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INVERSION OF ANALYTIC MATRICES AND LOCAL SOLVABILITY OF SOME INVARIANT DIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS

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1. INTRODUCTION

We shall prove a necessary and sufficient condition for the local solvability of some left invariant operators on a class of 2-step nilpotent groups including the Heisenberg groups. Our method involves the use of the Plancherel formula and the interpretation of the inverse of an analytic matrix as a matrix of distributions. Here we will say that a partial differential operator L is locally solvable at x_0 provided there is a neighborhood U_{x_0} of x_0 such that

(1.1) $$L\sigma = f$$

has a solution $\sigma \in C^\infty(U_{x_0})$ for every $f \in C_0^\infty(U_{x_0})$.

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ with $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $\mathfrak{g}_2$ contained in the center of $\mathfrak{g}$. Following [17] we shall say that $\mathfrak{g}$ is of type H if for every non-zero linear functional η on $\mathfrak{g}_2$ and basis $Y_1, \cdots, Y_{p_1}$ of $\mathfrak{g}_1$,

$$\det B_\eta = \det \eta\,([Y_i, Y_j])_{1 \le i, j \le p_1} \ne 0 \quad . \tag{1.2}$$

Let G be the simply connected Lie group corresponding to $\mathfrak{g}$ and $\mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra, which we identify with the set of all left invariant differential operators on G. The dilations δ_s defined by $\delta_s|_{\mathfrak{g}_1} = s \cdot I$ and $\delta_s|_{\mathfrak{g}_2} = s^2 \cdot I$, $s > 0$, form a family of automorphisms which extend to a family of dilations, again denoted δ_s, on $\mathfrak{U}(\mathfrak{g})$. An element $L \in \mathfrak{U}(\mathfrak{g})$ is homogeneous of degree d if $\delta_s(L) = s^d L$. Any such L can be written in the form

$$L = \sum_{|\alpha| = d} c_\alpha Y_{I_\alpha} \tag{1.3}$$

where Y_{I_α} can denote any $Y_{i_1} Y_{i_2} \cdots Y_{i_{|\alpha|}}$ and the c_α are constants. We shall assume that L is "elliptic in the generating directions", i.e., that

$$\sum_{|\alpha| = d} c_\alpha \xi^\alpha \ne 0, \quad 0 \ne \xi \in R^{p_1} \quad . \tag{1.4}$$

Let $\hat{G}$ be the set of all irreducible unitary representations of G. If $\pi \in \hat{G}$ we again denote by π the representation of $\mathfrak{U}(\mathfrak{g})$ on the space of C^∞ vectors. For $\lambda \in \mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, let π_λ be the corresponding representation. We may now state our main results.

Theorem I. *Let* $L \in \mathfrak{U}(\mathfrak{g})$, $\mathfrak{g}$ *of type* H, *be homogeneous and satisfy* (1.4). *Then the following are equivalent*:

i) L *is locally solvable*.

ii) *There is no open set* $U \subset \mathfrak{g}^*$ *such that* $\pi_\lambda(L^t)$ *has a non-trivial kernel (in* L^2*) for all* $\lambda \in U$.

iii) $\ker L^t \cap L^2(G) = \{0\}$.

Theorem II. *If* $L \in \mathfrak{U}(\mathfrak{g})$, $\mathfrak{g}$ *of type* H *with* $p_2 > 1$, *is homogeneous of degree* 2 *and satisfies* (1.4), *then* ker L *and* $\ker L^t$ *are trivial and thus* L *is locally solvable*.

Remark 1. The implications i) implies iii) and iii) implies ii) of Theorem 1 are contained in Corwin-Rothschild [3].

Remark 2. Theorem 1 was known for the Heisenberg group [6].

Remark 3. Theorem 2 improves the result of Lévy-Bruhl [13], who showed that i) and ii) are equivalent when deg (L) = 2. Results for other 2-step groups with deg (L) = 2 had previously been obtained by the first author [22].

The first example of an unsolvable linear differential operator, given by H. Lewy [14], may also be interpreted as a left invariant operator on the three dimensional Heisenberg algebra, homogeneous of degree 1. A representation-theoretic condition for hypoellipticity and local solvability for a related class of operators on the Heisenberg group was given by Folland-Stein [5]. Rockland [20] then proved that a homogeneous left invariant operator L on the Heisenberg group is hypoelliptic if and only if $\pi(L)$ has trivial kernel for every irreducible, nontrivial unitary representation π of G, and conjectured the validity of the statement for all nilpotent Lie groups with dilations. The general case was later proved by Helffer and Nourrigat [8]. Rockland also conjectured that the existence of a right inverse for $\pi(L)$ would give local solvability. This was proved by the first author [21] and Lion [15], independently, and generalized by Corwin [2].

For operators on the Heisenberg group which are elliptic in the generating directions, a very detailed analysis was made by Geller [6], who obtained as a side result the necessary and sufficient conditions of Theorem 1. (See also Greiner, Kohn and Stein [7].)

The most general results for local solvability on Lie groups have been obtained for bi-invariant operators, not necessarily homogeneous. The first result was proved by Raïs [17] (see also [24]), who used the method of division by analytic functions as in Atiyah [1] and the Plancherel formula to prove that bi-invariant differential operators on nilpotent Lie groups are locally solvable. The general result for any Lie group was later obtained by Duflo [4], who again relied on the resolution on singularities in [1].

The present work is inspired by that of Raïs, in that we make essential use of the division by analytic functions, and by that of Lévy-Bruhl [13], whose work on local solvability on groups of type H emphasized the simplicity of dealing with solvability in that case. In addition, our results extend some previously obtained by the first author [22] for second order operators.

2. Harmonic analysis on G.

We shall use some calculations given explicitly by Métivier in [16]. Recall that the Kirillov theory [12] identifies, up to unitary equivalence, $\hat{G}$ with the orbits of $\mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, under the action of the co-adjoint representation of G. If $\mathfrak{g}$ is of type H, the infinite dimensional representations of G may be parameterized by $\mathfrak{g}_2^* - \{0\}$. We introduce polar coordinates $\eta = (\rho, \omega)$ in $\mathfrak{g}_2^* - \{0\}$. Then there exists a local basis $\{X_i^\omega\}$ of $\mathfrak{g}_1$ with

(2.1) $$\eta[R_i, S_j] = \rho\,\delta_{ij}\ , \quad \eta[R_i, R_j] = \eta[S_i, S_j] = 0\ ,$$

where $R_i = X_i^\omega$ if $i \le p_1/2$ and $S_j = X^\omega_{j+(p_1/2)}$ if $j \le p_1/2$ and

(2.2) $$X_i^\omega = \sum \gamma_{ij}(\omega)\, Y_j$$

with $\gamma_{ij}(\omega)$ analytic in ω. Indeed, this is a simple application of the Gram-Schmidt orthogonalization process. Then we may write locally (and use a partition of unity in §3)

(2.3)
$$L = \sum_{|\alpha| = d} b_\alpha(\omega) X^\omega_{I_\alpha}$$

where the $b_\alpha(\omega)$ are analytic.

Now for $\eta = (\rho, \omega)$ we may define $\pi_{(\rho,\omega)} \in \hat{G}$ as follows. Let $T_1, \cdots, T_{p_2}$ be a basis of $\mathfrak{g}_2$ (independent of ω), and define global coordinates on G (for fixed ω) by

$$(r, s, t) \longleftrightarrow \exp(\rho^{-1/2}\, r\cdot R + \rho^{-1/2}\, s\cdot S + \rho^{-1}\, t\cdot T) ,$$

for r, s in $R^{p_1/2}$ and t in R^{p_2}, with $r\cdot R = \sum r_j R_j$, etc. Then put

(2.4)
$$\pi_{(\rho,\omega)}(r, s, t) f(u) = e^{i(t\cdot\eta + s\cdot r/2 + s\cdot u)} f(u+r) .$$

It follows that

$$\pi_{(\rho,\omega)}(R_j) = \rho^{1/2}\, \partial/\partial u_j , \quad \pi_{(\rho,\omega)}(S_j) = i\rho^{1/2} u_j ,$$

and

$$\pi_{(\rho,\omega)}(T_j) = i\eta_j .$$

We shall also need the Plancherel formula on G. For this, we define, for $\varphi \in C_0^\infty(G)$,

(2.5)
$$\pi_{(\rho,\omega)}(\varphi) = \int_G \varphi(g)\, \pi_{(\rho,\omega)}(g^{-1})\, dg ,$$

where dg is Haar measure (= Lebesgue measure) on G. Then the Plancherel formula for G, in polar coordinates, is (see, e.g. [16])

$$\varphi(0) = c_0 \int_0^\infty \int_{S^{p_2-1}} \operatorname{tr}(\pi_{(\rho,\omega)}(\varphi))(\det B_{\eta(\rho,\omega)})^{1/2} \rho^{p_2-1}\, d\omega\, d\rho, \tag{2.6}$$

where S^{p_2-1} denotes the unit sphere in R^{p_2}, c_0 is a constant, and tr denotes trace. An easy calculation, given in [16], shows that the distribution kernel $K(u,v)$ of the operator $\pi_{(\rho,\omega)}(\varphi)$ is given by

$$K(u,v) = (\det B_{\eta(\rho,\omega)})^{-1/2} (\varphi\cdot\exp)^{\wedge_{2,3}}(u-v, \tfrac{u+v}{2}, \eta(\rho,\omega)) \tag{2.7}$$

where exp denotes the exponential map $\exp: \mathfrak{g} \to G$ and $\wedge_{2,3}$ is the partial Fourier Transform in s and t. Finally, we shall need the general fact that for any $L \in \mathfrak{U}(\mathfrak{g})$, $\varphi \in C_0^\infty(G)$

$$\pi_{(\rho,\omega)}(L\varphi) = \pi_{(\rho,\omega)}(L)\pi_{(\rho,\omega)}(\varphi), \tag{2.8}$$

which follows easily from (2.5).

The Laplacian in $(t_1, \cdots, t_{p_2})$ will be written $\Delta_t = -\sum_1^{p_2} T_j^2$, and $\pi_{(\rho,\omega)}(\Delta_t) = \rho^2$.

For technical reasons involved in the proof of Theorem 1, we shall need to consider a more complicated operator.

Lemma 2.1. *Suppose that* L *satisfies* (ii) *of Theorem 1. Then the same is true of*

$$L' = LL^{*\,M} (\sum Y_j^2)^N$$

for any positive integer N . *Further, if* L' *satisfies* (i) *of the theorem, so does* L . *Thus it suffices to prove that* (ii) *implies* (i) *for the operator* L' .

Proof: Suppose $\pi((L')^*)f = 0$, $f \in L^2$. Then $\pi(\sum Y_j^2)^N \pi(LL^*) f = 0$, and we claim that also $\pi(LL^*)f = 0$. If so, then $\pi(L^*)f = 0$ also since the kernel of $\pi(LL^*)$ is in the Schwartz space $\mathcal{S}$ [8], so that (ii) holds also for L' . To prove that $\pi(LL^*) f = 0$, we show that $\int f(u)\pi(LL^*) h(u)\, du = 0$ for all h in $\mathcal{S}$. Since any such h can be written as $\pi(\sum Y_j^2)^N h'$ with h in $\mathcal{S}$ [13], the result follows. Finally, if $L' = LL''$ is locally solvable, it is clear that L is also.

3. *Main outlines of the proof.*

In view of Lemma 2.1, we may replace L by LL^* and from now on we shall assume that L is self-adjoint and that $L' = L(\sum Y_j^2)^N$.

The first key step in the proof of Theorem I is the following Proposition, whose proof is given in section 4.

Proposition 3.1. Fix ω_0. Then there exists a neighborhood, U_{ω_0} of ω_0 in S^{p_2-1} and $\varepsilon > 0$ and for each ω in U_{ω_0} an L^2 projection P_ω with range in $\mathrm{Dom}(\pi_{(\rho,\omega)}(L))$ such that

i) The spectrum of $\pi_{(1,\omega)}(L)|\mathrm{Im}(I-P_\omega) \subset (\varepsilon,\infty)$,

ii) The rank of P_ω is finite and constant in $\mathcal{U}_{\omega_0}$,

iii) $V_\omega = \mathrm{Image}\ P_\omega$ varies analytically with $\omega \in U_{\omega_0}$; that is, there exists a basis $\{e_i^\omega\}$ of V_ω with each e_i^ω strongly differentiable of all orders and $\omega \to (e_i^\omega, w)$ analytic for all w in $L^2(R^{p_1/2})$, and

iv) $\pi_{(1,\omega)}(L) : V_\omega \to V_\omega$ is given by an analytic matrix $(A_{ij}(\omega))$.

In those regions in ω-space where 0 is not in the spectrum of $\pi_{(1,\omega)}(L)$, P_ω is 0 and we may invert $\pi_{(1,\omega)}(L)$ boundedly in view of part i) of the Proposition. Near values of ω where P_ω is not trivial, we may still invert $\pi_{(1,\omega)}(L)$ boundedly on $\ker P_\omega$. Since $\ker(I-P_\omega)$ is finite dimensional, we are reduced to inverting an analytic matrix on a finite dimensional space. We do this by using the method of Lojasiewicz for inverting analytic functions as in [16]. By means of the Plancherel formula, these pieces will sum to give the desired solution.

Next, we observe that it is sufficient to show the local solvability of the problem

(3.2) $$L'\sigma = Zf \quad , \qquad f \in C_0^\infty(G)$$

where Z is a fixed, constant coefficient operator, since such Z are known to be locally solvable (cf. [10]). To solve (3.2), we shall construct a global distribution solution to the problem

(3.3) $$L'\sigma = Z\delta \ ,$$

δ the Dirac distribution. We shall put $Z = \Delta_t^k$ where k is an integer which will be chosen later.

By the compactness of S^{p_2-1}, we may choose a cover of finitely many open sets of the form U_{ω_0} whose existence is asserted by Proposition 3.1 and choose a partition of unity, $\{\psi_j(\omega)\}$, subordinate to $\{U_{\omega_j}\}$.

By using (i), the inverse of $M'_\omega = \pi_{(1,\omega)}(L)|\mathrm{Im}(I-P_\omega)$ is bounded, and so we may define the linear functional $\sigma_{j,2}$ on $C_0^\infty(G)$ by the formula

(3.4) $$\sigma_{j,2}(\varphi) = c_0\int_0^\infty\int_{S^{p_2-1}} \rho^{2k-\frac{d}{2}-N}\,\mathrm{tr}\,((I-P_\omega)\,\overline{M}_\omega'^{-1} \cdot (\pi_{(1,\omega)}(\sum Y_j^2)^{-N})\pi_{(\rho,\omega)}(\varphi))\psi_j(\omega)\,(\det B_{\eta(\rho,\omega)})^{\frac{1}{2}}\rho^{p_2-1}\,d\omega\,d\rho \ .$$

Then, since $L^t = \overline{L}$, for $\chi \in C_0^\infty(G)$,

$$\sigma_{j,2}(L'^t\chi) = c_0\int_0^\infty\int_{S^{p_2-1}} \rho^{2k}\,\mathrm{tr}\,((I-P_\omega)\overline{M}_\omega'^{-1}\pi_{(1,\omega)}(\sum Y_j^2)^{-N} \cdot$$

(3.5) $$\cdot\ \pi_{(1,\omega)}(L')^t\pi_{(\rho,\omega)}(X))\psi_j(\omega)\rho^{p_2-1}(\det B_{\eta(\rho,\omega)})^{\frac{1}{2}}\,d\omega\,d\rho$$

$$= c_0\int_0^\infty\int_{S^{p_2-1}}\rho^{2k}\,\mathrm{tr}((I-P_\omega)\pi_{(\rho,\omega)}(X))\psi_j(\omega)(\det B_{\eta(\rho,\omega)})^{\frac{1}{2}}\rho^{p_2-1}\,d\omega\,d\rho.$$

We will show in the section 6 that each $\sigma_{j,2}$ is actually a distribution.

Now we must invert $M''_\omega = \pi_{(1,\omega)}(L)|_{\mathrm{Im}\,P_\omega}$. According to the proposition, it is the matrix $A(\omega)$ which we must invert. By hypothesis $|A(\omega)| = \det A(\omega) \not\equiv 0$. When $|A(\omega)| \neq 0$ we have

$$A(\omega)^{-1} = |A(\omega)|^{-1}B(\omega)$$

where $B(\omega)$ is the cofactor matrix of $A(\omega)$, and hence is analytic even at points where $|A(\omega)| = 0$. The result of Lojasiewicz on division by analytic functions (discussed in the next section) allows us locally to construct a distribution $\tilde{A}$ on a neighborhood V_{ω_0} of ω_0 such that $\tilde{A}(A(\omega)\chi(\omega)) = \int \chi(\omega)\,d\omega$ for all $\chi \in C^\infty(V_{\omega_0})$ We then define linear functionals $\sigma^\ell_{j,1}$ on $C_0^\infty(G)$ by

(3.6) $$\sigma^\ell_{j,1}(\varphi) = c_0\int_0^\infty \rho^{2k-\frac{d}{2}}\tilde{A}((\pi_{(\rho,\omega)}(\varphi)e^\omega_\ell,\ \pi_{(\rho,\omega)}((\textstyle\sum Y_j^2)^{-N})\ \cdot$$
$$\cdot\ \overline{B}(\omega)e^\omega_\ell\psi_j(\omega))(\det B_{\eta(\rho,\omega)})^{\frac{1}{2}}\rho^{p_2}\,d\rho$$

where the inner product (in ω) is in $L^2(R^{p_1/2})$ and this definition is justified by the following

Proposition 3.2. For k sufficiently large, the linear functional defined by (3.6) exists; i.e., for $\varphi \in C_0^\infty(G)$ the function

$$(3.7)\qquad J_\ell(\rho,\omega) = (\pi_{(\rho,\omega)}(\varphi)e_\ell^\omega,\ \pi_{(\rho,\omega)}(\sum Y_j^2)^{-N}\bar{B}(\omega)\psi_j(\omega)e_\ell^\omega)_{L^2}$$

belongs to $C^\infty(S^{p_2-1})$ for each $\rho > 0$ and $\rho^{2k}\tilde{A}(J_\ell(\rho,\omega)) \in L^1(0,\infty)$.

In section 5 we shall prove Proposition 3.2 and show that the linear functional given by (3.6) defines a distribution on G, i.e., that for any compact K in G there exist C_K and N_K such that for all $\varphi \in C_0^\infty(K)$,

$$(3.8)\qquad \left|\int_0^\infty \rho^{2k-(d/2)+p_2-1}\tilde{A}(J_\ell(\rho,\omega))d\rho\right| \le C_K \sup_{|\alpha|\le N_K} |D^\alpha\varphi| .$$

Finally, we claim that when $\varphi = L'^t\chi$, $\chi \in C_0^\infty(G)$,

$$(3.9)\qquad \begin{aligned}\sigma_{j,1}^\ell(L'^t\chi) = c_0\int_0^\infty\int_{S^{p_2-1}} &\rho^{2k}(\pi_{(\rho,\omega)}(\chi)e_\ell^\omega,e_\ell^\omega)_{L^2}\psi_j(\omega)\,\cdot\\ &\cdot(\det B_{\eta(\rho,\omega)})^{1/2}\rho^{p_2-1}\,d\omega\,d\rho .\end{aligned}$$

Hence by summing over ℓ we obtain

$$\sum_\ell \sigma_{j,1}^\ell(L'^t\chi) = c_0\int_0^\infty\int_{S^{p_2-1}} \rho^{2k}\operatorname{tr}(P_\omega\pi_{(\rho,\omega)}(\chi)\psi_j(\omega))\rho^{p_2-1}\,\cdot$$

(3.10)

$$\cdot \, (\det B_{\eta(\rho,\omega)})^{1/2} \, d\omega \, d\rho \quad .$$

To prove (3.9), one merely checks that

$$(\pi_{(\rho,\omega)}(L'^{t})\pi_{(\rho,\omega)}(X)\, e_{\ell}^{\omega}, \ \pi_{(\rho,\omega)}(\sum Y_j^2)^{-N}\overline{B(\omega)}\, e_{\ell}^{\omega})$$

$$= \rho^{d/2} \, |A(\omega)| \, (\pi_{(\rho,\omega)}(X)\, e_{\ell}^{\omega}, e_{\ell}^{\omega}) \quad .$$

From (3.5) and (3.10) and the Plancherel formula it is clear that the distribution

(3.11) $$\sigma = \sum_{j} \sigma_{j,2} + \sum_{j,\ell} \sigma_{j,1}^{\ell}$$

satisfies

$$L' \sigma = Z \delta$$

where Z is the bi-invariant operator with $\pi_{(\rho,\omega)}(Z) = \rho^{2k}$.

4. Analytic families of differential operators.

We show in this section that the operator-valued function $\omega \to \pi_{(1,\omega)}(L)$ extends to a complex analytic family of unbounded operators from L^2 to itself, in the sense of Kato ([11], Ch. VII, Sec. 1.1), whose work follows that of Rellich [19] . This will allow us to define the projections P_{ω} introduced in the previous section and thereby prove Proposition 3.1.

The Sobolev space H^s , s a positive integer, is defined by

$$H^s = \{f \in L^2(R^{p_1/2}) : x^\alpha D^\beta f \in L^2 \text{ provided } |\alpha| + |\beta| \leq s\} .$$

Since $\pi_{(\rho,\omega)}(L)$ is a linear combination of monomials $x^\alpha D^\beta$ with $|\alpha| + |\beta| \leq d$, it maps H^d to L^2 boundedly. On L^2, however, the operator is unbounded, though clearly closable when initially defined on $C_0^\infty(R^{p_1/2})$. We denote the closure again by $\pi_{(\rho,\omega)}(L)$. Denoting the norm in H^s by $\| \ \|_s$, we have the estimate, for $v \in C_0^\infty(R^{p_1/2})$,

$$\|v\|_d \leq C(\omega)(\|\pi_{(1,\omega)}(L)v\|_0 + \|v\|_0), \quad v \in H^d . \tag{4.1}$$

This estimate is proved in [8] for ω real but persists into the complexes for $|\text{Im}\ \omega|$ small by (2.3). Hence the domain of each $\pi_{(1,\omega)} : L^2 \to L^2$ is exactly H^d. Thus $\{\pi_{(1,\omega)}(L)\}$ forms a "holomorphic family of Type (A)" in the terminology of [11, VII, §2.1].

Now we may define the V_ω. For this, we use the following result, see e.g., [11, III, 6.4, Thm 6.17].

Lemma 4.1. Let T be a self-adjoint operator with discrete spectrum consisting entirely of eigenvalues, and let Γ be a closed curve in $\mathbb{C}$ not meeting the spectrum of T. Then

$$P = -\frac{1}{2\pi i} \int_\Gamma (T-\xi)^{-1}\, d\xi$$

is the orthogonal projection onto the subspace spanned by the eigenvectors corresponding to the eigenvalues of T enclosed by Γ.

In our context, T will be $\pi_{(1,\omega)}(L)$ for certain values of ω. Let 0 be an eigenvalue of $\pi_{(1,\omega_0)}(L)$. Since the operators $\pi_{(1,\omega)}(L)$ form a holomorphic family, we may choose a smooth closed curve Γ which encloses 0 alone among points in the spectrum of $\pi_{(1,\omega_0)}(L)$ and meets the spectrum of no $\pi_{(1,\omega)}(L)$ for ω close enough to ω_0, say $\omega \in U_{\omega_0}$. Let V_ω denote the image of the projection P_ω where

$$P_\omega = -\frac{1}{2\pi i}\int_\Gamma (\pi_{(1,\omega)}(L) - \xi)^{-1}\, d\xi .$$

Then $\pi_{(1,\omega)}(L)P_\omega$ is the restriction of $\pi_{(1,\omega)}(L)$ to V_ω. The hypothesis (ii) of Theorem I implies that $\pi_{(1,\omega)}(L)$ is an invertible operator for most ω. The properties i) through iv) of Proposition 3.1 now follow from the properties of holomorphic families [11, VII, §7.3, Theorem 1.7].

In view of the definition of H^d, it is elementary to show that the injection of H^d into L^2 is compact for $d \geq 1$. We now prove

<u>Proposition 4.2</u>. <u>For any real</u> ω, $\pi_{(1,\omega)}(L)$ <u>has discrete spectrum consisting entirely of eigenvalues with finite multiplicity</u>.

<u>Proof</u>: Since $\pi_{(1,\omega)}(L)$ is self-adjoint, its spectrum is non-negative and thus $\ker(\pi_{(1,\omega)}(L) + \gamma) = 0$ for any $\gamma > 0$. Now condition (1.4) implies the estimate

(4.2) $$\|v\|_d \leq C(\|\pi_{(1,\omega)}(L)v\|_0 + \|v\|_0)$$

for all v in $C_0^\infty(R^{p_1/2})$ (see [8]). From (4.2) then

$$\|v\|_d \leq C'_\gamma(\|(\pi_{(1,\omega)}(L) + \gamma)v\|_0 + \|v\|_0)$$

for each fixed γ, and hence, using Proposition 4.1,

$$\|v\|_d \leq C''_\gamma(\|(\pi_{(1,\omega)}(L) + \gamma)v\|_0 + \|Q_\gamma v\|_0)$$

for γ fixed and all v in $C_0^\infty(R^{p_1/2})$, where Q_γ denotes the L^2 projection onto $\ker(\pi_{(1,\omega)}(L) + \gamma)$. If $\gamma > 0$, $Q_\gamma = 0$ and thus $(\pi_{(1,\omega)}(L) + \gamma)^{-1}$ exists and is compact. Now a standard well-known result (see, e.g., [11, III, 6.8, Theorem 6.29]) implies Proposition 4.2.

5. Application of the method of the division of distributions.

To prove Proposition 3.2 we write

$$J_\ell(\rho,\omega) = \rho^{-N}(\pi_{(\rho,\omega)}(\varphi)e_\ell^\omega, f_\ell^\omega)$$

with f_ℓ^ω independent of ρ, analytic in ω, and in L^2 together with its ω-derivatives. Thus

(5.1) $$J_\ell(\rho,\omega) = \rho^{-N}g_\ell^\varphi(\rho,\omega)$$

where, by (2.7),

(5.2) $$g_\ell^\varphi(\rho,\omega) = \iint (\varphi \circ \exp)^{\wedge_{2,3}}\left(u-v, \frac{u+v}{2}, \eta(\rho,\omega)\right)e_\ell^\omega(u)f_\ell^\omega(v)\,du\,dv.$$

Lemma 5.1. $g_\ell^\varphi(\rho,\omega)$ is in $C^\infty(R^+ \times S^{p_2-1})$ and for any compact set K in G, k' and α there exist $N = N_{K,k',\alpha}$ and $C = C_{N,k',\alpha}$ such that for all φ in $C_0^\infty(K)$

$$\sup_\omega |D_\omega^\alpha g_\ell^\varphi(\rho,\omega)| \le C(1+|\rho|^2)^{k'} \sup_{|\beta|\le N} |D_{r,s,t}^\beta \varphi| \ . \tag{5.3}$$

Proof: Since $\varphi \in C_0^\infty(G)$, $(\varphi \cdot \exp)^{\wedge 2,3} \in \mathcal{S}$ and hence $g_\ell^\varphi(\rho,\omega) = h_\ell(\omega)$ is infinitely differentiable in ω. But D_ω is a sum of vector fields (in the $\partial/\partial\eta_j$) with coefficients in $C^\infty(R^+ \times S^{p_2-1})$, homogeneous of degree 1 in ρ. Thus we may differentiate under the integral sign in (5.2) arbitrarily often in ρ,ω which proves that $g_\ell^\varphi(\rho,\omega)$ is C^∞ in $\rho \ne 0$ and ω.

Since for any j, $\rho^{2j} D_\omega^\alpha\, g_\ell^\varphi(\rho,\omega) = cD_\omega^\alpha g_\ell^{\Delta_t^j \varphi}(\rho,\omega)$ with $|c| = 1$, we may assume $k' \gg 0$.

In applying D_ω^α to $g_\ell^\varphi(\rho,\omega)$ in (5.2), ω-derivatives which fall on $e_\ell^\omega(u)$ and $f_\ell^\omega(v)$ yield functions still in L^2 together with their derivatives of any order:

$$\|D_\omega^\beta\, e_\ell^\omega(u)\|_{L^2} \le C_{\beta,\ell}$$

$$\|D_\omega^\beta\, f_\ell^\omega(v)\|_{L^2} \le C'_{\beta,\ell} \tag{5.4}$$

in view of the form of $f_\ell^\omega(v)$. The operator Φ_γ with kernel

$$K_\gamma(u, v, \eta(\rho, \omega)) = D_\omega^\gamma (\varphi \circ \exp)^{\wedge 2,3}(u-v, \frac{u+v}{2}, \eta(\rho, \omega)):$$

$$\Phi_\gamma e(v, \eta(\rho, \omega)) = \int_{R^{p_1/2}} K_\gamma(u, v, \eta(\rho, \omega))\, e(u)\, du$$

is bounded in L^2 with norm less than a constant times

$$(5.6) \quad (\sup_v \int |K_\gamma(u,v,\eta(\rho,\omega))|du)^{1/2} (\sup_u \int |K_\gamma(u,v,\eta(\rho,\omega))|\,dv)^{1/2}$$

(Young's inequality). Since D_ω is homogeneous in ρ of degree 1 and $\varphi \in \mathcal{S}$, (5.6) is bounded by $(1+|\rho|)^{|\gamma|}$ times a Schwartz seminorm of φ. The support of φ, however, is contained in a fixed compact subset of G, and an application of the Schwartz inequality in (5.2) yields (5.3). This proves Lemma 5.1.

Finally, it is now easy to see that $\sigma_{j,1}^{\ell}$ is actually a distribution. Since $\tilde{A}$ is a distribution, for any k'

$$|\tilde{A}(J(\rho,\omega))| \le \sup_{|\alpha| \le N_{\tilde{A}}} |D_\omega^\alpha J(\rho,\omega)|$$

$$\le C_{k'} \sup_{\substack{r,s,t \\ |\alpha| \le N_{\tilde{A}}}} |D_{r,s,t}^\alpha \varphi|(1+|\rho|)^{-2k'}$$

and so for all $\varphi \in C_0^\infty(K)$,

$$\| \rho^{2k-(d/2)+p_2-1} \tilde{A}(J(\rho,\omega))\|_{L^1(R^+)} \le C_K \sup_{|\alpha| \le N_K} |D^\alpha \varphi| .$$

6. Proof that the $\sigma_{j,2}$ are distributions.

Following [21] we write

$$(6.1)\qquad \sigma_{j,2}(\varphi) = c_0 \int_0^1 h(\rho)\, d\rho + c_0 \int_1^\infty h(\rho)\, d\rho \quad .$$

We first bound the second integral on the right as follows. By the Schwartz inequality, for any n,

$$(6.2)\qquad \left| \int_1^\infty h(\rho) d\rho \right| \le C \left\{ \int_1^\infty \rho^{2(2k-(d/2)-N+p_2-1-n)} d\rho \right\}^{1/2} \times$$

$$\left\{ \int_1^\infty \int_{S^{p_2-1}} \left| tr((I-P_\omega) \overline{M'_\omega}^{-1} \pi_{(1,\omega)} (\sum Y_j^2)^{-N} \pi_{(\rho,\omega)}(\varphi)) \right|^2 \right.$$

$$\left. |\det B_{\eta(\rho,\omega)}|\; |\psi_j(\omega)|^2\, d\omega\, \rho^{2n}\, d\rho \right\}^{1/2} \quad .$$

The first factor on the right of (6.2) is finite if $n \gg 0$. To bound the second factor we use the generalized Schwartz inequality, $|tr(AB)|^2 \le tr(AA^*)\, tr(BB^*)$, and follow [21]. Then

$$(6.3)\qquad \left| tr((I-P_\omega) \overline{M'_\omega}^{-1} \pi_{(1,\omega)} (\sum Y_j^2)^{-N} \pi_{(\rho,\omega)}(\varphi)) \right|^2$$

$$\le tr(\pi_{(\rho,\omega)}(\varphi) (\pi_{(\rho,\omega)}(\varphi))^*) \cdot tr(B_\omega B_\omega^*)$$

where $B_\omega = (I-P_\omega) \overline{M'_\omega}^{-1} \pi_{(1,\omega)} (\sum Y_j^2)^{-N}$.

Lemma 6.1. $\operatorname{tr}(B_\omega B_\omega^*) \le C$, independent of ω.

Proof: Since $\|(I-P_\omega)\overline{M'_\omega}^{-1}\|_{L^2} \le C_1$, independent of ω, and $\sum Y_j^2$ is self-adjoint,

$$(6.4) \qquad \operatorname{tr}(B_\omega B_\omega^*) \le C_1^2 \operatorname{tr}(\pi_{(1,\omega)}(\sum Y_j^2)^{-N\,2}) .$$

The eigenvalues of $\pi_{(1,\omega)}(\sum Y_j^2)$ are $\lambda_\alpha = \sum_{j=1}^{p_1/2} \tilde{\omega}_j(2\alpha_j+1)$; $\alpha = (\alpha_1, \cdots, \alpha_{p_1/2})$, each α_j a non-negative integer (see, e.g., [20]), where $\tilde{\omega}_j > 0$ and $\pm i\tilde{\omega}_j$ are the eigenvalues of the matrix $\eta([Y_j, Y_k])$, $\eta = \eta(1,\omega)$. Hence

$$(6.5) \qquad \operatorname{tr}((\pi_{(1,\omega)}(\sum Y_j^2)^{-N})^2) = \sum_\alpha |\lambda_\alpha|^{-2N} \le (C_2/2)\sum_\alpha |\alpha|^{-2N},$$

$|\alpha| = \sum_1^{p_1/2} \alpha_j$, since $|\lambda_\alpha| \ge 2C_2|\alpha|$ for

$$C_2 = \min_{\substack{\omega \in S^{p_2-1} \\ 1 \le j \le p_1/2}} |\tilde{\omega}_j| .$$

Now the Lemma follows from (6.4) and (6.5), provided N is sufficiently large.

Now by (6.3) and Lemma 6.1, the second integral on the right hand side of (6.2) is bounded by

$$(6.6) \qquad C\int_0^\infty\int_{S^{p_2-1}} \operatorname{tr}(\pi_{(\rho,\omega)}(\varphi)\pi_{(\rho,\omega)}(\varphi)^*)|\det B_{\eta(\rho,\omega)}| \cdot \cdot |\psi_j(\omega)|^2 \rho^{2n}\, d\omega\, d\rho$$

$$\le C'\|\Delta_t^{n/2}\varphi\|^2 ,$$

provided n is chosen to be even, the last inequality following from the Plancherel formula since $\pi_{(\rho,\omega)}(\Delta_t^{n/2}\varphi) = \rho^n \pi_{(\rho,\omega)}(\varphi)$. Hence

$$\int_1^\infty h(\rho)\, d\rho \le C'' \|\Delta_t^{n/2}\varphi\| \quad .$$

The estimate for $\int_0^1 h(\rho)\, d\rho$ may be obtained similarly, beginning with the estimate

$$|\int_0^1 h(\rho)\, d\rho| \le C\{\int_0^1 \rho^{2(2k-(d/2)-N)}\, d\rho\}^{1/2} \cdot$$

$$\cdot \{\int_0^1 |tr((I-P_\omega)\overline{M'_\omega}^{-1}(\pi_{(1,\omega)}(\sum Y_j^2)^{-N}\pi_{(\rho,\omega)}(\varphi)|^2 \cdot$$

$$\cdot |\det B_{\eta(\rho,\omega)}|\, |\psi_j(\omega)|^2 \rho^{2(p_2-1)}\, d\omega\, d\rho^{1/2} \quad .$$

The second integral on the right may be bounded as before. For the first, recall that N has been chosen independent of k; we may then choose k sufficiently large so that $2k-(d/2)-N \ge 0$, which guarantees the convergence of the first integral. This completes the proof that each $\sigma_{j,2}$ is a distribution. We remark that it is only for this result that it was necessary to replace L by $L(\sum Y_j^2)^N$, in order to make use of (6.5).

7. Proof of Theorem II.

Given $L = \sum a_{ij} Z_i Z_j$, where $\{Z_i\}$ forms a basis of $\mathfrak{g}_1$ with (a_{ij}) positive definite, there is a linear change of basis $Z_i \to Y_i$ for $\mathfrak{g}_1$ such that

(7.1) $$L = \sum Y_i^2 + i \sum c_j T_j$$

for some constants c_j, where the T_j form a basis of $\mathfrak{g}_2$. By direct calculation (see e.g. [13] or [23]) it can be shown that the eigenvalues of $\pi_\eta(L)$ are all of the form

(7.2) $$m_\alpha(\eta) = -\sum_{j=1}^{p_1/2} \rho_j(2\alpha_j+1) - \sum c_j \eta_j$$

where $\alpha = (\alpha_1, \cdots, \alpha_{p_1/2})$, α_j a non-negative integer, and the ρ_j, all positive, are $\pm i$ times the eigenvalues of $B_\eta(Y_i, Y_j) = \eta([Y_i, Y_j])$. In light of Theorem I (which in this case had been obtained previously by Lévy-Bruhl [13]), it suffices to show the following.

Proposition 7.1. For any constants c_j, and any fixed α, if $p_2 > 1$ $m_\alpha(\eta)$ does not vanish in any open set on the unit sphere in R^{p_2}.

Remark. Since the ρ_j are all bounded away from zero on the sphere, there are only finitely many α for which there exists non-zero η with $m_\alpha(\eta) = 0$.

Proof of the Proposition: Suppose that there exist $c = (c_j)$, α such that $m_\alpha(\omega)$ is identically zero in an open neighborhood U_{ω_0} of ω_0 in S^{p_2-1}. Let ω_1 be a point such that $c \cdot \omega_1 > 0$. Then $m_\beta(\omega_1) \leq -(c \cdot \omega_1)$ for all β. Since $p_2 > 1$, there exists a path $\gamma(t)$ on S^{p_2-1} connecting the points ω_0 and ω_1, i.e., $\gamma(0) = \omega_0$ and $\gamma(1) = \omega_1$. Let I be the set of all t in $(0,1]$ such that for $0 < s \leq t$, there exists α and ε such that $m_\alpha(\gamma(u)) = 0$ for $u \in [s-\varepsilon, s]$. By hypothesis, I is non empty and bounded. Let δ denote the least upper bound for I. δ cannot be equal to 1, since $\pi_{(1,\gamma(t))}(L)$ is invertible for t sufficiently close to 1 (since $\pi_{(1,\omega_1)}(L)$ is invertible). Now if $\delta < 1$, consider the point $\gamma(\delta)$. From Proposition 3.1, applied at $\gamma(\delta)$, $\det A_{ij}(\omega)$ is analytic near $\gamma(\delta)$, and by the construction of δ, $\det(A_{ij}(\gamma(t)) = 0$ for $\delta - \varepsilon_1 < t < \delta$, for some $\varepsilon_1 > 0$. But then this determinant must vanish identically for $\delta \leq t \leq \delta + \varepsilon_2$, for some $\varepsilon_2 > 0$, contradicting the hypothesis that δ is the least upper bound for I.

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