Iterated Segre mappings of real submanifolds in complex space and applications

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Abstract. This article is a survey of various applications of the method of iterated Segre mappings obtained by a number of mathematicians, including the author, over the past decade. This method is applied to various problems involving real submanifolds in complex space and their mappings. The article begins with a description of the iterated Segre mappings associated to generic submanifolds. The problems addressed concern transversality of holomorphic mappings, finite jet determination, local stability groups, and algebraicity of holomorphic mappings between real-algebraic manifolds.

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1. Introduction and notation

In this survey we consider real submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ through p and p' respectively and study local properties of germs of holomorphic mappings $H\colon (\mathbb{C}^N, p) \to (C^{N'}, p')$ such that $H(M) \subset M'$. The questions we consider are the following:

1. When is H determined by finitely many derivatives at p?
2. If $M = M'$ and $p = p'$, when does the set of all such H form a finite dimensional Lie group?
3. If M and M' are real-algebraic manifolds, when does it follow that H is necessarily an algebraic mapping?
4. When is H transversal to M' at p'?

The common thread of the approach to these problems is the use of the iterated Segre mappings, which represent a kind of blow-up of the complexification of a real submanifold. These mappings first appeared in the joint work of the author with

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Baouendi and Ebenfelt [2]. In what follows I shall assume, without loss of generality, that p and p' are both the origin. Then the real submanifold M is given near 0 by the vanishing of a system of equations $\rho = (\rho_1, \ldots, \rho_d) = 0$, where d is the codimension of M and the ρ_j are real-valued functions with linearly independent differentials at 0. If, in addition, the complex vectors $(\frac{\partial \rho_j}{\partial Z_1}, \ldots, \frac{\partial \rho_j}{\partial Z_N})(0)$, $1 \le j \le d$, are also linearly independent, the real submanifold M is called *generic*. In particular, the condition that M is generic insures that M is a *CR manifold*, i.e. that the space $T_q^{0,1}M$ of complex $(0,1)$ vectors at q that are tangent to M at q is of constant dimension for $q \in M$ near 0. The assumption that a CR submanifold of $\mathbb{C}^N$ is generic is not very restrictive, since any such manifold can be locally embedded as a generic submanifold in some complex manifold, possibly of lower dimension (see e.g. [6]).

For quite some time an important tool in the study of holomorphic mappings has been the complexification of the real manifolds M and M' (see e.g. Webster [48], [49], Diederich–Webster [25], Baouendi–Jacobowitz–Treves [12], Diederich–Pinchuk [24], Chern–Ji [19]). To simplify notation and statements, I shall assume, unless stated otherwise, that the real submanifolds M and M' are real-analytic, and hence their defining functions can be assumed to be real-analytic. Then we may regard the real vector valued function ρ as a convergent power series, $\rho(Z, \bar{Z})$, in Z and $\bar{Z}$, which may be complexified as a (germ at 0 of a) convergent power series $\rho(Z, \zeta)$ in $2N$ complex variables. The *complexification* $\mathcal{M}$ of M is defined to be the (germ at 0 of a) complex manifold $\mathcal{M}$ defined near $(0,0) \in \mathbb{C}^N \times \mathbb{C}^N$ by

$$\mathcal{M} = \{(Z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N : \rho(Z, \zeta) = 0\}. \tag{1.1}$$

(Here the real-analyticity of M and ρ allows the local complexification of ρ. By abuse of notation, I shall denote both ρ and its local complexification by the same letter.) Similarly, I denote by $\mathcal{M}'$ the complexification of M'. For $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ denote by $\mathcal{H} = (H, \bar{H})\colon (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^{N'} \times \mathbb{C}^{N'}, 0)$ the *complexification* of H. (Here $\bar{H}(\zeta) := \overline{H(\bar{\zeta})}$.) Using again real-analyticity, it is easy to see that

$$H(M) \subset M' \iff \mathcal{H}(\mathcal{M}) \subset \mathcal{M}'. \tag{1.2}$$

The submanifold M is said to be of *finite type* at 0 in the sense of Kohn [40] (and also of Bloom–Graham) if the (complex) Lie algebra $\mathfrak{g}_M$ generated by all smooth $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(0) = \mathbb{C}T_0M$, where $\mathbb{C}T_0M$ is the complex tangent space to M at 0. The method of Segre mappings is an important tool in the analysis of the equation on the right hand side of (1.2) in case M and M' are of finite type at the origin.

In this survey I shall focus mainly on results for the case $N = N'$. Also, for simplicity of notation, the (germ at 0 of a) holomorphic map $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ will be assumed to be finite, i.e. any $Z \in \mathbb{C}^N$ near 0 has only finitely many inverse images under H near 0. An equivalent algebraic condition is that the ideal $I(H)$ in $\mathbb{C}\{Z\}$ (the ring of convergent power series in $Z = (Z_1, \ldots, Z_N)$) generated by the components, $H_1(Z), \dots, H_N(Z)$, of $H(Z)$ is of finite codimension, i.e. the vector space $\mathbb{C}\{Z\}/I(H)$ is finite dimensional.

2. Iterated Segre mappings

Following the approach of [8], I shall give here an outline of the definition and main properties of the iterated Segre mappings for a generic submanifold M of codimension d in $\mathbb{C}^N$. Let $n := N - d$ and $\gamma\colon (\mathbb{C}^N \times \mathbb{C}^n, 0) \to (\mathbb{C}^N, 0)$ be a (germ of a) holomorphic mapping such that

$$\rho(\gamma(\zeta, t), \zeta) \equiv 0, \quad \operatorname{rk} \frac{\partial \gamma}{\partial t}(0,0) = n, \tag{2.1}$$

where $\zeta = (\zeta_1, \dots, \zeta_N)$, $t = (t_1, \dots, t_n)$. By the implicit function theorem, the existence of such γ follows from the genericity assumption on M. We define a sequence of germs of holomorphic mappings $v^j\colon (\mathbb{C}^{nj}, 0) \to (\mathbb{C}^N, 0)$, $j \geq 0$, called *the iterated Segre mappings* of M at 0 (relative to γ), inductively as follows:

$$\begin{aligned} &v^0 = 0, \\ &v^1(t^1) := \gamma(0, t^1), \\ &v^{j+1}(t^1, \dots, t^{j+1}) := \gamma\bigl(\overline{v^j}(t^1, \dots, t^j), t^{j+1}\bigr). \end{aligned} \tag{2.2}$$

Here $\overline{v^j}$ denotes the vector-valued convergent power series obtained from v^j by conjugating its coefficients.

For a (germ of a) holomorphic mapping $v\colon (\mathbb{C}^k, 0) \to (\mathbb{C}^l, 0)$, we denote by $\operatorname{Rk} v$ the generic rank of any representative of v in a sufficiently small neighborhood of 0. Some of the main properties of the Segre mappings are summarized in the following theorem.

Theorem 2.1 ([8]). *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through* 0 *of codimension d, and let γ, v^j be as above. Then:*

(i) *For $l = 1, 2, \dots$, the mapping $(v^l(t^1, \dots, t^l), \overline{v^{l-1}}(t^1, \dots, t^{l-1}))$ sends $\mathbb{C}^{nl}$ into $\mathcal{M}$.*

(ii) *There exists an integer k_0, $1 \leq k_0 \leq d + 1$, such that $\operatorname{Rk} v^j = \operatorname{Rk} v^{j+1}$ for $j \geq k_0$, and if $k_0 > 1$, then $\operatorname{Rk} v^j < \operatorname{Rk} v^{j+1}$ for $1 \leq j \leq k_0 - 1$.*

If k_0 is as in (ii), *the following hold.*

(iii) *The submanifold M is of finite type at* 0 *if and only if* $\operatorname{Rk} v^{k_0} = N$.

(iv) *There exists a (germ at* 0 *of a) submanifold $\Sigma \subset \mathbb{C}^{2nk_0}$, such that $v^{2k_0}(\Sigma) = \{0\}$ and v^{2k_0} achieves full rank (i.e.* $\operatorname{Rk} v^{k_0}$*) on some points on Σ (arbitrarily close to* 0*).*

Here are some observations about Theorem 2.1. Part (ii) shows that the generic ranks of the Segre mappings stabilize before $d+1$ iterations. Part (iii) gives a characterization of finite type in terms of the generic rank of the Segre mappings after stabilization. Part (iv) shows that after at most $2(d+1)$ iterations, the Segre mappings achieve maximal rank on a set that is mapped by v^{2k_0} to the origin in $\mathbb{C}^N$.

The Segre mappings can be used in conjunction with special choices of coordinates in $\mathbb{C}^N$ for a given generic submanifold M. The coordinates $(z, w) \in \mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^d$ are called *normal* if there exists a (germ of a) holomorphic mapping $Q(z, \chi, \tau)$, $Q\colon (\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$, with $Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau$, such that M is given near the origin by the vector equation

$$w = Q(z, \bar{z}, \bar{w}). \tag{2.3}$$

The existence of normal coordinates may be proved by the use of the implicit function theorem ([20], [6]). If $Z = (z, w)$ is a choice of normal coordinates for M given by (2.3), then the equation in (2.1) for $\gamma(\zeta, t) = (\gamma_1(\zeta, t), \gamma_2(\zeta, t)) \in \mathbb{C}^n \times \mathbb{C}^d$ becomes

$$\gamma_2(\zeta, t) \equiv Q(\gamma_1(\zeta, t), \zeta). \tag{2.4}$$

Hence we may choose

$$\gamma(\zeta, t) = ((\gamma_1(\zeta, t), \gamma_2(\zeta, t)) = (t, Q(t, \zeta)). \tag{2.5}$$

Now suppose $(z', w') \in \mathbb{C}^{N'} = (\mathbb{C}^{n'} \times \mathbb{C}^{d'})$ is a given set of normal coordinates for a generic submanifold $M' \subset \mathbb{C}^{N'}$ of codimension d'. Then if $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ we may write

$$H(Z) = (F(Z), G(Z)), \quad \text{where } z' = F(Z),\ w' = G(Z). \tag{2.6}$$

By (1.2), $H(M) \subset M'$ is equivalent to the holomorphic vector equation

$$G(Z) = Q'(F(Z), \bar{F}(\zeta), \bar{G}(\zeta)) \quad \text{for all } (Z, \zeta) \in \mathcal{M}. \tag{2.7}$$

By (i) of Theorem 2.1, if (2.7) holds, then for any integer $k \geq 0$ we may take $(Z, \zeta) = (v^{k+1}(t^1, \dots, t^{k+1}), \overline{v^k}(t^1, \dots, t^k))$ in equation (2.7) to obtain

$$G \circ v^{k+1} = Q'\big(F \circ v^{k+1}, \overline{F \circ v^k}, \overline{G \circ v^k}\big), \tag{2.8}$$

yielding a holomorphic vector equation in $n(k+1)$ free complex parameters. If M is of finite type at 0 and (2.8) holds for $k \geq d+1$, then by Theorem 2.1 (iii) it follows that (2.7) also holds. Hence (2.8) is equivalent to (2.7) in this case.

A construction analogous to that of the iterated Segre mappings was developed independently by Christ–Nagel–Stein–Wainger [21] in a different context.

3. Nondegeneracy conditions for generic submanifolds

In order to formulate nontrivial results concerning the questions posed in Section 1, we recall some nondegeneracy conditions that may be imposed on generic submanifolds. The best known (and strongest) of these conditions is that of Levi-nondegeneracy for a real hypersurface in $\mathbb{C}^N$. If M is a hypersurface given in normal coordinates by the (scalar) equation (2.3), then M is *Levi-nondegenerate* at 0 if the quadratic form

$$\mathbb{C}^{N-1} \times \mathbb{C}^{N-1} \ni (z, \chi) \mapsto Q^{(2)}(z, \chi) \tag{3.1}$$

is nondegenerate, where $Q^{(2)}(z, \chi)$ is the quadratic part of the Taylor expansion at 0 of $Q(z, \chi, 0)$. (Equivalent definitions independent of the choice of coordinates can be given for Levi-nondegeneracy, as well as for the other nondegeneracy conditions to be discussed here.) Germs of Levi-nondegenerate hypersurfaces have been completely classified up to local equivalence in the celebrated work of Chern and Moser [20] in the 1970s.

For a generic submanifold M of any codimension, given in normal coordinates by (2.3), a number of weaker geometric conditions can be given in terms of the vector-valued function $Q(z, \chi, \tau)$. We write $Q = (Q_1, \ldots, Q_d)$. The strongest of these conditions is finite nondegeneracy: M is *ℓ-nondegenerate* at 0 if the set of vectors

$$\frac{\partial^\alpha}{\partial \chi^\alpha}\left(\frac{\partial Q_j}{\partial z_1}, \ldots, \frac{\partial Q_j}{\partial z_n}\right)(0), \quad j = 1, \ldots, d, \ \alpha \in \mathbb{N}^n, \ |\alpha| \leq \ell, \tag{3.2}$$

spans $\mathbb{C}^n$, and ℓ is the smallest positive integer for which this is true. If M is ℓ-nondegenerate at 0 for some finite integer ℓ, then M is called *finitely nondegenerate*. A hypersurface is Levi-nondegenerate at 0 if and only if it is 1-nondegenerate.

A condition weaker than finite nondegeneracy is the following. Expand $Q(z, \chi, 0)$ as a Taylor series in χ,

$$Q(z, \chi, 0) = \sum Q^\alpha(z, 0, 0)\chi^\alpha, \tag{3.3}$$

and let I be the ideal in $\mathbb{C}\{z\}$ generated by all the $Q^\alpha(z, 0, 0)$. Then M is said to be *essentially finite* at 0 if the ideal I is of finite codimension in $\mathbb{C}\{z\}$. If so, the dimension of the vector space $\mathbb{C}\{z\}/I$ is called the *essential type* of M at 0, denoted $\operatorname{ess}_0 M$. It is easy to see that if M is finitely nondegenerate at 0, then it is essentially finite at 0. However, the converse is not true. For example, the hypersurface $M \subset \mathbb{C}^2$ given as $\{(z, w) \in \mathbb{C}\times\mathbb{C} : \operatorname{Im} w = |z|^4\}$ is essentially finite at 0, but not finitely nondegenerate. The precise relationship is the following (see [6], Proposition 11.8.27):

$$M \text{ is finitely nondegenerate at } 0 \iff M \text{ is essentially finite at } 0 \text{ with } \operatorname{ess}_0 M = 1. \tag{3.4}$$

The conditions of finite nondegeneracy and essential finiteness can be expressed invariantly, i.e. independently of any choice of defining function or coordinates (see [6], Chapter XI), but I shall not do so here. However, I want to point out that the condition of finite type, as defined in Section 1, cannot be easily expressed in terms of an arbitrary defining function, except in the case of hypersurfaces. This is one reason that Theorem 2.1 (iii) and (iv) turns out to be useful for the study of generic submanifolds of finite type.

Another condition, which is weaker than essential finiteness, is most easily expressed in terms of $(1,0)$ vector fields, i.e. sections of the vector bundle $T^{1,0}\mathbb{C}^N$ of $(1,0)$ vectors on $\mathbb{C}^N$. The generic submanifold M is said to be *holomorphically nondegenerate* at 0 if there is no nonzero $(1,0)$ vector field with holomorphic coefficients that is tangent to M in a whole neighborhood of 0.

It can be shown (see e.g. [6], Corollary 11.7.28) that holomorphic degeneracy of a connected generic submanifold M can also be described in terms of finite nondegeneracy or essential finiteness. In fact, if M is a connected, real-analytic generic submanifold of $\mathbb{C}^N$ with $0 \in M$, then the following conditions are equivalent:

(i) M is holomorphically nondegenerate at 0.

(ii) M is holomorphically nondegenerate at all $p \in M$.

(iii) M is essentially finite at some point in M.

(iv) M is essentially finite at all points outside a proper real-analytic subset of M.

(v) M is finitely nondegenerate at some point in M.

(vi) M is finitely nondegenerate at all points outside a proper real-analytic subset of M.

(vii) There is an integer $\ell \leq N-1$ such that M is ℓ-nondegenerate at all points outside a proper real-analytic subset of M.

The relation of the above conditions with finite type is more complicated. If a hypersurface is essentially finite at 0, it is necessarily of finite type at 0, but no implication holds for generic submanifolds of higher codimension.

In view of the equivalence of (i) and (ii), if M is holomorphically nondegenerate at 0, one simply says that M is holomorphically nondegenerate. If M is not holomorphically nondegenerate, then it is not essentially finite at any point. In this sense, holomorphic nondegeneracy is the weakest condition that guarantees the existence of some "good" points.

The notion of essential finiteness has been implicitly used in the work of Diederich–Webster [25] and was defined first in the work of Baouendi–Jacobowitz–Treves [12]. Holomorphic nondegeneracy was first introduced for hypersurfaces by Stanton [47] and for higher codimension in [2]. Finite nondegeneracy was implicitly used by Han [34] for hypersurfaces and defined in joint work with Baouendi and Huang [10]. An invariant definition for CR manifolds, not necessarily embedded in complex space, is due to Ebenfelt (see [6]).

4. Transversality of mappings

The notion of transversality of real differentiable mappings has been central in differential geometry. Recall that if $f: (\mathbb{R}^k, 0) \to (\mathbb{R}^\ell, 0)$ is a germ of a smooth mapping, and if $E \subset \mathbb{R}^\ell$ is a smooth manifold through 0, then f is called *transversal* to E at 0 if

$$T_0E + df(T_0\mathbb{R}^k) = T_0\mathbb{R}^\ell. \tag{4.1}$$

If $E \subset \mathbb{C}^\ell$ is a generic submanifold, and $H: (\mathbb{C}^k, 0) \to (\mathbb{C}^\ell, 0)$ is a (germ of a) holomorphic mapping, a stronger notion of transversality is needed, for example, to guarantee that $H^{-1}(E)$ is again a generic submanifold of $\mathbb{C}^k$. Also, for mappings in which the target is a real hypersurface, it is desirable to have a notion of transversality that guarantees the nonvanishing of a derivative of the transversal component. For generic submanifolds and holomorphic mappings as above, the appropriate notion is the following. The mapping H is said to be *CR transversal* to E at 0 if

$$T_0^{1,0}E + dH(T_0^{1,0}\mathbb{C}^k) = T_0^{1,0}\mathbb{C}^\ell. \tag{4.2}$$

Here $T_0^{1,0}E$ denotes the space of $(1,0)$ vectors tangent to M at 0. It is not hard to see that CR transversality implies transversality, but the converse is not necessarily true. However, we shall restrict ourselves here to the case $k = \ell$ and assume that H maps one generic submanifold into another. In this context it can be shown that the two notions of transversality coincide (see [30]).

Recently Ebenfelt and the author have obtained the following result.

Theorem 4.1 ([30]). *Let $M, M' \subset \mathbb{C}^N$ be real-analytic generic submanifolds of the same dimension through 0 such that either M or M' is of finite type at 0. Then any finite holomorphic mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ with $H(M) \subset M'$ is CR transversal to M' at 0.*

For the case where M and M' are assumed to be hypersurfaces, Theorem 4.1 was proved earlier by Baouendi and the author (see [14]). The case of generic submanifolds of higher codimension had remained an open problem since then.

Theorem 4.1 can be viewed as a kind of complex Hopf Lemma. For smooth mappings between hypersurfaces where the target has some convexity properties, results of this type were previously obtained by Fornaess [31], [32] for the case of pseudoconvex hypersurfaces (using the classical Hopf Lemma). Related results could also be found in the author's joint work with Baouendi [15], as well as with Baouendi and Huang [11].

I shall describe briefly here how the iterated Segre mappings can be used in the proof of Theorem 4.1. We begin with normal coordinates (z, w) and (z', w') for M and M' respectively, leading to the equation (2.7). The condition of CR transversality in these coordinates is equivalent to

$$\det \frac{\partial G}{\partial w}(0) \neq 0, \tag{4.3}$$

where $H = (F, G)$ as in (2.6). Now assume M is of finite type at 0 and let k_0 be given by Theorem 2.1 and take $k = 2k_0 - 1$ in (2.8) to obtain

$$G \circ v^{2k_0} = Q'\big(F \circ v^{2k_0}, \overline{F \circ v^{2k_0-1}}, \overline{G \circ v^{2k_0-1}}\big), \tag{4.4}$$

which is equivalent to (2.7) since M is of finite type at 0. Put $v^{2k_0}(t) = (t^{2k_0}, u^{2k_0}(t))$, with $t = (t^1, \ldots, t^{2k_0}) = (t', t^{2k_0}) \in \mathbb{C}^{2nk_0-n} \times \mathbb{C}^n$. If $\Sigma \subset \mathbb{C}^{2nk_0}$ is a submanifold given by Theorem 2.1 (iv), then necessarily $\Sigma \subset \mathbb{C}^{2nk_0-n} \times \{0\}$. Taking $t^{2k_0} = 0$ in (4.4), differentiating in t', and evaluating at any $s \in \Sigma$ and using the chain rule, the left hand side of (4.4) becomes

$$\frac{\partial G}{\partial w}(0)\frac{\partial u^{2k_0}}{\partial t'}(s), \tag{4.5}$$

since $v^{2k_0} \equiv 0$ on Σ. Now if (4.3) fails, there is a nonzero constant vector $V \in \mathbb{C}^d$ such that $V^\tau \dfrac{\partial G}{\partial w}(0) = 0$. Our proof proceeds by multiplying both sides of (4.4) by V^τ, differentiating in t' and restricting to Σ. We then show that under the hypotheses of the theorem, the right hand side cannot vanish identically. The details can be found in [30].

By using formal power series, instead of convergent ones, we can also prove Theorem 4.1 when M and M' are merely assumed to be smooth and H is replaced by a CR mapping that is assumed to be "formally" finite.

One may ask also when a finite holomorphic mapping H satisfying $H(M) \subset M'$ is not only transversal to M', but actually is a diffeomorphism at 0. If M is essentially finite at 0, then we prove, using Theorem 4.1, that M' is also essentially finite at 0 and

$$\mathrm{ess}_0\, M = \mathrm{mult}(H) \cdot \mathrm{ess}_0\, M', \tag{4.6}$$

where ess_0 denotes the essential type at 0 and $\mathrm{mult}(H)$ denotes the multiplicity of H as a finite holomorphic mapping. If M is finitely nondegenerate at 0, then as noted in (3.4), it follows that it is essentially finite at 0 with $\mathrm{ess}_0\, M = 1$. Since $\mathrm{ess}_0\, M'$ is a positive integer, the equation (4.6) implies that $\mathrm{mult}(H) = 1$, i.e., H is a diffeomorphism. Hence we have the following consequence of Theorem 4.1:

Theorem 4.2 ([30]). *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of finite type and finitely nondegenerate at* 0. *If $M' \subset \mathbb{C}^N$ is a real-analytic generic submanifold of the same dimension and $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is a finite holomorphic mapping with $H(M) \subset M'$, then H is a local biholomorphism at* 0.

Although this paper is mostly focused on the case where the manifolds M and M' are equidimensional and contained in the same complex space, I should mention that in recent years there have been a number of related results for nonequidimensional hypersurfaces. In particular, without striving for completeness, I would like to mention the work of D'Angelo [23], Forstnerič [33], Huang [35], Huang–Ji [36], Ebenfelt–Huang–Zaitsev [27], Baouendi–Huang [9].

5. Finite jet determination

One of the most basic questions to be asked about a (germ of a) holomorphic mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is to find the least data to determine H. Clearly H is determined by $\partial^\alpha H(0)$ for all $\alpha \in \mathbb{Z}^N$. If $H(M) \subset M'$ for real analytic generic submanifolds M and M', then H also satisfies a system of real analytic equations when restricted to M, and under favorable circumstances may be determined by finitely many derivatives at the origin. We denote by $j_p^k H$ the *k-jet* of H at p, i.e.

$$j_p^k H := (\partial^\alpha H)(p)_{\{|\alpha| \le k\}}. \tag{5.1}$$

It follows from the work of Chern–Moser [20] that if H is a diffeomorphism and M and M' are both Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$ through 0, then H is actually determined by $j_0^2 H$.

A striking theorem of finite determination for hypersurfaces in $\mathbb{C}^2$ is the following.

Theorem 5.1 (Ebenfelt–Lamel–Zaitsev, [29]). *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface of finite type at 0. Then if $H^1, H^2\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ are (germs of) holomorphic diffeomorphisms both sending M into itself and satisfying $j_0^2 H^1 = j_0^2 H^2$, then $H^1 = H^2$.*

The weakest known conditions, for finite determination, that can be imposed on M and M' in higher codimenson is that of finite type and holomorphic nondegeneracy at 0. The following result is joint work with Baouendi and Mir [13].

Theorem 5.2 ([13]). *Let M and M' be real-analytic generic submanifolds of $\mathbb{C}^N$ through 0 of the same dimension, with M of finite type and holomorphically nondegenerate at 0. Let $H^0\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a finite holomorphic mapping with $H^0(M) \subset M'$. Then there exists an integer K such that if $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is any finite holomorphic mapping sending M into M' with $j_0^K H = j_0^K H^0$, it follows that $H = H^0$.*

The techniques of the proof do not provide an explicit integer K nor give any kind of dependence of this integer on the base point. Theorem 5.2 can be strengthened by merely assuming M and M' to be smooth and by a slight weakening of the assumption that H be finite (see [13]).

For more explicit results in $\mathbb{C}^N$, $N > 2$, the condition of finite nondegeneracy might need to be imposed. This condition enters as follows. For simplicity, let us assume that $M = M'$. We wish to describe those mappings $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ that are diffeomorphisms at 0 and satisfy $H(M) \subset M$. If M is ℓ_0-nondegenerate at 0, by taking appropriate derivatives of (2.7) and applying the implicit function theorem, the following can be shown (see [6]). For $\ell \ge 0$, there exists a holomorphic mapping Ψ_ℓ defined in a neighborhood of $(0, 0, j_0^{\ell_0+\ell}\mathrm{Id})$ in $\mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^J$ (for an appropriate integer J) such that one has the following "basic identity". For every $(Z, \zeta) \in \mathcal{M}$

near 0 and every H sending M into itself with $j_0^{\ell_0+\ell}H$ sufficiently close to $j_0^{\ell_0+\ell}\mathrm{Id}$, the following holds:

$$(Z, j_Z^{\ell}H) = \Psi_\ell\big(Z, (\zeta, j_\zeta^{\ell_0+\ell}\overline{H})\big). \tag{5.2}$$

We now use (5.2), by taking $(Z,\zeta) = (v^k(t^1,\dots,t^k), \overline{v^{k-1}}(t^1,\dots,t^{k-1})) \in \mathcal{M}$ (as defined in Section 2). In this way, we express all derivatives of H of length $\le \ell$ on the image of v^k in terms of the derivatives of $\overline{H}$ of length $\le \ell_0+\ell$ on the image of $\overline{v^{k-1}}$. We begin by taking $(Z,\zeta) = (v^1(t^1), \overline{v^0}) = ((t^1,0),(0,0)) \in \mathcal{M}$ in (5.2) to obtain

$$\big((t^1,0), j_{(t^1,0)}^{\ell}H\big) = \Psi_\ell\big((t^1,0),((0,0), j_0^{\ell_0+\ell}\overline{H})\big). \tag{5.3}$$

Hence all derivatives of H up to order ℓ on v^1 are determined by the $\ell_0+\ell$ jet of $\overline{H}$ at 0. Taking $(Z,\zeta) = (v^2(t^1,t^2), \overline{v^1}(t^1))$, we find that all derivatives of H up to order ℓ on v^2 are determined by the $\ell_0+\ell$ jet of $\overline{H}$ on $\overline{v^1}$. Since the latter is determined by $j_0^{2\ell_0+\ell}H$ by the first step (after complex conjugation), we observe that $j_{v^2}^{\ell}H$ is determined by $j_0^{2\ell_0+\ell}H$. Now let k_0 be as in Theorem 2.1. By using inductively the above argument, we conclude that $j_{v^{k_0}(t)}^0 H$ is determined by $j_0^{k_0\ell_0}H$. If M is of finite type at 0, then by Theorem 2.1 (iii), H is completely determined by the values of $j_{v^{k_0}(t)}^0 H$ as t varies in $\mathbb{C}^{nk_0}$. Since $k_0 \le d+1$, the above argument gives an outline of a proof of the following.

Theorem 5.3 ([4]). *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of finite type and ℓ_0-nondegenerate at 0. If $H^i\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$, $i = 1,2$ are holomorphic diffeomorphisms both sending M into itself and satisfying $j_0^{(d+1)\ell_0}H^1 = j_0^{(d+1)\ell_0}H^2$, it follows that $H^1 = H^2$.*

The results stated in this paper are given for (germs of) holomorphic mappings, but many can also be formulated for (germs of) smooth generic submanifolds and smooth CR mappings that are diffeomorphisms. Since the methods described above rely on unique determination by the Taylor series at 0, different techniques must be used to prove finite determination for mappings that are merely smooth. By using the method of complete systems, Ebenfelt [26] proved unique determination results for smooth CR mappings between finitely nondegenerate hypersurfaces. These were later generalized by Ebenfelt–Lamel [28]. General results for smooth CR diffeomorphisms between finitely nondegenerate smooth generic submanifolds of any codimension were recently obtained by Kim–Zaitsev [39]. The reader is referred to the survey article of Zaitsev [52] for a more detailed discussion of this topic.

6. Stability groups

In this section I shall discuss questions concerned with the structure of the group of holomorphic mappings that send a generic submanifold into itself. Let $M \subset \mathbb{C}^N$

be (the germ at 0 of) a real-analytic generic submanifold. The *holomorphic stability group*, $\mathrm{Hol}\,(M,0)$ of M at 0 is defined as the set of all (germs at 0) of holomorphic diffeomorphisms $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ such that $H(M) \subset M$, where the group structure is given by composition. The group $\mathrm{Hol}\,(M,0)$ is equipped with its natural inductive limit topology. In particular, a sequence $\{H^j\}$ in $\mathrm{Hol}\,(M,0)$ converges to H if there exists a compact neighborhood $\overline{U}$ of 0 such that each H^j has a representative holomorphic in an open neighborhood of $\overline{U}$ and H^j converges uniformly to H on $\overline{U}$.

One may ask whether $\mathrm{Hol}\,(M,0)$ may be given the structure of a finite dimensional Lie group compatible with its topology. For any integer $\ell \geq 0$, let $G^\ell(\mathbb{C}^N)$ denote the set of all ℓ-jets of invertible holomorphic mappings, i.e.

$$G^\ell(\mathbb{C}^N) := \{j_0^\ell H : H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0),\ H \text{ biholomorphic}\}. \tag{6.1}$$

The set $G^\ell(\mathbb{C}^N)$ has a finite dimensional Lie group structure with the multiplication defined by $(j_0^\ell H^1)\cdot(j_0^\ell H^2) := j_0^\ell(H^1 \circ H^2)$. It is easy to check that this multiplication is independent of the choice of representatives H^1 and H^2. If M is holomorphically nondegenerate and of finite type at 0, then by taking $M = M'$ and $H^0 = \mathrm{Id}$ in Theorem 5.2, it follows that there exists an integer $K > 0$ such that the mapping $\mathrm{Hol}\,(M,0) \ni H \mapsto j_0^K H \in G^K(C^N)$ is continuous and injective. Hence $\mathrm{Hol}\,(M,0)$ may be identified with a subgroup of $G^K(\mathbb{C}^N)$ in this case. However, to show that $\mathrm{Hol}\,(M,0)$ is a Lie group one must show that its image is closed in $G^K(\mathbb{C}^N)$. This was proved for any finitely nondegenerate hypersurface M by the author with Baouendi and Ebenfelt [3] and later by Zaitsev [50] for any finitely nondegenerate generic submanifold of finite type in higher codimension. The following sharper result was given in [5].

Theorem 6.1 ([5]). *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, which is ℓ_0-nondegenerate and of finite type at* 0. *Then the mapping*

$$\mathrm{Hol}\,(M,0) \ni H \mapsto j_0^{\ell_0(d+1)} H \in G^{\ell_0(d+1)}(\mathbb{C}^N),$$

taking a germ of a local biholomorphism at 0 *to its $\ell_0(d+1)$-jet, gives a diffeomorphism of* $\mathrm{Hol}\,(M,0)$ *onto a real-algebraic Lie subgroup of* $G^{\ell_0(d+1)}(\mathbb{C}^N)$.

It should be noted here that Theorem 6.1 for the case of a Levi-nondegenerate hypersurface (i.e. $\ell_0 = d = 1$) follows from the work of Chern–Moser [20] and Burns–Shnider [18].

Recent work of Kim–Zaitsev [38] gives a construction of a smooth hypersurface $M \subset \mathbb{C}^N$, finitely nondegenerate at 0, for which $\mathrm{Hol}\,(M,0)$ is not a Lie group, although it is contained as a subgroup of a finite dimensional Lie group. In attempting to generalize Theorem 6.1 in another direction, one may ask whether the condition that M be finitely nondegenerate can be weakened (see the equivalences (i)–(vii) in Section 3). As observed above, in the case of a holomorphically nondegenerate generic submanifold of finite type there is an integer K for which the mapping

$\mathrm{Hol}(M, 0) \ni H \mapsto j_0^K H \in G^K(\mathbb{C})$ is an injection, but it is not known if its image is closed. However, very recently Lamel and Mir [42] gave a positive answer in the slightly more restrictive case of a real-analytic generic submanifold that is essentially finite and of finite type at 0.

Although global questions are outside the announced scope of this paper, I would like to mention here a recent result joint with Baouendi, Winkelmann, and Zaitsev [16] in which we prove that the global automorphism group of a CR manifold that is finitely nondegenerate and of finite type at every point has the structure of a finite dimensional Lie group. This work uses the above mentioned results of Kim–Zaitsev [39].

7. Algebraicity of mappings

Recall that a real submanifold is *real-algebraic* if it is contained in a real-algebraic subset of the same dimension. A (germ at 0 of a) holomorphic mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is *complex-algebraic* if its graph is a (germ at 0 of a) complex algebraic variety of $\mathbb{C}^N \times \mathbb{C}^N$. One may ask under what conditions a mapping that sends one real-algebraic generic submanifold into another is necessarily complex-algebraic. Very early results on this question are contained in the work of Poincaré [46], who proved that a local biholomorphism between two pieces of spheres in $\mathbb{C}^2$ must be a rational mapping. In 1977 Webster [48] solved the above problem for the case when M and M' are algebraic hypersurfaces that are Levi-nondegenerate, proving that any such local biholomorphism is necessarily complex-algebraic. The following result for higher codimensional submanifolds was obtained in joint work of the author with Baouendi and Ebenfelt [2].

Theorem 7.1 ([2]). *Let $M, M' \subset \mathbb{C}^N$ be two real-algebraic generic submanifolds of the same dimension through* 0. *Assume that M is connected and of finite type and holomorphically nondegenerate at some point. Then if $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is a germ of a holomorphic mapping with $H(M) \subset M'$ satisfying* $\mathrm{Jac}\, H \not\equiv 0$ *then H is complex-algebraic.*

If the condition that $\mathrm{Jac}\, H \not\equiv 0$ is not assumed, or if M and M' are generic submanifolds in complex spaces of different dimensions, a stronger assumption must be imposed on the target space. For the case of strongly pseudoconvex hypersurfaces of different dimensions, see the work of Huang [37], see also [10]. One of the most general results in this direction is due to Zaitsev:

Theorem 7.2 (Zaitsev [51]). *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-algebraic submanifolds through* 0. *Then all (germs at* 0 *of) local holomorphic maps $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ with $H(M) \subset M'$ are complex-algebraic if and only if the following are satisfied:*

(i) *M is generic and of finite type on a dense subset;*

(ii) *M' contains no analytic discs.*

8. Concluding remarks

Although I have attempted to survey a number of recent results in CR geometry that make use of the Segre mappings (as defined in Section 2), I have omitted a number of other interesting questions to which this method has also been applied. In particular, I would like to mention one such area of current research, namely the study of the convergence of formal mappings sending one generic submanifold into another. Here "formal" means that the mappings consist of formal power series, rather than convergent ones. In addition to the method of Segre mappings, the celebrated Artin Approximation Theorem [1] has been an important tool in this area of research. For some recent results in this direction I refer the reader to [7], [44], [45], [41], [13], [17], [43]. With these references, as well as with all the other references given here, I apologize in advance for any omissions.

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