

LOCAL SOLVABILITY OF LEFT INVARIANT DIFFERENTIAL OPERATORS ON THE HEISENBERG GROUP

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ABSTRACT. Representation-theoretic criteria are given for the local solvability of left invariant differential operators homogeneous under dilations.

In this note we prove that a sufficient condition for the local solvability of a homogeneous left invariant differential operator on the Heisenberg group is that the image of the operator under any infinite dimensional irreducible unitary representation be invertible. This result was conjectured by Rockland [7], and the proof given here is based mainly on his approach.

Let H^n be the Heisenberg group of dimension $2n + 1$; i.e. H^n is the simply connected nilpotent Lie group whose Lie algebra \mathfrak{h}^n is spanned by $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n, T$, satisfying $[X_j, Y_k] = \delta_{jk}$, with all other commutators zero. A differential operator D on H^n is *left invariant* if $D({}^g f) = {}^g(Df)$, for any $g \in H^n$ and any function f ; here ${}^g f$ is the function $x \rightarrow f(gx)$. By the exponential map any $Z \in \mathfrak{h}^n$ may be identified with a left invariant differential operator:

$$(Zf)(g) = \left[\frac{d}{dt} f(g \exp tZ) \right]_{t=0}.$$

Conversely, any left invariant differential operator on H^n may be written as a polynomial in the elements of \mathfrak{h}^n with constant coefficients.

The algebra \mathfrak{h}^n carries a family of automorphisms δ_r , defined for $r > 0$ by

$$\delta_r(X_j) = rX_j, \quad \delta_r(Y_j) = rY_j, \quad j = 1, 2, \dots, n, \text{ and } \delta_r(T) = r^2T.$$

δ_r may be extended to polynomials on \mathfrak{h}^n by defining

$$\delta_r(Z_1 Z_2 \cdots Z_k) = \delta_r(Z_1) \delta_r(Z_2) \cdots \delta_r(Z_k).$$

δ_r is well defined by the Birkhoff-Poincaré-Witt theorem (e.g. [2]). A left invariant differential operator is *homogeneous* of degree d if $\delta_r(D) = r^d D$.

Every irreducible unitary representation π of H^n on a Hilbert space V determines an irreducible representation of \mathfrak{h}^n , also denoted π , by

$$\pi(Z)v = \left[\frac{d}{dt} \pi(\exp tZ)v \right]_{t=0}, \quad Z \in \mathfrak{h}^n, v \in V.$$

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The irreducible unitary representation of H^n are of two types (e.g. [5]):

(i) Finite-dimensional: $V = \mathbb{C}$, and the representations are parametrized by $(a, b) \in \mathbb{R}^{2n}$ with

$$\pi_{(a,b)}(X_j) = ia_j, \quad \pi_{(a,b)}(Y_j) = ib_j, \quad \text{and } \pi(T) = 0.$$

(ii) Infinite-dimensional: $V = L^2(\mathbb{R}^n)$, and the representations are parametrized by $\lambda \in \mathbb{R} - \{0\}$ with

$$\pi_\lambda(X_j) = |\lambda|^{1/2} \frac{\partial}{\partial x_j}, \quad \pi_\lambda(Y_j) = i|\lambda|^{1/2}(\text{sgn } \lambda)x_j,$$

$$j = 1, 2, \dots, n \quad \text{and} \quad \pi_\lambda(T) = i\lambda. \quad (1)$$

A differential operator D is *locally solvable* at a point x_0 if there is a neighborhood U of x_0 such that $DC^\infty(U) \subset C_0^\infty(U)$, where C^∞ and C_0^∞ denote smooth functions and smooth functions with compact support, respectively. For left invariant differential operators it suffices to consider $x_0 = 0$, the identity. The reader is referred to the survey article by Helgason [3] on local solvability for general Lie groups and to [6] or [8] for local solvability of bi-invariant differential operators.

The result may now be stated precisely.

THEOREM. *Let P be a homogeneous left invariant differential operator on the Heisenberg group H^n . Then P is locally solvable if $\pi(P)$ has a bounded left inverse for every infinite dimensional irreducible unitary representation π of H^n .*

LEMMA 1. *It suffices to prove that there exists $N > 0$ such that for every $v \in C_0^\infty(H^n)$*

$$Pu = T^N v \quad (2)$$

has a solution $u \in C^\infty(H^n)$.

PROOF. Since $T = \partial/\partial t$ in suitable coordinates, one may obtain $v_1 \in C^\infty$ satisfying

$$T^N v_1 = f$$

by integrating f N times in t . Now if $U \subset H^n$ has compact closure, choose $v \in C_0^\infty$ such that $v \equiv v_1$ on U . Then if $u \in C^\infty$ satisfies (2),

$$Pu = T^N v_1 = f \quad \text{in } U. \quad (3)$$

We now prove the theorem by constructing a solution of (2), using the Plancherel Theorem for H^n (e.g. [5]). If $\varphi \in C_0^\infty$, and π_λ is an infinite dimensional irreducible unitary representation of H^n , then the operator

$$\pi_\lambda(\varphi) = \int_{H^n} \varphi(x) \pi_\lambda(x) dx$$

is of trace class. In particular, if $\{e_i\}$ is any orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\text{tr}(\pi_\lambda(\varphi)) = \sum_i (\pi_\lambda(\varphi)e_i, e_i) < \infty.$$

The Plancherel Theorem for H^n states that for $\varphi \in C_0^\infty(H^n)$,

$$\|\varphi\|_{L^2} = \int_{\mathbf{R} - \{0\}} \operatorname{tr}(\pi_\lambda(\varphi)\pi_\lambda(\varphi)^*)|\lambda|^n d\lambda \quad (4)$$

where $\pi_\lambda(\varphi)^*$ is the formal adjoint, and

$$\varphi(0) = \int_{\mathbf{R} - \{0\}} \operatorname{tr}(\pi_\lambda(\varphi))|\lambda|^n d\lambda. \quad (5)$$

Now suppose P is homogeneous of degree $\alpha > 0$. Then

$$\pi_\lambda(P) = \begin{cases} |\lambda|^{\alpha/2}\pi_1(P), & \lambda > 0, \\ |\lambda|^{\alpha/2}\pi_{-1}(P), & \lambda < 0, \end{cases} \quad (6)$$

by (1).

The hypothesis of the theorem is then that $\pi_1(P)$ and $\pi_{-1}(P)$ have bounded inverses, to be denoted B_1 and B_{-1} . Following [7], we first reduce to the case where B_1 and B_{-1} are of trace class. This is accomplished by replacing P by PQ , where

$$Q = \left(\sum X_j^2 + Y_j^2 \right)^{N'}$$

for large N' . Indeed, it is shown in [7, Lemma 3.3] that $\pi_1(Q)^{-1}$ is of trace class. Since the product of a bounded operator and an operator of trace class is again of trace class, the inverse of $\pi_1(PQ) = \pi_1(P)\pi_1(Q)$ is of trace class. PQ again satisfies the hypotheses of the theorem and the local solvability of PQ implies that of P . Replacing P by PQ if necessary, we may assume that B_1 and B_{-1} are of trace class.

To solve (2), let u be the linear mapping $C_0^\infty(H^n) \rightarrow \mathbf{C}$ defined by

$$\varphi \rightarrow u(\varphi) = \int (i\lambda)^N \operatorname{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)(\pi_\lambda(P))^{-1})|\lambda|^n d\lambda; \quad (7)$$

for some large N , where $\bar{v}(x) = v(x^{-1})$. We shall first show that u is an L^2 function by proving

$$|u(\varphi)| < C\|\varphi\|_{L^2}. \quad (8)$$

By (6), the absolute value of the right-hand side of (7) is bounded by

$$\begin{aligned} & \left| \int_{\lambda > 0} |\lambda|^{N-\alpha/2} |\operatorname{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_1)| |\lambda|^n d\lambda \right| \\ & + \left| \int_{\lambda < 0} |\lambda|^{N-\alpha/2} |\operatorname{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_{-1})| |\lambda|^n d\lambda \right|. \end{aligned} \quad (9)$$

If $\alpha > 0$ is arbitrary, then

$$\begin{aligned} & \left| \int_{\lambda > 0} |\lambda|^{N-\alpha/2} |(\text{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_1))| |\lambda|^n d\lambda \right| \\ & \leq \left| \int_{0 < \lambda < \alpha} |\lambda|^{N-\alpha/2} |(\text{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_1))| |\lambda|^n d\lambda \right| \\ & \quad + \left| \int_{\alpha < \lambda < \infty} |\lambda|^{N-\alpha/2} |(\text{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_1))| |\lambda|^n d\lambda \right|. \quad (10) \end{aligned}$$

The generalized Schwartz inequality for arbitrary operators C, D of trace class states that

$$|\text{tr}(CD)| \leq (\text{tr}(CC^*))^{1/2} (\text{tr}(DD^*))^{1/2}. \quad (11)$$

The first integral on the right of (10) is bounded by

$$\left[\int_0^\alpha |\lambda|^{2(N-\alpha/2)+n} d\lambda \right]^{1/2} \left[\int_0^\alpha |\text{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_1)|^2 |\lambda|^n d\lambda \right]^{1/2} \quad (12)$$

by the standard Schwartz inequality. If $2(N - \alpha/2) + n > -1$ i.e. if

$$N > -(n+1) + \alpha/2,$$

then the first factor in (12) is finite, say bounded by $C > 0$. By (11), (12) is bounded by

$$C [\text{tr}(B_1 B_1^*)]^{1/2} \left[\int_0^\alpha \text{tr}(\pi_\lambda(\bar{v})\pi_\lambda(\varphi)\pi_\lambda(\varphi)^*\pi_\lambda(\bar{v})^*) |\lambda|^n d\lambda \right]^{1/2}. \quad (13)$$

Now recall that the group convolution of, say, two functions $\psi, \chi \in C_0^\infty$ is defined as

$$(\psi * \chi)(g) = \int \psi(x^{-1})\chi(xg) dx. \quad (14)$$

It is well known, and easy to check, that

$$\pi_\lambda(\psi * \chi) = \pi_\lambda(\psi)\pi_\lambda(\chi).$$

Hence (13) is bounded by

$$C' \left| \int_0^\alpha \text{tr}(\pi_\lambda(\bar{v} * \varphi)\pi_\lambda(\bar{v} * \varphi)^*) |\lambda|^n d\lambda \right|^{1/2} = C' \|\bar{v} * \varphi\|_{L^2}$$

by (4). By the general Hausdorff-Young inequality

$$\|\bar{v} * \varphi\|_{L^2} \leq \|\bar{v}\|_{L^1} \|\varphi\|_{L^2} \leq C' \|\varphi\|_{L^2}. \quad (15)$$

Estimation of the second integral on the right-hand side of (10) is similar except that (12) is replaced by

$$\left[\int_a^\infty |\lambda|^{2(N-\alpha/2)+n-N'} d\lambda \right]^{1/2} \left[\int_0^\lambda |\text{tr}(\lambda^{N'}\pi_\lambda(\bar{v})\pi_\lambda(\varphi)B_1)| |\lambda|^n d\lambda \right]^{1/2}. \quad (16)$$

Now $(i\lambda)^{N'} = \pi_\lambda(T^{N'})$ and $\pi_\lambda(T^{N'})\pi_\lambda(\bar{v}) = \pi_\lambda(\widetilde{T^{N'}\bar{v}})$. If N' is chosen so large

that the first factor is finite, (16) is bounded by

$$C'' \left\{ \int_0^\lambda \operatorname{tr}(\pi_\lambda(\widetilde{T}^{N'}v)\pi_\lambda(\varphi)\pi_\lambda(\varphi)^*\pi_\lambda(\widetilde{T}^{N'}v)^*)|\lambda|^n d\lambda \right\}^{1/2} \\ = C'' \|\widetilde{T}^{N'}v * \varphi\| < C''' \|\varphi\|_{L^2}.$$

Combining these estimates and similar ones for the other integral in (9) gives (8). This argument is very similar to that given in [7].

To show that the L^2 function u is actually smooth it suffices, by Sobolev's Lemma, to show that all distributional derivatives of u are in L^2 . We shall show that if D is any left invariant differential operator on H^n , then

$$|u(D\varphi)| < C_D \|\varphi\|_{L^2}, \quad \varphi \in C_0^\infty(H^n). \quad (17)$$

In proving (17) it will be convenient to use right invariant differential operators. If $Z \in \mathfrak{h}^n$, then there is a right invariant vector field Z' determined by Z , i.e.

$$Z'f(g) = \left[\frac{d}{dt} f(\exp tZ \cdot g) \right]_{t=0}.$$

If $D = Z_1 Z_2 \cdots Z_k$ is a left invariant differential operator with $Z_j \in \mathfrak{h}^n$, $1 < j < k$, denote by D' the right invariant vector field $D' = Z_k' Z_{k-1}' \cdots Z_1'$.

LEMMA 2. $D\psi * \chi = \psi * D'\chi$ for any left invariant differential operator D and any $\psi, \chi \in C_0^\infty(H^n)$.

PROOF. It suffices to assume $D = Z \in \mathfrak{h}^n$. Then

$$Z\psi * \chi = \int \left[\frac{d}{dt} (\psi(gx^{-1}\exp tZ)\chi(x)) \right]_{t=0} dx \\ = \int \left[\frac{d}{dt} \psi(gx^{-1})\chi((\exp tZ)x) \right]_{t=0} dx = \psi * Z'\chi.$$

(The second equality is obtained by making the change of variables $x \rightarrow (\exp tZ)x$ and using the fact that the measure dx is left invariant.)

To prove the inequality (17), one proceeds as above, but with φ replaced by $D\varphi$. Since

$$\operatorname{tr}(\pi_\lambda(\bar{v})\pi_\lambda(D\varphi)\pi_\lambda(D\varphi)^*\pi_\lambda(\bar{v})^*) = \operatorname{tr}(\pi_\lambda(D\varphi)^*\pi_\lambda(\bar{v})\pi_\lambda(\bar{v})^*\pi_\lambda(D\varphi)), \\ \int_{0 < \lambda < a} |\lambda|^N \operatorname{tr}(\pi_\lambda(\bar{v})\pi_\lambda(D\varphi)(\pi_\lambda(P))^{-1})|\lambda|^n d\lambda \\ < C \|D\varphi * \bar{v}\|_{L^2} = C \|\varphi * D'\bar{v}\|_{L^2} < C' \|\varphi\|_{L^2}.$$

Similarly,

$$\int_{0 < \lambda < \infty} |\lambda|^N \operatorname{tr}(\pi_\lambda(\bar{v})\pi_\lambda(D\varphi)(\pi_\lambda(P))^{-1})|\lambda|^n d\lambda \\ < C \|D\varphi * T^{N'}\bar{v}\|_{L^2} = C \|\varphi * D'T^{N'}\bar{v}\|_{L^2} < C' \|\varphi\|_{L^2}.$$

The estimates for $\lambda < 0$ are the same. This completes the proof that u is a C^∞ function. Finally, it is a simple formal calculation to show that u satisfies (2) i.e. that $u(P^\tau\varphi) = T^N v(\varphi)$ for all $\varphi \in C_0^\infty(H^n)$, where P^τ is the formal transpose of P . Indeed, since $\pi_\lambda(P^\tau\varphi) = \pi_\lambda(\varphi)\pi_\lambda(P)$, and $(i\lambda)^N \pi_\lambda(\tilde{v}) = \pi_\lambda(\tilde{v})\pi_\lambda(T^N) = \pi_\lambda(\widetilde{T^N v})$,

$$\begin{aligned} u(P^\tau\varphi) &= \int (i\lambda)^N \operatorname{tr}(\pi_\lambda(v)\pi_\lambda(P^\tau\varphi)(\pi_\lambda(P))^{-1})|\lambda|^N d\lambda \\ &= \int \operatorname{tr}(\pi_\lambda(\widetilde{T^N v})\pi_\lambda(\varphi))|\lambda|^N d\lambda \\ &= \int \operatorname{tr}(\widetilde{T^N v} * \varphi)|\lambda|^N d\lambda \\ &= (\widetilde{T^N v} * \varphi)(0), \text{ by the Plancherel theorem (5),} \\ &= T^N v(\varphi), \text{ by (14).} \end{aligned}$$

This completes the proof of the theorem.

REMARKS. 1. Rockland [7] proves that if $\pi(P)$ and $\pi(P^*)$ have trivial L^2 kernels for all irreducible unitary representations, then P is hypoelliptic and locally solvable. Using pseudodifferential operators, Miller [4] was able to prove this theorem assuming the hypothesis only for $\pi(P)$ has no rapidly decreasing eigenfunction of eigenvalue zero, for all π . The possible relationship between hypoellipticity and invertibility of representations on H^n was first observed by Folland and Stein [1]. (See also [9].)

2. The author has proved local solvability for some classes of homogeneous left invariant differential operators on more general nilpotent Lie groups. The proofs involve more detailed harmonic analysis [10].

ADDED IN PROOF. These results have been proved independently by G. Lion.

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