 
\cdot (\det B_{\eta(p,\omega)})^{1/2} \int d\omega \, dp 
\]

To prove (3.9), one merely checks that

\[ 
(\pi_{(p,\omega)}(L^t)) \pi_{(p,\omega)}(X) e^\omega_{\ell}, \pi_{(p,\omega)}(\sum Y_j^2 - \overline{N_{B(\omega)}}) e^\omega_{\ell} 
\]

\[ 
= \rho^{d/2} |A(\omega)| (\pi_{(p,\omega)}(X) e^\omega_{\ell}, e^\omega_{\ell}) .
\]

From (3.5) and (3.10) and the Plancherel formula it is clear that the distribution

(3.11) \[ 
\sigma = \sum_j \sigma_{j,2} + \sum_{j,\ell} \sigma_{j,1}^\ell
\]

satisfies

\[ 
L' \sigma = Z \delta
\]

where \( Z \) is the bi-invariant operator with \( \pi_{(p,\omega)}(Z) = \rho^{2k} \).


We show in this section that the operator-valued function

\[ 
\omega \mapsto \pi_{(1,\omega)}(L)
\]

extends to a complex analytic family of unbounded operators from \( L^2 \) to itself, in the sense of Kato ([11], Ch. VII, Sec. 1.1), whose work follows that of Rellich [19]. This will allow us to define the projections \( P_{\omega} \) introduced in the previous section and thereby prove Proposition 3.1.

The Sobolev space \( H^s \), \( s \) a positive integer, is defined
by

\[ H^s = \{ f \in L^2(\mathbb{R}^{p_1/2}) : x^\alpha D^\beta f \in L^2 \text{ provided } |\alpha| + |\beta| \leq s \} . \]

Since \( \pi(\rho, \omega)(L) \) is a linear combination of monomials \( x^\alpha D^\beta \) with \( |\alpha| + |\beta| \leq d \), it maps \( H^d \) to \( L^2 \) boundedly. On \( L^2 \), however, the operator is unbounded, though clearly closable when initially defined on \( C_0^\infty(\mathbb{R}^{p_1/2}) \).

We denote the closure again by \( \pi(\rho, \omega)(L) \). Denoting the norm in \( H^s \) by \( \| \cdot \|_s \), we have the estimate, for \( v \in C_0^\infty(\mathbb{R}^{p_1/2}) \),

\[ (4.1) \quad \| v \|_d \leq C(\omega)(\| \pi(1, \omega)(L)v \|_0 + \| v \|_0), \quad v \in H^d . \]

This estimate is proved in [8] for \( \omega \) real but persists into the complexes for \( |\text{Im } \omega| \) small by (2.3). Hence the domain of each \( \pi(1, \omega) : L^2 \rightarrow L^2 \) is exactly \( H^d \). Thus \( \{ \pi(1, \omega)(L) \} \) forms a "holomorphic family of Type (A)" in the terminology of [11, VII, §2.1].

Now we may define the \( V_\omega \). For this, we use the following result, see e.g., [11, III, 6.4, Thm 6.17].

**Lemma 4.1.** Let \( T \) be a self-adjoint operator with discrete spectrum consisting entirely of eigenvalues, and let \( \Gamma \) be a closed curve in \( \mathbb{C} \) not meeting the spectrum of \( T \). Then

\[ P = -\frac{1}{2\pi i} \int_{\Gamma} (T-\xi)^{-1} \, d\xi \]

is the orthogonal projection onto the subspace spanned by the eigenvectors corresponding to the eigenvalues of \( T \) enclosed by \( \Gamma \).
In our context, \( T \) will be \( \pi_{(1, \omega)}(L) \) for certain values of \( \omega \). Let 0 be an eigenvalue of \( \pi_{(1, \omega_0)}(L) \). Since the operators \( \pi_{(1, \omega)}(L) \) form a holomorphic family, we may choose a smooth closed curve \( \Gamma \) which encloses 0 alone among points in the spectrum of \( \pi_{(1, \omega_0)}(L) \) and meets the spectrum of no \( \pi_{(1, \omega)}(L) \) for \( \omega \) close enough to \( \omega_0 \), say \( \omega \in U_{\omega_0} \). Let \( V_\omega \) denote the image of the projection \( P_\omega \) where

\[
P_\omega = -\frac{1}{2\pi i} \int_\Gamma (\pi_{(1, \omega)}(L) - \xi)^{-1} \, d\xi.
\]

Then \( \pi_{(1, \omega)}(L) P_\omega \) is the restriction of \( \pi_{(1, \omega)}(L) \) to \( V_\omega \).

The hypothesis (ii) of Theorem I implies that \( \pi_{(1, \omega)}(L) \) is an invertible operator for most \( \omega \). The properties i) through iv) of Proposition 3.1 now follow from the properties of holomorphic families [11, VII, §7.3, Theorem 1.7].

In view of the definition of \( H^d \), it is elementary to show that the injection of \( H^d \) into \( L^2 \) is compact for \( d \geq 1 \). We now prove

**Proposition 4.2.** For any real \( \omega \), \( \pi_{(1, \omega)}(L) \) has discrete spectrum consisting entirely of eigenvalues with finite multiplicity.

**Proof:** Since \( \pi_{(1, \omega)}(L) \) is self-adjoint, its spectrum is non-negative and thus \( \ker (\pi_{(1, \omega)}(L) + \gamma) = 0 \) for any \( \gamma > 0 \). Now condition (1.4) implies the estimate
\begin{equation}
\|v\|_d \leq C \left( \|\pi_{1,\omega}(L)v\|_0 + \|v\|_0 \right)
\end{equation}

for all \( v \) in \( C_0^\infty(\mathbb{R}^{p_1/2}) \) (see [8]). From (4.2) then
\[ \|v\|_d \leq C_{\gamma} \left( \|\pi_{1,\omega}(L) + \gamma\|_0 + \|v\|_0 \right) \]

for each fixed \( \gamma \), and hence, using Proposition 4.1,
\[ \|v\|_d \leq C_{\gamma}^{''} \left( \|\pi_{1,\omega}(L) + \gamma\|_0 + \|Q_{\gamma}v\|_0 \right) \]

for \( \gamma \) fixed and all \( v \) in \( C_0^\infty(\mathbb{R}^{p_1/2}) \), where \( Q_{\gamma} \) denotes the \( L^2 \) projection onto \( \ker (\pi_{1,\omega}(L) + \gamma) \). If \( \gamma > 0 \), \( Q_{\gamma} = 0 \) and thus \( (\pi_{1,\omega}(L) + \gamma)^{-1} \) exists and is compact.

Now a standard well-known result (see, e.g., [11, III, 6.8, Theorem 6.29]) implies Proposition 4.2.

5. Application of the method of the division of distributions.

To prove Proposition 3.2 we write
\[ J_\xi(\rho, \omega) = \rho^{-N}(\pi_{(\rho, \omega)}(\varphi)e_{\omega}^\omega, f_{\xi}^\omega) \]

with \( f_{\xi}^\omega \) independent of \( \rho \), analytic in \( \omega \), and in \( L^2 \) together with its \( \omega \)-derivatives. Thus
\begin{equation}
J_\xi(\rho, \omega) = \rho^{-N}g_{\xi}^\varphi(\rho, \omega)
\end{equation}

where, by (2.7),
\begin{equation}
g_{\xi}^\varphi(\rho, \omega) = \int \int (\varphi \exp)^2, 3, (u-v, \frac{u+v}{2}, \eta(\rho, \omega))e_{\xi}^\omega(u)f_{\xi}^\omega(v)dudv.
\end{equation}
Lemma 5.1. \( g^\varphi_{\ell}(\rho, \omega) \) is in \( C^\infty(\mathbb{R}^+ \times S^{p_2 - 1}) \) and for any compact set \( K \) in \( G \), \( k' \) and \( \alpha \) there exist \( N = N_{K, k', \alpha} \) and \( C = C_{N, k', \alpha} \) such that for all \( \varphi \) in \( C^\infty_0(K) \)

\[
(5.3) \quad \sup_{\omega} |D^\alpha_\omega g^\varphi_{\ell}(\rho, \omega)| \leq C(1 + |\rho|^2)^{k'} \sup_{|\xi| \leq N} |D^\beta_{r, s, t} \varphi|.
\]

**Proof:** Since \( \varphi \in C^\infty_0(G) \), \( (\varphi \cdot \exp)^{\wedge 2, 3} \in \mathfrak{g} \) and hence \( g^\varphi_{\ell}(\rho, \omega) = h^\varphi_{\ell}(\omega) \) is infinitely differentiable in \( \omega \). But \( D^\alpha_\omega \) is a sum of vector fields (in the \( \partial/\partial \eta_j \)) with coefficients in \( C^\infty(\mathbb{R}^+ \times S^{p_2 - 1}) \), homogeneous of degree 1 in \( \rho \). Thus we may differentiate under the integral sign in (5.2) arbitrarily often in \( \rho, \omega \) which proves that \( g^\varphi_{\ell}(\rho, \omega) \) is \( C^\infty \) in \( \rho \neq 0 \) and \( \omega \).

Since for any \( j \), \( \rho^{2j} D^\alpha_\omega g^\varphi_{\ell}(\rho, \omega) = cD^\alpha_\omega g^\varphi_{\ell}(\rho, \omega) \) with \( |c| = 1 \), we may assume \( k' \gg 0 \).

In applying \( D^\alpha_\omega \) to \( g^\varphi_{\ell}(\rho, \omega) \) in (5.2), \( \omega \)-derivatives which fall on \( e^\omega_\ell(u) \) and \( f^\omega_\ell(v) \) yield functions still in \( L^2 \) together with their derivatives of any order:

\[
\|D^\beta_\omega e^\omega_\ell(u)\|_{L^2} \leq C_{\beta, \ell}
\]

(5.4)

\[
\|D^\beta_\omega f^\omega_\ell(v)\|_{L^2} \leq C_{\beta, \ell}
\]

in view of the form of \( f^\omega_\ell(v) \). The operator \( \tilde{\varphi}_{\gamma} \) with kernel
\[ K_\gamma(u, v, \eta(\rho, \omega)) = D_\omega^\gamma (\varphi \circ \exp)^{\Lambda_2, 3} (u-v, \frac{u+v}{2}, \eta(\rho, \omega)) : \]

\[ \varphi \circ \exp(v, \eta(\rho, \omega)) = \int_{p_{1/2}} K_\gamma(u, v, \eta(\rho, \omega)) \, e(u) \, du \]

is bounded in \( L^2 \) with norm less than a constant times

\[(5.6) \quad (\sup_v \int |K_\gamma(u, v, \eta(\rho, \omega))| \, du)^{1/2} \leq (\sup_u \int |K_\gamma(u, v, \eta(\rho, \omega))| \, dv)^{1/2} \]

(Young's inequality). Since \( D_\omega \) is homogeneous in \( \rho \) of degree 1 and \( \varphi \in \mathcal{S} \), (5.6) is bounded by \((1 + |\rho|)^{\gamma}\) times a Schwartz seminorm of \( \varphi \). The support of \( \varphi \), however, is contained in a fixed compact subset of \( G \), and an application of the Schwartz inequality in (5.2) yields (5.3). This proves Lemma 5.1.

Finally, it is now easy to see that \( \sigma^\ell_{j, 1} \) is actually a distribution. Since \( \tilde{A} \) is a distribution, for any \( k' \)

\[ |\tilde{A}(J(\rho, \omega))| \leq \sup_{|\alpha| \leq N_A} |D_\omega^{\alpha} J(\rho, \omega)| \]

\[ \leq C_{k'} \sup_r, s, t |D_{r, s, t}^{\alpha} \varphi| (1 + |\rho|)^{-2k'} |\alpha| \leq N_{\tilde{A}} \]

and so for all \( \varphi \in C_0^\infty(K) \),

\[ \| \rho^{2k-(d/2)+p_2-1} \tilde{A}(J(\rho, \omega)) \|_{L^1(R^+)} \leq C_K \sup_{|\alpha| \leq N_K} |D_\alpha \varphi| . \]
6. Proof that the $\sigma_{j,2}$ are distributions.

Following [21] we write

$$
(6.1) \quad \sigma_{j,2}(\phi) = c_0 \int_0^1 h(\rho) \, d\rho + c_0 \int_1^{\infty} h(\rho) \, d\rho .
$$

We first bound the second integral on the right as follows. By the Schwartz inequality, for any $n$,

$$
(6.2) \quad \left| \int_1^{\infty} h(\rho) \, d\rho \right| \leq C \left\{ \int_1^{\infty} \rho^{2(2k-(d/2)-N+p_2-1-n)} \, d\rho \right\}^{1/2} \times \left\{ \int_1^{\infty} \rho^{2n} \, d\rho \right\}^{1/2}.
$$

The first factor on the right of (6.2) is finite if $n \gg 0$. To bound the second factor we use the generalized Schwartz inequality, $|\text{tr}(AB)|^2 \leq \text{tr}(AA^*) \text{tr}(BB^*)$, and follow [21]. Then

$$
(6.3) \quad \left| \text{tr} \left( (I-P_\omega) \overline{M_\omega}^{-1} \pi_{(1,\omega)} \left( \sum Y_j^2 \right)^{-N} \pi_{(\rho,\omega)}(\phi) \right) \right|^2
$$

$$
\leq \text{tr} \left( \pi_{(\rho,\omega)}(\phi) (\pi_{(\rho,\omega)}(\phi))^* \right) \cdot \text{tr} (B_\omega B_\omega^*)
$$

where $B_\omega = (I-P_\omega) M_\omega \pi_{(1,\omega)} \left( \sum Y_j^2 \right)^{-N}.$
Lemma 6.1. \( \text{tr} (B_\omega B_\omega^*) \leq C, \) independent of \( \omega \).

**Proof:** Since \( \| (I-P_\omega) M_{\omega}^{-1} \| _{L^2} \leq C_1 \), independent of \( \omega \), and \( \sum Y_j^2 \) is self-adjoint,

\[
(6.4) \quad \text{tr} (B_\omega B_\omega^*) \leq C_1^2 \text{tr} (\pi(1, \omega)(\sum Y_j^2)^{-N/2})
\]

The eigenvalues of \( \pi(1, \omega)(\sum Y_j^2) \) are \( \lambda_\alpha = \sum_{j=1}^{\lceil p_1/2 \rceil} \tilde{\omega}_j (2\alpha_j + 1) \); \( \alpha = (\alpha_1, \ldots, \alpha_{\lceil p_1/2 \rceil}) \), each \( \alpha_j \) a non-negative integer (see, e.g., [20]), where \( \tilde{\omega}_j > 0 \) and \( \pm \tilde{\omega}_j \) are the eigenvalues of the matrix \( \eta([Y_j, Y_k]) \), \( \eta = \eta(1, \omega) \). Hence

\[
(6.5) \quad \text{tr} ((\pi(1, \omega)(\sum Y_j^2)^{-N/2})) = \sum_{\alpha} |\lambda_\alpha|^{-N} \leq (C_2/2) \sum_{\alpha} |\alpha|^{-2N},
\]

\[
|\alpha| = \sum_{1}^{p_1/2} \alpha_j, \quad \text{since} \quad |\lambda_\alpha| \geq 2C_2 |\alpha| \quad \text{for}
\]

\[
C_2 = \min_{\omega \in S} \min_{1 \leq j \leq p_1/2} |\tilde{\omega}_j|.
\]

Now the Lemma follows from (6.4) and (6.5), provided \( N \) is sufficiently large.

Now by (6.3) and Lemma 6.1, the second integral on the right hand side of (6.2) is bounded by

\[
(6.6) \quad C \int_{0}^{\infty} \int_{S^{p_2-1}} \text{tr} (\pi(\rho, \omega)(\varphi) \pi(\rho, \omega)(\varphi)^*) |\det B(\eta(\rho, \omega))| \cdot
\]

\[
\cdot |\psi_j(\omega)|^2 \rho^{2n} d\omega d\rho
\]

\[
\leq C' \| \Delta_t^{n/2} \varphi \|^2,
\]
provided \( n \) is chosen to be even, the last inequality following from the Plancherel formula since \( \pi_{(\rho, \omega)}(\Delta_{t}^{n/2} \varphi) = \rho^{n} \pi_{(\rho, \omega)}(\varphi) \). Hence

\[
\int_{1}^{\infty} h(\rho) \, d\rho \leq C' \| \Delta_{t}^{n/2} \varphi \|
\]

The estimate for \( \int_{0}^{1} h(\rho) \, d\rho \) may be obtained similarly, beginning with the estimate

\[
| \int_{0}^{1} h(\rho) \, d\rho | \leq C \left\{ \int_{0}^{1} \rho^{2(2k-(d/2)-N)} \, d\rho \right\}^{1/2}
\]

\[
\times \left\{ \int_{0}^{1} | \text{tr} \left( (I-P_{\omega}) M_{\omega}^{-1} \right) (\pi_{(1, \omega)}(\sum_{j} Y_{j}^{2})^{-N} \pi_{(\rho, \omega)}(\varphi)) |^{2} \right\}
\]

\[
\times \left| \det B_{\eta(\rho, \omega)} \right| |\psi_{j}(\omega)|^{2} \rho^{2(p_{2}-1)} d\omega d\rho^{1/2}
\]

The second integral on the right may be bounded as before. For the first, recall that \( N \) has been chosen independent of \( k \); we may then choose \( k \) sufficiently large so that \( 2k-(d/2)-N \geq 0 \), which guarantees the convergence of the first integral. This completes the proof that each \( \sigma_{j,2} \) is a distribution. We remark that it is only for this result that it was necessary to replace \( L \) by \( L(\sum_{j} Y_{j}^{2})^{N} \), in order to make use of (6.5).
7. **Proof of Theorem II**.

Given \( L = \sum a_{ij} Z_i Z_j \), where \( \{Z_i\} \) forms a basis of \( g_1 \) with \( (a_{ij}) \) positive definite, there is a linear change of basis \( Z_i \rightarrow Y_i \) for \( g_1 \) such that

\[
L = \sum Y_i^2 + i \sum c_j T_j
\]

for some constants \( c_j \), where the \( T_j \) form a basis of \( g_2 \).

By direct calculation (see e.g. [13] or [23]) it can be shown that the eigenvalues of \( \pi_\eta(L) \) are all of the form

\[
m_\alpha(\eta) = -\sum_{j=1}^{p_1/2} \rho_j (2\alpha_j+1) - \sum c_j \eta_j
\]

where \( \alpha = (\alpha_1, \cdots, \alpha_{p_1/2}) \), \( \alpha_j \) a non-negative integer, and the \( \rho_j \), all positive, are \( \pm i \) times the eigenvalues of \( B_\eta(Y_i, Y_j) = \eta([Y_i, Y_j]) \). In light of Theorem I (which in this case had been obtained previously by Lévy-Bruhl [13]), it suffices to show the following.

**Proposition 7.1.** For any constants \( c_j \), and any fixed \( \alpha \), if \( p_2 > 1 \), \( m_\alpha(\eta) \) does not vanish in any open set on the unit sphere in \( \mathbb{R}^{p_2} \).

**Remark.** Since the \( \rho_j \) are all bounded away from zero on the sphere, there are only finitely many \( \alpha \) for which there exists non-zero \( \eta \) with \( m_\alpha(\eta) = 0 \).
Proof of the Proposition: Suppose that there exist \( c = (c_j) \), \( \omega \) such that \( m_\omega(\omega) \) is identically zero in an open neighborhood \( U_{\omega_0} \) of \( \omega_0 \) in \( S^{p_2-1} \). Let \( \omega_1 \) be a point such that \( c \cdot \omega_1 > 0 \). Then \( m_\beta(\omega_1) \leq -(c \cdot \omega_1) \) for all \( \beta \). Since \( p_2 > 1 \), there exists a path \( \gamma(t) \) on \( S^{p_2-1} \) connecting the points \( \omega_0 \) and \( \omega_1 \), i.e., \( \gamma(0) = \omega_0 \) and \( \gamma(1) = \omega_1 \). Let \( I \) be the set of all \( t \) in \( (0, 1] \) such that for \( 0 < s \leq t \), there exists \( \alpha \) and \( \varepsilon \) such that \( m_\alpha(\gamma(u)) = 0 \) for \( u \in [s-\varepsilon, s] \). By hypothesis, \( I \) is non empty and bounded.

Let \( \delta \) denote the least upper bound for \( I \). \( \delta \) cannot be equal to 1, since \( \pi(1, \gamma(t))(L) \) is invertible for \( t \) sufficiently close to 1 (since \( \pi(1, \omega_1)(L) \) is invertible). Now if \( \delta < 1 \), consider the point \( \gamma(\delta) \). From Proposition 3.1, applied at \( \gamma(\delta) \), \( \det A_{ij}(\omega) \) is analytic near \( \gamma(\delta) \), and by the construction of \( \delta \), \( \det (A_{ij}(\gamma(t)) = 0 \) for \( \delta - \varepsilon_1 < t < \delta \), for some \( \varepsilon_1 > 0 \). But then this determinant must vanish identically for \( \delta \leq t \leq \delta + \varepsilon_2 \), for some \( \varepsilon_2 > 0 \), contradicting the hypothesis that \( \delta \) is the least upper bound for \( I \).

We are grateful to the referee for many suggestions which have improved the exposition, especially in the proof of the above proposition.

ACKNOWLEDGMENTS

The work of both authors was partially supported by grants from the N.S.F. The first author was partially supported by a fellowship from the Alfred P. Sloan Foundation.
REFERENCES


[23] __________ and E. M. Stein, Hypoelliptic

[24] F. Rouvière, Sur la résolubilité locale des opérateurs
bi-invariants, Ann. Scuola Norm. Sup. Pisa, Ser. IV,

Received September 1980