L^2 ESTIMATES FOR THE BOUNDARY LAPLACIAN OPERATOR ON HYPERSURFACES

By ALAIN GRIGIS and LINDA PREISS ROTHSCHILD*

1. Introduction. We prove a necessary condition for maximal estimates for the boundary Laplacian $\square_b$ on a smooth hypersurface M in C^{n+1}, not necessarily pseudoconvex. For certain classes of hypersurfaces we show that our condition is also sufficient.

We let $Z_1, Z_2, \ldots, Z_n$ be an orthonormal basis of holomorphic tangential vector fields on M, so that the Z_i's together with $\overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_n, T$ form a local orthonormal frame for the tangent bundle. If $\bar\partial_b$ is the tangential Cauchy Riemann operator acting from $(0, q)$ forms to $(0, q + 1)$ forms, then $\square_b$ is defined as the Laplacian on q forms $\bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$, where * denotes the adjoint with respect to the fixed metric. See e.g. [2] and [3] for precise definitions and useful calculations. Then $\square_b$ is a second order operator whose leading term is the scalar operator

$$\left(\sum_{j=1}^{n} X_j^2 + Y_j^2\right)\cdot I,$$

where $Z_j = X_j + iY_j$, and I denotes the identity operator on $(0, q)$ forms. As in [6] we shall say that $\square_b$ satisfies *maximal estimates* (or $\square_b$ is *maximally hypoelliptic*) in an open set U if there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \|X_j^2 u\| + \sum_{j=1}^{n} \|Y_j^2 u\| \le C(\|\square_b u\| + \|u\|) \tag{1.1}$$

for all $u \in C_0^\infty(U)$.

Kohn [7] introduced the following condition $Y(q)$ for hypoellipticity of $\square_b$ with loss of one derivative.

Manuscript received June 1, 1985; revised September 26, 1985.
*Partly supported by a grant from the NSF.
American Journal of Mathematics 110 (1988), 577-593.

The Levi form is the Hermitian form defined by choosing τ dual to T and setting

$$\mathcal{L}_\tau(Z_i, Z_j) = \frac{1}{2} i\langle \tau, [Z_i, \bar{Z}_j] \rangle. \tag{1.2}$$

Let n_+, n_0, n_- be the number of positive, zero, and negative eigenvalues of the Levi form, with multiplicity. With this notation we write

$$Y(q) \Leftrightarrow q \notin \{n^+, n^+ + 1, \ldots, n^+ + n_0\} \cup \{n^-, n^- + 1, \ldots, n^- + n_0\}$$

Rothschild and Stein [13] proved that if $Y(q)$ is satisfied at any point of an open set U, $\Box_b$ satisfies maximal estimates in U.

To formulate our condition, we first give an equivalent version of $Y(q)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the Levi matrix computed with multiplicity. Define the nonnegative numbers σ^+ and σ^- by

$$\sigma^+ + \sigma^- = \sum_{j=1}^{n} |\lambda_j|, \qquad \sigma^+ - \sigma^- = \sum_{j=1}^{n} \lambda_j.$$

Denote by S_q any of the $\binom{n}{q}$ sums $\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_q}$, $1 \leq j_1 < j_2 \cdots < j_q \leq n$. We have

$$Y(q) \Leftrightarrow -\sigma^- < S_q < \sigma^+, \tag{1.3}$$

for any S_q.

We may now state our condition. We shall say that M satisfies $D(q)$ in U if for any compact $K \subset U$ there exists a constant $\epsilon_k > 0$ such that

$$-\sigma^- + \epsilon_K(\sigma^+ + \sigma^-) \leq S_q \leq \sigma^+ - \epsilon_K(\sigma^+ + \sigma^-) \tag{1.4}$$

for any S_q at any point of K. This definition was inspired by that of Derridj [1] for the case $q = 1$. In Section 2 we shall show that $D(q)$ is satisfied if $\Box_b$ satisfies the maximal estimate (1.1). This generalizes an example of the second author [12]. (See also Maire [11].) In Section 3 we prove that for a restricted class of hypersurfaces $D(q)$ is sufficient for maximal hypoellipticity.

Remark. Since $\sigma^+ + \sigma^-$ is a norm of the Levi matrix $D(q)$ reduces to $Y(q)$ at any point where the Levi matrix is not identically vanishing.

These results have subsequently been obtained also by Helffer-Nourrigat [7], who have in fact proved a more general version of our sufficiency theorem, using their more general set-up. Our proof is self-contained and therefore of independent interest, we hope.

We wish to thank the referee for his many suggestions.

2. A necessary condition for maximal hypoellipticity. We shall prove the following.

THEOREM 1. *Let M be a smooth real hypersurface of $\mathbf{C}^{n+1}$ and $m_0 \in M$. Then if $\Box_b^q$ satisfies the maximal estimate* (1.1) *near m_0 the eigenvalues of the Levi form must satisfy $D(q)$ in a neighborhood of m_0.*

We shall prove Theorem 1 through a number of reductions very similar to those used by Helffer-Nourrigat [5] and [6]. However, our proof does not rely on their general machinery.

Now fix a point m_1 near m_0, so that $\Box_b^q$ satisfies (1.1) in a neighborhood of m_1 also. As in [6] and [13] we shall choose local coordinates $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), t$ around m_1 so that $m_1 \leftrightarrow (0, 0, 0)$, and define the dilations $\delta_r(x, y, t) = (r \cdot x, r \cdot y, r^2 t)$. These extend to a notion of homogeneity for polynomials by writing

$$\delta_r(x^\alpha y^\beta t^n) = r^{|\alpha| + |\beta| + 2n} x^\alpha y^\beta t^n \tag{2.1}$$

where α, β are multi-indices. We write $\mathcal{O}^i$ for any polynomial whose Taylor series around m_1 begins with polynomials of homogeneous degree $\geq i$.

The following is proved in [6] (see also Rothschild-Stein [13, Section 19].)

LEMMA 2.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Levi form at m_1. Assume at least one $\lambda_j(m_1) \neq 0$. Then one may choose an orthonormal basis $W_1, W_2, \ldots, W_n$ for the tangential holomorphic vector fields and local coordinates (x, y, t) so that if $W_j = U_j + iV_j$ then*

$$U_j = \frac{\partial}{\partial x_j} - \frac{\lambda_j}{2} y_j \frac{\partial}{\partial t} + \mathcal{O}^1 \frac{\partial}{\partial x} + \mathcal{O}^1 \frac{\partial}{\partial y} + \mathcal{O}^2 \frac{\partial}{\partial t} \tag{2.3}$$

$$V_j = \frac{\partial}{\partial y_j} + \frac{\lambda_j}{2} x_j \frac{\partial}{\partial t} + \mathcal{O}^1 \frac{\partial}{\partial x} + \mathcal{O}^1 \frac{\partial}{\partial y} + \mathcal{O}^2 \frac{\partial}{\partial t}. \tag{2.4}$$

Now we may apply the dilations δ_r to any vector field X by using (2.1) and setting

$$\delta_r X = \sum_j r^{-1}\left((\delta_r f_j)\frac{\partial}{\partial x_j} + (\delta_r g_i)\frac{\partial}{\partial y_i}\right) + r^{-2}(\delta_r h)\frac{\partial}{\partial t}$$

if

$$X = \Sigma\left(f_j\frac{\partial}{\partial x_j} + g_j\frac{\partial}{\partial y_j}\right) + h\frac{\partial}{\partial t},$$

with $f_j, g_j, h \in C^\infty(\mathbf{R}^{2n+1})$. Let

$$\tilde{U}_j = \lim_{r\to 0} r(\delta_r U_j) \tag{2.5}$$

$$\tilde{V}_j = \lim_{r\to 0} r(\delta_r V_j) \tag{2.6}$$

$$\tilde{\Box}_b = \lim_{r\to 0} r^2(\delta_r \Box_b). \tag{2.7}$$

LEMMA 2.8. *If for some $C > 0$ there exists a neighborhood Ω of m_1 such that*

$$\Sigma\,\|U_j^2 u\| + \Sigma\,\|V_j^2 u\| \le C(\|\Box_b u\| + \|u\|) \tag{2.8}$$

for all $u \in C_0^\infty(\Omega)$, then also

$$\Sigma\,\|\tilde{U}_j^2 v\| + \Sigma\,\|\tilde{V}_j^2 v\| \le C\|\tilde{\Box}_b v\| \tag{2.9}$$

for all $v \in C_0(\mathbf{R}^{2n+1})$ with the same constant C.

Proof. Use (2.8) with the function $u = h \circ \delta_{r^{-1}}$ which, for r sufficiently small, has support in Ω. Multiply both sides by r^2 to obtain

$$r^2(\Sigma\,\|U_j^2(h \circ \delta_{r^{-1}})\| + \Sigma\,\|V_j^2(h \circ \delta_{r^{-1}})\|) \le C(\|r^2\Box_b(h \circ \delta_{r^{-1}})\| + \|r^2(h \circ \delta_{r^{-1}})\|). \tag{2.10}$$

Now $\tilde{U}_j^2(h \circ \delta_{r^{-1}}) = r^{-2}(\tilde{U}_j^2 h) \circ \delta_{r^{-1}}$ with similar equations holding for $\tilde{V}_j^2$ and $\tilde{\Box}_b$. Next, it is easy to check using (2.3) and (2.4) that

$$(r^{2n+2})^{-1/2}\|r^2(U_j^2 - \tilde{U}_j^2)(h \circ \delta_{r^{-1}})\| \to 0 \quad \text{as} \quad r \to 0, \tag{2.11}$$

with analogous statements for $V_j^2 - \tilde{V}_j^2$ and $\square_b - \tilde{\square}_b$. Now the proof of Lemma 2.8 is complete if we multiply both sides of (2.10) by $(r^{2n+2})^{-1/2}r^2$ and let r tend to zero.

Now for any $\tau \in \mathbf{R}-\{0\}$ we let $\hat{U}_j = (\partial/\partial x_j)$, $\hat{V}_j = (\partial/\partial y_j) + i\lambda_j x_j \tau$ and $\hat{\square}_b$ be the operator on $(0, q)$ forms obtained by substituting $\hat{U}_j$ and $\hat{V}_j$ for U_j and V_j in the formula for $\square_b$. (For this we need a more specific choice of the form of $\square_b$. See (3.18) below and the remarks there.)

LEMMA 2.12. *If* (2.8) *holds for all* $v \in C_0^\infty(\mathbf{R}^{2n+1})$ *then for the same constant* C,

$$\Sigma \|\hat{U}_j^2 w\| + \Sigma \|\hat{V}_j^2 w\| \le C\|\hat{\square}_b w\| \tag{2.13}$$

for all $w \in C_0^\infty(\mathbf{R}^{2n})$.

Proof. We first make a change of coordinates so that $\tilde{U}_j$, $\tilde{V}_j$ become $\tilde{U}_j=(\partial/\partial x_j')$, $\tilde{V}_j=(\partial/\partial y_j')+\lambda_j x_j(\partial/\partial t')$. Now $\hat{U}_j$ and $\hat{V}_j$ are obtained by taking the Fourier transform in t' and dropping the primes. Then (2.13) follows by an elementary argument such as that given in Helffer-Nourrigat [5], Lemma 2.2]. Now let

$$\hat{\hat{U}}_j=\frac{\partial}{\partial x_j}, \qquad \hat{\hat{V}}_j=i\lambda_j x_j \tau \tag{2.14}$$

and write $\hat{\hat{\square}}_b$ for the operator constructed by replacing U_j and V_j by $\hat{\hat{U}}_j$ and $\hat{\hat{V}}_j$ in the formula for $\square_b$ (with the choices made as above.)

LEMMA 2.15. *If* (2.13) *holds then also*

$$\Sigma \|\hat{\hat{U}}_j^2 f\| + \Sigma \|\hat{\hat{V}}_j^2 f\| \le C(\|\hat{\hat{\square}}_b f\|^2) \tag{2.16}$$

for all $f \in C_0^\infty(\mathbf{R}_x^n)$.

Proof. That (2.13) implies (2.16) follows by a simple argument as in Helffer-Nourrigat [5].

We may now prove Theorem 1. For any m_1 sufficiently close to m_0, maximal hypoellipticity implies the conclusion of Lemma (2.15), where the λ_j in (2.14) are the eigenvalues of the Levi form at m_1, and τ is taken to be ± 1. The vector $(e^{-\Sigma(\lambda_j x_j)^2/2}, 0, \ldots, 0)$ is a common eigenvector of $(-\Sigma \hat{\hat{U}}_j^2 + \hat{\hat{V}}_j^2)$ and $\hat{\hat{\square}}_b$ corresponding to the lowest eigenvalue of each. Indeed note that $-\Sigma \hat{\hat{U}}_j^2 + \hat{\hat{V}}_j^2$ and $\hat{\hat{\square}}_b$ are the same modulo *a* scalar matrix. Now since

$$\|(\Sigma \hat{\hat{U}}_j^2 + \hat{\hat{V}}_j^2)f\| \le \Sigma \|\hat{\hat{U}}_j^2 f\| + \Sigma \|\hat{\hat{V}}_j^2 f\|$$

we have, by (2.16), with the same constant C,

$$|\rho_1| \le C|\rho_2|$$

where ρ_1 is the lowest eigenvalue of $-\Sigma\,(\hat{U}_j^2 + \hat{V}_j)$ and ρ_2 is the lowest eigenvalue of $\hat{\Box}_b$. By calculating these for $\tau = \pm 1$ (see e.g. [3] or [4]) we obtain

$$(2.17)\qquad \frac{1}{2}(\sigma^+ + \sigma^-)|\tau| \le C\left(\frac{1}{2}(\sigma^+ + \sigma^-)|\tau| + S_q\tau - \frac{1}{2}(\sigma^+ - \sigma^-)\tau\right)$$

where C is the same constant as that which occurs in Lemmas (2.8) to (2.15) and S_q is any one of the sums $\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_q}$, $0 \le j_q \le n$. From (2.17) for $\tau = \pm 1$ we obtain

$$S_q \ge -\sigma^- + \frac{1}{2C}(\sigma^+ + \sigma^-), \qquad \text{and}$$

$$S_q \le \sigma^+ - \frac{1}{2C}(\sigma^+ + \sigma^-)$$

from which $D(q)$ follows with $\epsilon = 1/2C$.

3. Sufficiency of $D(q)$ for maximal hypoellipticity of $\Box_b$ on some hypersurfaces. Our main result is the following.

THEOREM 2. *Let M be the hypersurface in $\mathbf{C}^{n+1}$ given by* $\operatorname{Im} z_{n+1} = p(x_1, \ldots, x_n, y_1, \ldots, y_n)$ *where p is a homogeneous polynomial. Suppose that after a linear change of coordinates one has $(x, y) = (x_1', \ldots, x_K', y_1', \ldots, y_J', x_1'', \ldots, x_{n-K}'', y_1'', \ldots, y_{n-J}'')$ so that $p(x, y) = \tilde{p}(x_1', \ldots, x_K', y_1', \ldots, y_J')$ where $\tilde{p}$ is a nonzero homogeneous polynomial whose Levi form $\partial_{z\bar{z}}\tilde{p}$ has all nonnegative eignevalues which simultaneously vanish only where $x' = 0$ and $y' = 0$. Then $\Box_b^q$ satisfies maximal estimates if $D(q)$ is satisfied.*

Remark. In the context of Theorem 2, $D(q)$ is equivalent to $Y(q)$ being satisfied outside $x' = 0, y' = 0$.

Example 3.1. Let $M = \{\operatorname{Im} z_{n+1} = (\Sigma_{k=1}^n y_k^2)^{2r}\}$. Then one may take $x' = 0, x'' = x, y' = y, y'' = 0$. Since $D(q)$ is satisfied if $q \ne 0, n$, $\Box_b^q$ is maximally hypoelliptic.

In what follows we shall always assume that $\tilde{p}$ is homogeneous of degree $r > 2$; the case $r = 2$ corresponds to a nondegenerate Levi form for which the result is known ([3], [13]). Our proof will use some ideas of Helffer-Nourrigat ([5], [6]). Note that the origin is a point of type r.

We introduce the coordinates (x', x'', y', y'', t) on M by writing $t = \operatorname{Re} Z_{n+1}$, and write $P^{\wedge}(\tau)$ and $P^{\hat{\wedge}}(\gamma, \tau)$ respectively for the partial Fourier transforms in t and (x'', y'', t) for a differential operator P whose coefficients are independent of x'', y'' and t.

Now let $X_j, Y_j, j = 1, 2, \ldots, n$ be as in Section 1, and let A_j denote either X_j or Y_j. In the coordinates (x, y, t) given above we may write

$$X_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial y_j}\frac{\partial}{\partial t}\right), \quad \text{and}$$

$$Y_j = \frac{1}{2}\left(\frac{\partial}{\partial y_j} + \frac{\partial p}{\partial x_j}\frac{\partial}{\partial t}\right),$$

$j = 1, 2, \ldots, n$.

We write H for the closure of $C_0(M)$ in the norm

$$\|f\|_{H^\ell} = \sum_{s \le \ell} \|A_{j_1} A_{j_2} \cdots A_{j_s} f\|_{L^2}.$$

Similarly we define $H^\ell(\tau)$ in the same way, but with A_j replaced by $A_j^{\wedge}(\tau)$ and $H^\ell(\gamma, \tau)$ for the space obtained when A_j is replaced by $A_j^{\hat{\wedge}}(\gamma, \tau)$. We write $\Delta = \sum_{j=1}^{n} X_j^2 + Y_j^2$, and we shall prove there exists $C > 0$ such that

$$\|\Delta u\| \le C(\|\Box_b u\| + \|u\|) \tag{3.3}$$

for all $u \in C_0^\infty(M)$. Since Δ is known to satisfy maximal estimates [13], Theorem 3.1 follows from (3.3).

Before beginning the proof of Theorem 3, we note that because a basis for the antiholomorphic vector fields takes a particularly simple form here we can choose the dual frame to be constant i.e. $\bar{\partial}_b$ closed. Hence $\Box_b^q$ may be written in the form (3.18) below with no error term.

We begin with the following simple observation.

Lemma 3.4. *For any $\tau \in R$ there exists a constant C_τ such that*

$$\|\Delta^{\wedge}(\tau) v\| \le C_\tau(\|\Box_b^{\wedge}(\tau) v\| + \|v\|) \tag{3.5}$$

for all $v \in C_0^\infty(\mathbf{R}^{2n})$ with $\operatorname{supp} v \subset \{z\colon |x'| + |y'| < 2\}$.

Proof. $\Box_b(\tau) = -\Delta^\wedge(\tau)\cdot I + m^q\cdot\tau$ where m^q is a matrix whose entries are polynomials in x', y', from which the result is obvious.

PROPOSITION 3.6. *There exists $C > 0$ such that*

$$\|\Delta u\| \le C(\|\Box_b u\| + \|u\|.) \tag{3.7}$$

for all $u \in C_0(M_1)$, where $M_1 = \{(z,t): |x'| + |y'| > 1\}$.

The proof of Proposition 3.6, which uses the fact that away from $x' = 0$ and $y' = 0$, M has nondegenerate Levi matrix, will be given in Section 5. We now assume Proposition 3.6 and continue the proof of Theorem 2.

LEMMA 3.8. *There exists C_τ' such that for all $\gamma = (\xi'', \eta'')$*

$$\|\hat{\Delta}^\wedge(\gamma,\tau)w\| \le C_\tau'(\Box_b^\wedge(\gamma,\tau)w\| + \|w\|) \tag{3.9}$$

for all $w \in C_0^\infty(R^{J+K})$.

Proof. It follows from (3.5) that

$$\|\hat{\Delta}^\wedge(\gamma,\tau)w\| \le C_\tau(\|\Box_b^\wedge(\gamma,\tau)w\| + \|w\|) \tag{3.10}$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$ with supp $w \subset \{|x'| + |y'| < 2\}$. (See e.g. Helffer-Nourrigat [5, Lemma 2.2] for the details of an elementary argument.) Similarly, from (3.7) one obtains

$$\|\hat{\Delta}^\wedge(\gamma,\tau)w\| \le C(\|\Box_b^\wedge(\gamma,\tau)w\| + \|w\|) \tag{3.11}$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$ with supp $w \subset \{|x'| + |y'| > 1\}$. Now (3.9) follows from (3.10) and (3.11).

We need to eliminate the last term on the right hand side in (3.9).

LEMMA 3.12. *For any $\tau \neq 0$ and any $\gamma = (\xi'', \eta'')$ the inclusion*

$$H^r(\gamma,\tau) \hookrightarrow L^2(\gamma,\tau) \tag{3.13}$$

is compact if $r > 2$.

Proof. Since $\partial_{z\bar{z}}\tilde{p} = a_{ij}$ is homogeneous of degree $r - 2$ and nonvanishing away from $x' = 0, y' = 0$, it follows that $\Sigma\,|a_{ij}| \ge ((|x'| + |y'|)^{r-2}$. Then

$$\|w\|_{H^2(\gamma,\tau)} \geq C'\|(|x'| + |y'|)^{-2}w\|_{L^2}. \tag{3.14}$$

On the other hand on any bounded open set $\Omega \subset \mathbf{R}^n$ the inclusion (3.13) is compact, by Rellich's Lemma, since

$$\|w\|_{H^2(\gamma,\tau)} \geq C''' \Sigma \|A_j^\wedge(\gamma, \tau)w\|. \tag{3.15}$$

Now the compactness of (3.13) may be proved by writing $w = \phi_N w + (1 - \phi_N)w$, where $\operatorname{supp} \phi_N \subset \{|x'| + |y| < N\}$, and using a standard diagonal sequence type argument.

In order to eliminate the second term on the right in (3.9) by functional analysis type arguments we need to know that $\hat{\Box}_b^q(\gamma, \tau)$ is injective.

LEMMA 3.16. *For any (γ, τ), $\tau \neq 0$, the operator $\hat{\Box}_b^q(\gamma, \tau)$ extends to a bounded linear mapping from $H^2(\gamma, \tau)$ to $L^2(\mathbf{R}^{J+K})$ which is injective if $D(q)$ is satisfied.*

Proof. Recall that $([Z_j, \overline{Z}_k]) = i\mathcal{L}^{(1)}(\partial/\partial t)$, where $\mathcal{L}^{(1)}$ is the Levi matrix. Now, a straightforward calculation (see e.g. Grigis [4]) shows that:

$$\Box_b^q = -(\Sigma Z_j\overline{Z}_j)\cdot I + i\mathcal{L}^{(q)}\frac{\partial}{\partial t}, \tag{3.17}$$

where $\mathcal{L}^{(q)}$ is a matrix whose $\binom{n}{q}$ eigenvalues are all the possible sums of q eigenvalues of $\mathcal{L}^{(1)}$. (Recall that the dual frame to $\overline{Z}_j$ is assumed to be $\overline{\partial}_b$ closed. Hence there is no error term in the above.)

For any (γ, τ), $((-\Sigma_{j=1}^n Z_j\overline{Z}_j)^\wedge(\gamma, \tau)w, w) \geq 0$. If $\tau < 0$, then

$$\left(\left(i\mathcal{L}^{(q)}\frac{\partial}{\partial t}\right)^\wedge w, w\right) = -\tau(\mathcal{L}^{(q)}w, w) > 0,$$

since the eigenvalues of $\mathcal{L}^{(1)}$ are nonnegative and vanish simultaneously only where $x' = 0$ and $y' = 0$. Hence $(\mathcal{L}^{(q)}w, w) > 0$ if $w \neq 0$.

If $\tau > 0$ we write, using (3.17),

$$\Box_b^q = -\sum_{j=1}^n \overline{Z}_jZ_j - \frac{1}{2}\Sigma\,[\overline{Z}_j, Z_j]\cdot I + i\mathcal{L}^q\frac{\partial}{\partial t}. \tag{3.18}$$

Now $\Sigma_{j=1}^{n} [Z_j, \overline{Z}_j] = i(\Sigma_{j=1}^{n} \lambda_j)(\partial/\partial t)$, where the λ_j are the eigenvalues of the Levi form. Hence the eigenvalues of the matrix $(-\Sigma_{j=1}^{n} [Z_j, \overline{Z}_j])\hat{}(\gamma, \tau) - \mathfrak{L}^{(q)}\tau$ are all sums of the form $(\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{n-q}})\tau$ of $n - q$ eigenvalues of the Levi form, times τ. Since $\tau > 0$ we are done.

LEMMA 3.19. *If* (3.11) *holds and* $\hat{\Box}_b(\gamma, \tau)$ *is injective on* $H^2(\gamma, \tau)$ *then there exists* $C_{(\gamma,\tau)}$ *such that*

$$\|\hat{\Delta}(\gamma, \tau)w\| \leq C_{(\gamma,\tau)}\|\hat{\Box}_b(\gamma, \tau)w\| \tag{3.20}$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$. *In particular*, (3.20) *holds if* $D(q)$ *is satisfied.*

Proof. Since $H^2(\gamma, \tau)$ is compact in $L^2(\gamma, \tau)$ and $\hat{\Box}_b(\gamma, \tau)$ is injective, one may apply Peetre's Lemma (see e.g. [12, Theorem 5.17]) to eliminate the second term on the right in (3.11). This proves the first statement. For the second, note that if $D(q)$ holds, the above lemmas show that the hypotheses for the first statement in Lemma 3.19 are satisfied.

LEMMA 3.21. *Suppose for each* τ *there exists* C_τ *such that*

$$\|\hat{\Delta}(\tau)w\| \leq C_\tau\|\hat{\Box}_b(\tau)w\| \tag{3.22}$$

for all $w \in C_0^\infty(\mathbf{R}^{2n})$. *Then there exists* $C > 0$ *so that*

$$\|\hat{\Delta}(\tau)u\| \leq C\|\hat{\Box}_b(\tau)u\|. \tag{3.23}$$

Proof. This follows from the homogeneity of the operators $\Box_b$ and Δ, by using a change of variables.

Finally, if (3.23) holds with C independent of τ, then we may conclude

$$\|\Delta v\| \leq C\|\Box_b v\| \tag{3.24}$$

for all $v \in C_0^\infty(M)$, which, since Δ itself satisfies maximal estimates [13], will prove (3.3) and hence Theorem (3.1). We shall prove (3.22) in Section 4 and give the proof of Proposition 3.6 in Section 5.

4. Estimates independent of the dual variables to x'', y''. We shall prove the following.

PROPOSITION 4.1. *If* $D(q)$ *is satisfied then there is a constant* C_τ, *independent of* γ, *such that for each* $\gamma = (\xi'', \eta'')$ *one has*

(4.2) $$\|\Delta\hat{}(\gamma, \tau)w\| \leq C_\tau \|\hat{\Box}_b(\gamma, \tau)w\|$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$.

Since (3.22) is an immediate corollary of (4.2), the proof of Proposition 4.1 will complete that of Theorem 3.1, modulo Proposition 3.6.

In the spirit of [5], we shall show first that the estimate (4.2) holds for $|\gamma|$ large, and then show that (3.20) can be perturbed in γ.

LEMMA 4.3. *Let* $\tau \neq 0$ *be fixed. For any* $\epsilon > 0$ *there exists* N, *depending only on* τ, *such that if* $|\xi''| + |\eta''| > N$ *and* $\gamma = (\xi'', \eta'')$ *then*

$$\|w\|_{L^2} \leq \epsilon \|w\|_{H^{1/2}(\gamma,\tau)}$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$.

Proof. Recall that $\hat{A}_q(\gamma, \tau) = i(\gamma_q + p_q\tau)$, where $\gamma_j = \xi_j$ or η_j, and p_j is homogeneous of degree $r - 1$ in x' and y'. Then

(4.4) $$\|w\|^2_{L^2} + \|w\|^2_{H^{1/2}(\gamma,\tau)}$$
$$\geq \int |\gamma_q + p_q\tau| w\bar{w}\,dx'dy' \geq \int |\gamma_q| w\bar{w}\,dx'dy'$$
$$- \int |p_q\tau| w\bar{w}\,dx'dy'.$$

Now if the Levi form $\mathcal{L}\hat{}_{(\tau)}$ is $(a_{ij}\tau)$ then

(4.5) $$\|w\|^2_{H^{1/2}(\gamma,\tau)} \geq C \sum_{i,j} \|a_{ij}w\|^2 |\tau|^2,$$

for all $w \in C_0^\infty$.

Furthermore, since $\mathcal{L}\hat{}(\tau)$ is nonvanishing for $|x'| + |y'| \neq 0$, we may find N_1 and C so that if $|x'| + |y'| > N_1$, then

$$C \sum a_{ij}^2 |\tau| - |p_q| > 0$$

for all q. (Recall that degree of p_q is $r - 1$, while that of a_{ij}^2 is $2(r - 2)$.) Now, using this as well as (4.5) in (4.4) we conclude

$$(C+2)\|w\|^2_{H^2(\gamma,\tau)} \geq \int |\gamma_q| w\bar{w}dx'dy' - \int |p_q||\tau| w\bar{w}dx'dy'$$

$$|x'|^2 + |y|^2 < N_1.$$

Now if $|p_q||\tau| < K$ for $|x'|^2 + |y'|^2 < N_1$, then we obtain

$$(C+2)\|w\|^2_{H^2(\gamma,\tau)} \geq \int (|\gamma_q| - K)w\bar{w}dx'dy' = (|\gamma_q| - K)\|w\|^2.$$

Now if $|\xi''| + |\eta''| > N$ we may assume that for some q $|\gamma_q|$ is also large, so the Lemma is proved.

LEMMA 4.6. *For each τ there exists C_τ and $N(\tau)$ such that*

$$\|\Delta^{\wedge}(\gamma,\tau)w\| \leq C_\tau'\|\Box^{\wedge}(\gamma,\tau)w\| \tag{4.7}$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$, and all γ for which $|\gamma| > N(\tau)$.

Proof. This is immediate from (3.12) and Lemma (4.3).

Finally, to handle the case where $|\gamma|$ is small, we shall show that the constant $C_{(\gamma,\tau)}$ in the inequality (3.20) may be chosen to be locally bounded in γ.

LEMMA 4.8. *Let $\tau \neq 0$ be fixed. There exists C_τ such that for any A_j, A_k and γ, γ' with $|\gamma - \gamma'| < 1$ one has*

(4.9)

$$\|(\hat{A}_j(\gamma,\tau)\hat{A}_k(\gamma,\tau) - \hat{A}_j(\gamma',\tau)\hat{A}_k(\gamma',\tau))w\| \leq C_\tau|\gamma - \gamma'|\,\|w\|_{H^1(\gamma,\tau)}.$$

Proof. Since $\hat{A}_j(\gamma,\tau)$ is linear in γ we have $|\hat{A}_j(\gamma,\tau) - \hat{A}_j(\gamma',\tau)| \leq C_\tau|\gamma - \gamma'|$. Also, $\|\hat{A}_j(\gamma,\tau)w\| \leq \|w\|_{H^1(\gamma,\tau)}$ by definition of H^1.

Also,

$$\begin{aligned}\|\hat{A}_j(\gamma',\tau)w\| &\leq \|(\hat{A}_j(\gamma',\tau) - \hat{A}_j(\gamma,\tau))w\| + \|\hat{A}_j(\gamma,\tau)w\| \\ &\leq C_\tau|\gamma - \gamma'|\,\|w\|_{H^1(\gamma,\tau)} + \|w\|_{H^1(\gamma,\tau)}.\end{aligned}$$

Now (4.9) is easily proved by bounding the left-hand side by

$$\|[\hat{A}_j(\gamma, \tau)\hat{A}_k(\gamma, \tau) - \hat{A}_j(\gamma', \tau)\hat{A}_k(\gamma, \tau)]w\|$$
$$+ \|[\hat{A}_j(\gamma', \tau)\hat{A}_k(\gamma, \tau) - \hat{A}_j(\gamma', \tau)\hat{A}_k(\gamma', \tau)]w\|.$$

We may now complete the proof of (4.2). First we show that for any (γ, τ) there exists $C'_{\gamma\tau}$ such that

$$\|w\|_{H^1(\gamma,\tau)} \le C'_{\gamma,\tau}\|\hat{\Box}_b(\gamma, \tau)w\| \tag{4.10}$$

for all $w \in C_0^\infty(\mathbf{R}^{J+K})$. Indeed, since

$$\|w\|_{H^1(\gamma,\tau)} \le C''_{\gamma,\tau}(\|\hat{\Delta}(\gamma, \tau)w\| + \|w\|) \le C''_{\gamma,\tau}(\|\hat{\Box}_b(\gamma, \tau)w\| + \|w\|$$

(by 3.20), (4.10) is proved in the same way as Lemma 3.19. By Lemmas 3.19 and 4.8 we have (4.2) provided that γ varies in a compact set. On the other hand, from Lemma 4.6 it follows that (4.2) also holds for $|\gamma|$ large.

5. Uniform maximal estimates away from $x' = 0$, $y' = 0$. We shall prove Proposition 3.6 here.

LEMMA 5.1. *There exists $C > 0$ such that*

$$\|\Delta u\|_{L^2} \le C(\|\Box_b u\|_{L^2} + \|u\|_{L^2}) \tag{5.2}$$

for all $u \in C_0^\infty(\Omega)$, where

$$\Omega = \left\{\frac{1}{2} < |x'| + |y'| < 4, 0 \le |x''| < 1,\right.$$
$$\left. 0 \le |y''| < 1, 0 \le |t| < 1\right\}.$$

Proof. Since the Levi form is nonvanishing near Ω and the condition $D(q)$ then implies $Y(q)$, it follows from [13] that $\Box_b$ satisfies maximal estimates i.e. (5.2) holds on any set with compact closure, which proves the lemma.

Next we shall show that (5.2) holds on the unbounded annulus $\Omega' = \{1/2 < |x'| + |y'| < 4\}$. For any function $f = f(x'', y'', t)$, and any $w_j = (x_j'', y_j'', t_j) \in \mathbf{R}^{2n-J-K+1}$ let f_{w_j} denote the translate of f by w_j i.e.

$$f_{w_j}(x'', y'', t) = f(x'' + x_j'', y'' + y_j'', t + t_j).$$

The following is classical and easy to prove.

LEMMA 5.3. *There exists* $\phi \in C_0^\infty(\mathbf{R}^{2n-J-K+1})$ *with* $0 \leq \phi \leq 1$, supp $\phi \subset \{0 \leq |x''| < 1, 0 < |y''| < 1, 0 < |t| < 1\}$ *and a sequence* $w_j = (x_j'', y_j'', \tau_j) \in \mathbf{R}^{2n-J-K+1}$ *such that*

$$\sum_{j=1}^{\infty} (\phi_{w_j}(x'', y'', t))^2 = 1, \tag{5.4}$$

$$\sup_{|\alpha| \leq 2} \sum_{j=1}^{\infty} |\phi_{w_j}^{(\alpha)}(x'', y'', t)| \leq C \tag{5.5}$$

for some constant $C > 0$, all $(x'', y'', t) \in \mathbf{R}^{2n-J-K+1}$.

LEMMA 5.6. *There exists* $C' > 0$ *such that*

$$\|\Delta u\|_{L^2} \leq C'(\|\Box_b u\|_{L^2} + \|u\|_{L^2})$$

for all $u \in C_0^\infty(\Omega')$, *where*

$$\Omega' = \left\{(z, t) \in M: \frac{1}{2} < |x'| + |y'| < 4\right\}.$$

Proof. First note that for any $w_0 = (x_0'', y_0'', t_0) \in \mathbf{R}^{2n-J-K+1}$ we obtain

$$\|\Delta u\|_{L^2} \leq C\|\Box_b u\|_{L^2} + \|u\|_{L^2} \tag{5.7}$$

for all $u \in C_0^\infty(\Omega_{w_0})$, *where*

$$\Omega_{w_0} = \left\{(z, t) \in M: \frac{1}{2} < |x'| + |y'| < 4, |t - t_0| < 1, \right.$$

$$\left. |x'' - x_0''| < 1, |y'' - y_0''| < 1\right\}.$$

Indeed, $\|f_{-w_0}\| = \|f\|_{L^2}$, and Δ and $\Box_b$ are invariant by translation by (x_0'', y_0'', t), since these variables do not appear as coefficients in the vector fields X_j and Y_j.

Now from (5.7) we obtain

$$(5.8) \qquad \|\Delta\phi_{w_j}u\|^2 \leq C_2(\|\Box_b\phi_{w_j}u\|^2 + \|u\|_{L^2}^2)$$

for $w_j = (x_j'', y_j'', \tau_j)$, all $u \in C_0^\infty(\Omega')$. Next, using (5.4), (5.5) and (5.7), we obtain

$$(5.9) \qquad \|\Delta u\|^2 = \sum_j \|\phi_{w_j}\Delta u\|^2 \leq \sum_j \|\Delta\phi_{w_j}u\|^2 + \sum \|[\Delta, \phi_{w_j}]u\|^2$$
$$\leq C_4 \sum_j (\|\Box_b\phi_w u\|^2 + \|\phi_{w_j}u\|^2 + \|[\Delta, \phi_{w_j}]u\|^2.$$

For the third term on the right in (5.9) we note that since the variables x'', y'', t do not appear as coefficients in the X_k and Y_k there exists C such that

$$|W\phi_{w_j}| \leq C \sup_{|s| \leq 1} |\phi^{(s)}(x'', y'', t)|$$

for all $(z, t) \in \Omega'$, where W represents either X_k or Y_k. Hence by (5.4) we obtain

$$(5.10) \qquad \|\Delta u\|^2 \leq C_5(\|\Box_b u\|^2 + \|u\|_{H^1}^2)$$

for all $u \in C_0^\infty(\Omega')$, some $C_5 > 0$.

Next, we consider the family of dilations δ_s on M given by

$$\delta_s(z, t) = (sz, s^r t).$$

From (5.10) we obtain

$$(5.11) \qquad \|\Delta(u \circ \delta_s)\|^2 \leq C_5(\|\Box_b(u \circ \delta_s)\|^2 + \|u \circ \delta_s\|_{H^1}^2)$$

for all $u \in C_0^\infty(\Omega_s')$, where $\Omega_s' = \{(z, t): s/2 < |x'| + |y'| < 4s\}$. Now since $X_j(u \circ \delta_s) = s(X_j u) \circ \delta_s$, and similarly for Y_j, while $\Box_b(u \circ \delta_s) = s^2(\Box_b u) \circ \delta_s$, and similarly for Δ, we obtain from (5.11)

$$\|\Delta u\|^2 \leq C_6(\|\Box_b u\|^2 + s^{-2}\|u\|^2 + s^{-1} \sup \|Wu\|^2)$$

for all $u \in C_0^\infty(\Omega_s')$, where W denotes either X_j or Y_j. In particular we have

$$\|\Delta u\|^2 \leq C_7(\|\Box_b u\|^2 + \|u\|_{H^1}^2) \tag{5.12}$$

for all $u \in C_0^\infty(\Omega_s')$, all $s \geq 1$.

LEMMA 5.13. *There is a sequence of functions* $\psi_i \in C_0^\infty(\mathbf{R}^{J+K}_{x',y'})$, $0 \leq \psi_i(x', y') \leq 1$ *satisfying*

(i) $\operatorname{supp} \psi_i \subset \{i/2 < |x'| + |y'| < 4i\}$

(ii) $\sup_{(z,t)\in M} \sum_{j=1}^{n} |X_j\psi_i(x', y')|^2 + |Y_j\psi_i(x', y')|^2 < \infty$

(iii) $\Sigma\, \psi_i^2(x', y') = 1 \quad$ for $\quad |x'| + |y'| > 1$.

Proof. The construction is classical once we know there exists $C > 0$ so that for all i, j

$$|X_j\psi_i(x', y')| + |Y_j\psi_i(x', y')| \leq C \sup_{|\alpha|\leq 1} |D^\alpha\psi_i(x', y')|.$$

This is clear since ψ_i is a function of (x', y') only, and the only variable coefficients in the X_j and Y_j appear as coefficients of $\partial/\partial t$.

Now we proceed as before. Recall that $M_1 = \{(z, t) \in M: |x'| + |y'| > 1\}$. From (5.12) we have

(5.14)

$$\|\Delta u\|^2 = \Sigma\, \|\psi_i\Delta u\|^2 \leq C_7(\sum_i \|\Box_b\psi_i u\|^2 + \|\psi_i u\|_{H^1}^2 + \|[\psi_i, \Box_b]u\|^2)$$

for all $u \in C_0^\infty(M_1)$. By using Lemma 5.13 we have $\Sigma\, \|\psi_i u\|_{H^1}^2 \leq C_8\|u\|_{H^1}$, for all $u \in C_0^\infty(M_1)$. Similarly,

$$\sum_i \|[\psi_i, \Box_b]u\|^2 \leq C_9\|u\|_{H^1}.$$

Now we obtain the estimate (3.7) from (5.12). Hence Proposition 3.6 is proved, and the proof of Theorem 2 is complete.

Acknowledgment. The authors wish to thank the Institute for Advanced Study for its hospitality.

ECOLE POLYTECHNIQUE
UNIVERSITY OF CALIFORNIA, SAN DIEGO

REFERENCES

[1] M. Derridj, Régularité pour $\bar{\partial}$ dans quelques domaines faiblement pseudo-convexes, *J. Diff. Geom.*, 13 (1978), 559-576.

[2] G. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy-Riemann Complex*, Princeton Univ. Press (1972).

[3] ——— and E. M. Stein, Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.*, 27 (1974), 429-522.

[4] A. Grigis, Propagation des singularités au bord d'ouverts de C^n, *Comm. P.D.E.*, 6 (1981), 6, 689-717.

[5] B. Helffer and J. Nourrigat, Caractérisations des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe nilpotent gradué, *Comm. P.D.E.*, 4(8) (1979), 899-958.

[6] ———, Approximations d'un système de champs de vecteurs et applications à l'hypoellipticité, *Ark. Mat. 17*, 2 (1979), 237-254.

[7] ———, *Hypoellipticité Maximale pour des Opérateurs Polynômes de Champs de Vecteurs*, Birkhauser, Boston (1985).

[8] L. Hörmander, Hypoelliptic operators with double characteristics, *Math. Ann.*, 217, 2 (1975), 175-188.

[9] J. J. Kohn, Boundaries of complex manifolds, *Proc. Conference on Complex Manifolds*, Minneapolis (1964), 81-94.

[10] J. Lions and E. Magènes, *Problèmes aux Limites Nonhomogènes*, I Dunod, Paris (1968).

[11] H. M. Maire, Hypoelliptic overdetermined systems of partial differential equations, *Comm. P.D.E.*, 5 (1980), 331-380.

[12] L. P. Rothschild, Nonexistence of optimal L^2 estimates for the boundary Laplacian operator on certain weakly pseudo-convex domains, *Comm. P.D.E.*, 5(8) (1980), 897-912.

[13] ——— and E. M. Stein, Hypoelliptic differential operators and nilpotent groups, *Acta Math.*, 137 (1976), 247-320.