

Mappings between real submanifolds in complex space

Linda Preiss Rothschild

To Robert E. Greene on the occasion of his 60th birthday

1. Introduction

In this survey we shall discuss a number of questions and results on approximation, convergence, and finite determination of mappings between real submanifolds in complex space. We begin with some notation. Let M and M' be connected smooth real submanifolds of \mathbb{C}^N and $\mathbb{C}^{N'}$ respectively, with $p \in M$ and $p' \in M'$. We write $H : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ for a germ of a holomorphic mapping at p of \mathbb{C}^N into $\mathbb{C}^{N'}$ with $H(p) = p'$. We shall say that such a germ H sends M into M' if there is a neighborhood U of p in \mathbb{C}^N with $H(M \cap U) \subset M'$.

We shall consider here three questions concerning mappings sending one real submanifold into another. The first question (and easiest to state) is that of "finite determination."

Question 1. Suppose $H : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ is a germ of a holomorphic mapping at p sending M into M' . Is there a positive integer K (depending on M, M', p, p' and possibly also on H) such that if \tilde{H} is another such germ of a holomorphic mapping sending M into M' and satisfying

$$(1.1) \quad \partial^\alpha H(p) = \partial^\alpha \tilde{H}(p), \quad \forall \alpha \in \mathbb{N}^N, |\alpha| \leq K,$$

then $H = \tilde{H}$.

It is easy to see that the answer to Question 1 is "no" in general, even if $N = N'$, M is real analytic, and H is a biholomorphism. Indeed, one may take $N = 1$, $M = M' = \mathbb{R}$, and $p = p' = 0$. In that case, any convergent power series $\sum_{n=1}^{\infty} a_n z^n$ with a_n real for all n and $a_1 \neq 0$ defines a germ of a holomorphic mapping at 0 sending M into M' . However, it follows from the ground breaking work of Chern and Moser [CM74] in the mid '70's that the answer to Question 1 is "yes" when $N = N'$, M and M' are Levi nondegenerate hypersurfaces, and H is invertible i.e. the Jacobian of H is invertible at p . In fact, in this case it is shown that one may take $K = 2$. In this survey we shall give some recent generalizations of this result and state some open problems related to Question 1.

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In order to state two other questions, we introduce the notion of a formal mapping. For $p \in \mathbb{C}^N$ and $p' \in \mathbb{C}^{N'}$, a *formal mapping* $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ is an N' -vector of formal power series in N complex variables,

$$(1.2) \quad F = (F_1, \dots, F_{N'}), \quad F_j(Z) = \sum_{\alpha \in \mathbb{N}^N} a_{j\alpha} (Z - p)^\alpha,$$

with $(Z - p)^\alpha = (Z_1 - p_1)^{\alpha_1} \dots (Z_N - p_N)^{\alpha_N}$ and $F(p) = p'$.

Since a formal mapping is not necessarily a holomorphic mapping, one must redefine what it means to send one manifold into another. Suppose that (M, p) and (M', p') are germs of real analytic, real submanifolds in \mathbb{C}^N and $\mathbb{C}^{N'}$ respectively, given by the vanishing of real-analytic (vector valued) real local defining functions $\rho(Z, \bar{Z})$ and $\rho'(Z', \bar{Z}')$ near $p \in M$ and $p' \in M'$ respectively. We shall say that a formal mapping F as above sends M into M' if

$$\rho'(F(Z(x)), \overline{F(Z(x))}) = 0$$

in the sense of formal power series in x for some (and hence for any) real-analytic parametrization $x \mapsto Z(x)$ of M near $p = Z(0)$. (Here $x \in \mathbb{R}^{\dim M}$.)

Our next question concerns convergence of formal mappings.

Question 2. Suppose $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ is a formal mapping taking M into M' , where M and M' are real analytic submanifolds of \mathbb{C}^N and $\mathbb{C}^{N'}$ respectively. Is F necessarily convergent?

Again the case $N = N' = 1$, $M = M' = \mathbb{R}$, and $p = p' = 0$ is a simple counterexample. Indeed, any formal series $\sum_{n=1}^\infty a_n Z^n$ with $a_n \in \mathbb{R}$ for all n represents a formal mapping sending M into M' . Nevertheless, as in the case of finite determination, it was proved in [CM74] that if M and M' are Levi nondegenerate real analytic hypersurfaces in \mathbb{C}^N and if F is invertible (i.e. $\text{Jac } F(p) \neq 0$), then F must be convergent.

The third question we shall consider is more subtle. Recall that one version of the Artin approximation theorem [Art68] implies that if $F : (\mathbb{C}^j, 0) \rightarrow (\mathbb{C}^N, 0)$ is any formal mapping and $h_\alpha : (\mathbb{C}^j, 0) \rightarrow (\mathbb{C}^q, 0)$ a family of holomorphic mappings such that $h_\alpha \circ F = 0$ for all α , then for any integer $k > 0$ there is a holomorphic mapping $H^k : (\mathbb{C}^j, 0) \rightarrow (\mathbb{C}^N, 0)$ such that the Taylor series of H^k agrees with that of F up to order k , and $h_\alpha \circ H^k = 0$ for all α . Question 3 asks whether there is an analogous theorem for formal mappings sending one submanifold into another.

Question 3. Suppose $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ is a formal mapping sending M into M' , where M and M' are real analytic, real submanifolds of \mathbb{C}^N and $\mathbb{C}^{N'}$ respectively. Can F be approximated by convergent mappings sending M into M' ? More precisely, given any $k > 0$ is there a holomorphic mapping $H^k : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' such that

$$(1.3) \quad \partial^\alpha H^k(p) = \partial^\alpha F(p), \quad \forall \alpha \in \mathbb{N}^N, |\alpha| \leq k?$$

Returning to our example $N = N' = 1$, $M = M' = \mathbb{R}$, and $p = p' = 0$, any formal mapping $F : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ sending \mathbb{R} into itself is given by a power series with real coefficients. Hence it may be approximated by a convergent series to any order by simply truncating the series. Nevertheless, the answer to Question 3 is also "no" in general. Indeed, in their study of real surfaces in \mathbb{C}^2 , Moser and Webster

[MW83] exhibited pairs of real analytic surfaces M and M' through 0 for which there exists a formal invertible mapping $F: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ sending M into M' , but no germ of an invertible holomorphic mapping $H: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ sending M into M' . In this situation one says briefly that M and M' are formally, but not biholomorphically, equivalent at 0.

Note that Question 1, that of finite determination, still makes sense if the holomorphic mapping H in that question is assumed only to be formal. Before giving some recent results, we will explore the relationship between answers to Questions 1, 2, and 3. It is clear that a positive answer to Question 2 for a fixed triplet $((M, p), (M', p'), F)$, gives a positive answer to Question 3 for that triplet. Now suppose that the answers to Questions 1 and 3 are positive for a fixed pair of germs of real analytic, real submanifolds (M, p) and (M', p') in \mathbb{C}^N and for all invertible formal mappings sending p to p' and M to M' . We claim that this implies that every such mapping is necessarily convergent. Indeed, for any invertible formal mapping F sending p to p' and M to M' , by the positive answer to Question 1, there exists $K > 0$ such that if \tilde{F} is another such mapping and satisfies

$$(1.4) \quad \partial^\alpha F(p) = \partial^\alpha \tilde{F}(p), \quad \forall \alpha \in \mathbb{N}^N, |\alpha| \leq K,$$

then $\tilde{F} = F$. By the approximation of F given by the positive answer to Question 3 with $k = K$, there is a germ of a holomorphic mapping $H: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$ sending M into M' and satisfying (1.4) with \tilde{F} replaced by H . In that case it follows that $H = F$ and hence F is convergent.

In the rest of this paper I shall state some recent results related to Questions 1, 2, and 3 and also mention some open problems and conjectures. In the last section I will discuss some aspects of the techniques used in the proofs.

2. Approximation

I shall begin with the following approximation result, which is a partial answer to Question 3. If F and F' are formal mappings at p we shall write $F - F' = O(|Z - p|^K)$ if the series of the components of F and F' agree up to order $K - 1$ at p .

THEOREM 2.1. (Baouendi, Rothschild, & Zaitsev [BRZ01]) *Let $M \subset \mathbb{C}^N$ be a connected real analytic submanifold. Then there exists a closed proper real analytic subvariety $V \subset M$ such that for every $p \in M \setminus V$, every real analytic submanifold $M' \subset \mathbb{C}^N$ with $\dim_{\mathbb{R}} M' = \dim_{\mathbb{R}} M$, every $p' \in M'$, every integer $K > 1$, and every invertible formal mapping $F: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$ sending M into M' , there exists a holomorphic mapping $H: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$ sending M into M' with $H(Z) - F(Z) = O(|Z - p|^K)$.*

As a consequence of this approximation theorem, one may conclude in particular that for all points on a given submanifold outside a proper real analytic subvariety, formal equivalence to another submanifold is the same as local biholomorphic equivalence. In the absence of counterexamples, one may conjecture the following generalization of Theorem 2.1 in which the assumptions that M and M' are equidimensional submanifolds in the same complex space and the invertibility of the formal map F are dropped.

CONJECTURE 2.2. *Let $M \subset \mathbb{C}^N$ be a connected real analytic submanifold. Then there exists a closed proper real analytic subvariety $V \subset M$ such that for every $p \in$*

$M \setminus V$, every positive integer N' , every real analytic submanifold $M' \subset \mathbb{C}^{N'}$, every $p' \in M'$, every integer $K > 1$, and every formal mapping $F: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' , there exists a holomorphic mapping $H: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' with $H(Z) - F(Z) = O(|Z - p|^K)$.

A recent result in this direction was proved by Meylan, Mir, and Zaitsev [MMZ03]. To state their result we need to recall some definitions. Let $M \subset \mathbb{C}^N$ be a real submanifold of codimension d and $p_0 \in M$. Let $\rho = (\rho_1, \dots, \rho_d)$ be a set of real valued defining functions for M near p_0 with linearly independent differentials. For $p \in M$ near p_0 , let

$$(2.1) \quad r(p) := d - \dim \operatorname{span}_{\mathbb{C}} \{ \rho_{j,z}(p, \bar{p}) : 1 \leq j \leq d \}.$$

Here $\rho_{j,z} = (\partial \rho_j / \partial Z_1, \dots, \partial \rho_j / \partial Z_N) \in \mathbb{C}^N$ denotes the complex gradient of ρ_j with respect to $Z = (Z_1, \dots, Z_N)$. The point $p_0 \in M$ is called CR if the mapping $p \mapsto r(p)$ is constant for p in a neighborhood of p_0 in M . (It is easy to see that if M is real analytic then the set of points at which M is not CR is a proper real analytic subvariety of M .) The submanifold M is called CR if it is CR at all its points. If M is CR, the set $\mathcal{V} := T^{(0,1)}(\mathbb{C}^N) \cap \mathbb{C}TM$ of antiholomorphic vectors tangent to M form a bundle, as does the set of holomorphic vectors $\bar{\mathcal{V}} := T^{(1,0)}(\mathbb{C}^N) \cap \mathbb{C}TM$. The bundle \mathcal{V} is called the CR bundle of M . The CR submanifold M is said to be of finite type at p (in the sense of Kohn [Koh72], Bloom-Graham [BG77]) if the span at p of the Lie algebra generated by the sections of the bundles \mathcal{V} and $\bar{\mathcal{V}}$ equals $\mathbb{C}T_p M$.

THEOREM 2.3. (Meylan, Mir, & Zaitsev [MMZ03]) *Let $M \subset \mathbb{C}^N$ be a real analytic CR submanifold, $p \in M$ a point of finite type, and K a positive integer. Then for any real algebraic subset $M' \subset \mathbb{C}^{N'}$, any $p' \in M'$, and any formal mapping $F: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' , there exists a germ of a holomorphic mapping $H: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' with $H(Z) - F(Z) = O(|Z - p|^K)$.*

Although the hypothesis that M' be algebraic is essential for the proof of Theorem 2.3 as stated, one can conjecture the following.

CONJECTURE 2.4. *The conclusion of Theorem 2.3 still follows if M' is assumed to be a real analytic set, rather than an algebraic set.*

Another direction in which Theorem 2.1 might be generalized is to reduce the exceptional subvariety V (at which mappings may not be approximable) to the points at which M is not CR. Since the only known examples of germs (M, p) and (M', p') of real analytic submanifolds of \mathbb{C}^N which are formally equivalent but not biholomorphically equivalent occur in cases where M and M' are not CR at p and p' respectively, the following conjecture seems reasonable.

CONJECTURE 2.5. *Let $M \subset \mathbb{C}^N$ be a real analytic CR submanifold. Then for any $p \in M$, any real analytic real submanifold $M' \subset \mathbb{C}^N$ with $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} M'$, and $p' \in M'$, any formal invertible mapping $F: (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$ sending M into M' can be approximated by holomorphic ones, as in the conclusion of Theorem 2.1.*

In fact, when M is assumed to be of finite type at p , Baouendi, Mir, and the author [BMR02] have recently proved this conjecture.

3. Convergence of formal mappings

In this section we shall discuss results and conjectures concerning the convergence of formal mappings which take one real analytic submanifold into another. That is, we shall seek sufficient conditions for germs (M, p) and (M', p') of real analytic submanifolds of \mathbb{C}^N and $\mathbb{C}^{N'}$ and classes \mathcal{F} of formal mappings sending M into M' which guarantee that every element of \mathcal{F} is convergent, i.e. holomorphic.

To describe the results and conjectures, we first recall that a real CR manifold M is called *generic* at p if the number $r(p)$ defined in (2.1) is 0. (Any real hypersurface is a generic submanifold.) If M is CR but $r := r(p) > 0$, then after a local change of holomorphic coordinates in a neighborhood of p , one can assume $p = 0$ and write $M = M_1 \times 0 \subset \mathbb{C}^{N-r} \times \mathbb{C}^r$, where M_1 is a real analytic generic submanifold of \mathbb{C}^{N-r} . (See e.g. [BER99b] for details.) In this case, as is shown in Example 3.1 below, there are invertible formal mappings taking (M, p) into itself which are not convergent. It is also clear that near p , M may be identified with the generic submanifold M_1 of \mathbb{C}^{N-r} . For this reason we shall often restrict our attention to generic submanifolds.

EXAMPLE 3.1. Let $M = M_1 \times \{0\} \subset \mathbb{C}^{N-s} \times \mathbb{C}^s$, where s is a positive integer and M_1 is a real submanifold through 0 in \mathbb{C}^{N-s} . Let e_N be the last basis vector in \mathbb{C}^N . Then the function $F : Z \mapsto Z + g(Z_N)e_N$ maps $(M, 0)$ to itself and is invertible for any formal series $g(Z_N)$ satisfying $g(0) = 0$ and $g'(0) = 0$. If g is not convergent, neither is F . Furthermore, since g is arbitrary, F is not determined by any finite number of its derivatives at 0

Before stating a theorem, I will introduce some nondegeneracy conditions for a generic submanifold. Suppose first that M is a real hypersurface in \mathbb{C}^N given near $p \in M$ by $\rho(Z, \bar{Z}) = 0$, where ρ is a real-valued function with nonvanishing differential. As above, let ρ_Z be the vector $(\partial\rho/\partial Z_1, \dots, \partial\rho/\partial Z_N)$ and let L_1, \dots, L_{N-1} be a basis of $(0, 1)$ vector fields tangent to M near p . The hypersurface M is *Levi-nondegenerate* at p if and only if

$$(3.1) \quad \text{span}_{\mathbb{C}}\{\rho_Z(p), L_1\rho_Z(p), \dots, L_{N-1}\rho_Z(p)\} = \mathbb{C}^N.$$

We introduce a generalization of Levi-nondegeneracy not only for hypersurfaces but also for generic submanifolds of higher codimension. A generic submanifold M of codimension d is called *finitely nondegenerate* at p if there is an integer k for which

$$(3.2) \quad \text{span}_{\mathbb{C}}\{\rho_{j,Z}(p), L^\alpha \rho_{j,Z}(p), \alpha \in \mathbb{N}^{N-d}, 1 \leq j \leq d, |\alpha| \leq k\} = \mathbb{C}^N,$$

where $\rho = (\rho_1, \dots, \rho_d)$ are defining functions of M near p , $L^\alpha = L_1^{\alpha_1} \dots L_d^{\alpha_{N-d}}$, $|\alpha| = \alpha_1 + \dots + \alpha_{N-d}$. Here, as in the case of a hypersurface, L_1, \dots, L_{N-d} is a basis of $(0, 1)$ vector fields tangent to M near p . If k is the smallest integer such that (3.2) holds, then M is said to be *k-nondegenerate* at p .

The case of Levi degenerate, but finitely nondegenerate, hypersurfaces was considered in the joint work of Baouendi, Ebenfelt and the author [BER97]. In that work it was proved that any invertible formal mapping sending one finitely nondegenerate hypersurface in \mathbb{C}^N , $N > 1$, onto another is convergent. For hypersurfaces in \mathbb{C}^N , $N \geq 2$, the condition of finite nondegeneracy at a point can be satisfied only if the hypersurface is already of finite type at that point, but for generic submanifolds of higher codimension the two conditions are independent. (In \mathbb{C} any real analytic hypersurface (i.e. curve) is finitely nondegenerate at all points but

not of finite type at any point, since there are no nontrivial $(0,1)$ vector fields tangent to a curve.) In [BER99a] the above authors extended their results to prove that an invertible formal mapping $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$ sending a generic submanifold (M, p) onto (M', p') is necessarily convergent provided that M is finitely nondegenerate and of finite type at p .

Another condition that plays an important role for these questions is that of *holomorphic nondegeneracy*, which is weaker than finite nondegeneracy. A generic submanifold $M \subset \mathbb{C}^N$ is *holomorphically nondegenerate* at a point p if there is no nontrivial holomorphic vector field (with holomorphic coefficients) tangent to M in a neighborhood of p . The condition was first introduced for hypersurfaces by N. Stanton [Sta96] for studying infinitesimal automorphisms, and more generally later by the author in joint work with Baouendi and Ebenfelt [BER96] in their study of algebraicity of mappings. The following gives some basic properties of holomorphic nondegeneracy. See e.g. [BER99b] for proofs.

PROPOSITION 3.1. *Let $M \subset \mathbb{C}^N$ be a connected, real analytic, generic submanifold, and $p \in M$. The following are equivalent:*

- (1) M is holomorphically nondegenerate at p ;
- (2) M is holomorphically nondegenerate at all $q \in M$;
- (3) M is finitely nondegenerate on a dense subset of points.

In light of Proposition 3.1, for a connected, real analytic, generic submanifold M , one may assume that M is either holomorphically nondegenerate at no points, in which case we shall say that M is holomorphically degenerate, or at all points, in which case we shall say that M is holomorphically nondegenerate. In the first case, one may use the implicit function theorem to show that there is a dense open set $U \subset M$ such for any $p \in U$, after a change of local holomorphic coordinates in \mathbb{C}^N near p , one may assume $p = 0$ and $M = M_1 \times \mathbb{C}^s$, where M_1 is a holomorphically nondegenerate generic submanifold of \mathbb{C}^{N-s} , with s a positive integer (see [BRZ01]). Thus if M is connected and holomorphically degenerate, near all points $p \in U$, after a change of holomorphic coordinates near p , one may assume that $p = 0$ and M is given by a defining function which is independent of one of the coordinates, say Z_N . In this case we may construct invertible formal mappings $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p)$ which send M into itself but are not convergent by using the mappings given in Example 3.1 above.

One can also show (see e.g. [BER97]) that such mappings as those of Example 3.1 exist at all $p \in M$ in the holomorphically degenerate case. Hence another necessary condition for positive answers to either Questions 1 or 2 is that the target manifold M' be holomorphically nondegenerate.

Recall that a formal mapping F from (\mathbb{C}^N, p) to itself is called *finite* if the ideal generated by its components is of finite codimension in the ring, $\mathbb{C}[[Z - p]]$, of all formal power series in \mathbb{C}^N at p . The following theorem, which was first proved by Mir [Mir00] in the case of hypersurfaces, is the most general to date for finite formal mappings from (\mathbb{C}^N, p) to (\mathbb{C}^N, p') .

THEOREM 3.2. (Baouendi, Mir, & Rothschild [BMRO2]) *Let M and M' be connected real analytic generic submanifolds of \mathbb{C}^N of the same dimension, and $p \in M$ a point of finite type. Let $p' \in M'$ and suppose that M' is holomorphically nondegenerate. Then any formal finite mapping $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$ which sends M into M' is convergent.*

It is clear that even under the hypotheses on M , M' and p in Theorem 3.2, one must still impose stringent conditions on a formal mapping F in order to assure convergence, as the following example shows.

EXAMPLE 3.2. Let $M = M' \subset \mathbb{C}^3$ be the hypersurface through $p = p' = 0$ defined by

$$(3.3) \quad M = \{Z = (Z_1, Z_2, Z_3) \in \mathbb{C}^3 : \text{Im } Z_3 = |Z_1|^2 - |Z_2|^2\},$$

and let $F_1(Z) : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ be any divergent formal mapping. Then the formal mapping $F = (F_1, F_1, 0)$ sends M into itself, but is not convergent.

In Example (3.2) the target hypersurface M' contains the holomorphic subvariety $V = \{Z : Z_1 = Z_2\}$ and the formal map F sends M into V . If neither of the hypersurfaces M nor M' contains a nontrivial holomorphic subvariety through 0, then both are of finite type at 0 and holomorphically nondegenerate. In this case, it is possible to drop all conditions on the formal mapping F , as is shown by the following theorem.

THEOREM 3.3. (Baouendi, Ebenfelt, & Rothschild [BER00]) *Let M and M' be real analytic hypersurfaces through 0 in \mathbb{C}^N , $N \geq 2$, and suppose that neither M nor M' contains a nontrivial holomorphic variety through 0. Then any formal mapping $F : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^N, 0)$ sending M into M' is convergent.*

Even for real analytic connected hypersurfaces in \mathbb{C}^2 , necessary and sufficient conditions to guarantee the convergence of all invertible formal mappings sending one into another are not known. In this case, finite type at any point implies holomorphic nondegeneracy. A reasonable conjecture is the following.

CONJECTURE 3.4. *Let M and M' be connected real analytic hypersurfaces through 0 in \mathbb{C}^2 , and suppose that M is of finite type at some point. Then any invertible formal mapping $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ sending M into M' is convergent.*

If M is of finite type at 0, then the conclusion of Conjecture 3.4, is true; this has been proved in [BER00]. Hence the conjecture addresses the situation in which M is of finite type only at a dense set of points and M is not of finite type at 0.

The situation for generic submanifolds in complex spaces of different dimensions is much more complicated. The following result has been proved for the case where the target manifold is assumed to be real algebraic, i.e. contained in a real algebraic subvariety of the same dimension.

THEOREM 3.5. (Meylan, Mir, & Zaitsev [MMZ03]) *Let $M \subset \mathbb{C}^N$ be a real analytic generic submanifold and $p \in M$ a point of finite type. If $M' \subset \mathbb{C}^{N'}$ is a real algebraic set which contains no complex subvariety, then for any $p' \in M'$, any formal mapping $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' is convergent.*

It is reasonable to conjecture that the condition of algebraicity in Theorem 3.5 is superfluous. Hence the following.

CONJECTURE 3.6. *If M and p are as in Theorem 3.5 and $M' \subset \mathbb{C}^{N'}$ is a real analytic set containing no nontrivial holomorphic varieties, then any formal mapping $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M to M' is convergent.*

In fact Conjecture 3.6 is still open even for embeddings into strictly pseudoconvex hypersurfaces (which necessarily cannot contain nontrivial holomorphic

varieties). Recently, an advance for mappings between strictly pseudoconvex hypersurfaces was proved by Mir.

THEOREM 3.7. ([Mir02]) *Any formal embedding sending a real-analytic strictly pseudoconvex hypersurface $M \subset \mathbb{C}^N$ into another real-analytic strictly pseudoconvex hypersurface $M' \subset \mathbb{C}^{N+1}$ is convergent.*

In light of this result, the following special case of Conjecture 3.6 seems more accessible.

CONJECTURE 3.8. *For any integers $N, N', 2 \leq N \leq N'$, any formal embedding sending a real-analytic strictly pseudoconvex hypersurface $M \subset \mathbb{C}^N$ into another real-analytic strictly pseudoconvex hypersurface $M' \subset \mathbb{C}^{N'}$ is convergent.*

Recall that for $N' = N$ the conclusion of Conjecture 3.8 holds by the results in [CM74] mentioned above. Theorem 3.7 above shows that the conjecture is true for $N' = N + 1$.

Theorem 3.7 is a consequence of a more general result for which the target is assumed to be only Levi nondegenerate, rather than strictly pseudoconvex. (In this case M' may contain a nontrivial holomorphic subvariety, or even be foliated by complex subvarieties, as in Example 3.2.) To state the more general result we need another definition. A formal mapping $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p')$ sending M into M' is called *CR transversal* if $dF(p)(CT_p M) \not\subset \mathcal{V}'_{p'} \oplus \overline{\mathcal{V}'_{p'}}$, where \mathcal{V}' is the CR bundle of M' , and $\mathcal{V}'_{p'}$ is the fiber of that bundle at p' . It is known that any formal embedding from one strictly pseudoconvex hypersurface into another is necessarily CR transversal (see Lamel [Lam01], and Ebenfelt-Lamel [EL02]). Hence the following result implies Theorem 3.7.

THEOREM 3.9. (Mir [Mir02]) *Any formal CR transversal mapping sending a real analytic Levi nondegenerate hypersurface $M \subset \mathbb{C}^N$ into another such hypersurface $M' \subset \mathbb{C}^{N+1}$ is convergent.*

When M and M' are not strictly pseudoconvex, Theorem 3.10 is sharp in the sense that $N + 1$ cannot be replaced by any larger integer N' , as shown by the following example (see [Lam01]).

EXAMPLE 3.3. Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N+2}$ be given respectively by

$$(3.4) \quad \text{Im } Z_N = \sum_{j=1}^{N-1} |Z_j|^2, \quad \text{Im } Z'_{N+2} = \sum_{j=1}^N |Z'_j|^2 - |Z'_{N+1}|^2,$$

and consider the formal mapping $F: (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^{N+2}, 0)$ given by

$$F(Z) := (Z_1, \dots, Z_{N-1}, f(Z), f(Z), Z_N),$$

where $f(Z)$ is a divergent series with $f(0) = 0$. It is easy to see that F maps M into M' and is CR transversal. Here M is strictly pseudoconvex, but M' is only Levi nondegenerate and contains the complex curve given by the equations $Z'_{N+2} = Z'_1 = Z'_2 = \dots = Z'_{N-1} = 0, Z'_N = Z'_{N+1}$.

For formal embeddings (not necessarily CR transversal) for which the target manifold contains no nontrivial complex curves, Mir has proved the following, which also implies Theorem 3.7, since a strictly pseudoconvex hypersurface cannot contain a complex curve.

THEOREM 3.10. [Mir02] *Any formal embedding sending a real analytic Levi nondegenerate hypersurface $M \subset \mathbb{C}^N$ into another such hypersurface $M' \subset \mathbb{C}^{N+1}$ is convergent if M' contains no nontrivial complex curves.*

4. Finite determination of mappings: results and conjectures

In this section we return to variations of Question 1. As mentioned above, if M and M' are real analytic, one may ask the analogue of Question 1 when H is assumed to be a formal mapping rather than a convergent one. Since the publication of [CM74] in the mid '70's, a number of results were obtained concerning invertible mappings sending one finitely nongenerate generic submanifold in \mathbb{C}^N into another. In [BER98] Baouendi, Ebenfelt, and the author proved that if M is a real analytic generic submanifold of codimension $d > 0$ and $p \in M$ is a point of finite type at which M is k -nondegenerate, then any germ at p of an invertible holomorphic mapping sending M into itself is determined by its derivatives at p of order $k(d+1)$. This theorem has since been generalized in several directions (see e.g. [BER00]), and a number of authors have also proved stronger results in special cases.

For smooth CR submanifolds in \mathbb{C}^N an interesting class of mappings between two such manifolds consists of the CR mappings, since they are exactly the ones that preserve the CR structures. Recall that for CR submanifolds $M \subset \mathbb{C}^N, M' \subset \mathbb{C}^{N'}$, a differentiable mapping $h : M \rightarrow M'$ is called CR if $h_*\mathcal{V}_p \subset \mathcal{V}'_{h(p)}$ for all $p \in M$, where \mathcal{V} and \mathcal{V}' denote the CR bundles of M and M' respectively. The restriction of a holomorphic mapping is always CR, but when the submanifolds are both real analytic, a smooth CR mapping is the restriction of a holomorphic mapping if and only if it is real analytic.

The following generalization to the CR case of the theorem in [BER98] cited above was proved first by Ebenfelt [Ebe01] for the case of hypersurfaces and by Kim and Zaitsev [KZ01] for higher codimension.

THEOREM 4.1. (Ebenfelt [Ebe01], Kim & Zaitsev [KZ01]) *Let $M, M' \subset \mathbb{C}^N$ be smooth CR manifolds of codimension d with M k -nondegenerate and of finite type at a point p . If h_1 and h_2 are smooth invertible CR mappings of M into M' such that*

$$(4.1) \quad \partial^\alpha h_1(p) = \partial^\alpha h_2(p), \quad \forall \alpha, |\alpha| \leq k(d+1)$$

then $h_1 = h_2$ in a neighborhood of p .

In the case of formal or holomorphic mappings there are some results in which the condition of finite nondegeneracy in Theorem 4.1 can be weakened. The notion of holomorphic nondegeneracy at a point, as given in Section 3, can be extended to smooth manifolds by considering formal holomorphic vector fields (see e.g. [BER99b]). For holomorphically nondegenerate generic submanifolds the following was proved in [BMR02].

THEOREM 4.2. (Baouendi, Mir, & Rothschild [BMR02]) *Let $M \subset \mathbb{C}^N$ be a smooth generic submanifold through p , and assume that M is of finite type and holomorphically nondegenerate at p . Then there exists a positive integer K such that for any smooth generic submanifold $M' \subset \mathbb{C}^N$ with $\dim M = \dim M'$, if $H^j :$*

$(\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p')$, $j = 1, 2$, are invertible formal mappings sending M into M' and satisfying

$$(4.2) \quad \partial^\alpha H^1(p) = \partial^\alpha H^2(p) \quad \forall \alpha, |\alpha| \leq K,$$

then $H^1 = H^2$.

For the case where the point p is assumed to be essentially finite (a condition weaker than finite nondegeneracy, but stronger than holomorphic nondegeneracy, see e.g. [BER99b]) this result was previously proved in [BER00].

For a hypersurface in \mathbb{C}^2 Ebenfelt, Lamel & Zaitsev [ELZ01] have recently found optimal results for finite determination of biholomorphisms.

THEOREM 4.3. ([ELZ01]) *Let (M, p) and (M', p') be germs of real analytic hypersurfaces in \mathbb{C}^2 which are not Levi flat. Then there is an integer $K \geq 2$ such that if $H^1, H^2 : (\mathbb{C}^2, p) \rightarrow (\mathbb{C}^2, p')$ are germs of biholomorphisms sending M into M' with*

$$(4.3) \quad \partial^\alpha H^1(p) = \partial^\alpha H^2(p) \quad \forall \alpha, |\alpha| \leq K,$$

it follows that $H^1 = H^2$. Moreover, if M is of finite type at p , the conclusion holds with $K = 2$.

Theorem 4.3 is sharp. Indeed, as mentioned above, for a Levi flat hypersurface (i.e. given by $\text{Im } Z_2 = 0$), no finite K exists for which the conclusion of Theorem 4.3 holds. In addition, Kowalski [Kow02] has given a class of non-Levi flat (and hence holomorphically nondegenerate) hypersurfaces (M, p) and (M', p') in \mathbb{C}^2 for which the best integer K as in Theorem 4.3 can be arbitrarily large.

For hypersurfaces in \mathbb{C}^N , $N > 2$, finite type no longer implies holomorphic nondegeneracy and is not sufficient to guarantee finite determination of biholomorphisms, even between real analytic hypersurfaces. Indeed, the mappings $(Z_1, Z_2, Z_3) \mapsto (Z_1, Z_2, Z_3 + Z_3^K)$, $K = 1, 2, \dots$, are biholomorphisms at the origin in \mathbb{C}^3 sending the hypersurface $M = \{(Z_1, Z_2, Z_3) : \text{Im } Z_2 = |Z_1|^2\}$ into itself; no finite K will determine all such mappings. Indeed, in this case the defining function of M is independent of Z_3 (so that M is holomorphically degenerate) and the given mapping is a special case of that of Example 3.1. Recently it has been discovered that in \mathbb{C}^N , $N > 2$, there are real analytic hypersurfaces which are of finite type and holomorphically nondegenerate at all points, but for which self mappings fixing a point cannot be determined by derivatives of order 2.

A survey including other results on finite determination can be found in the recent article by Zaitsev [Z02].

5. Segre mappings and the characterization of finite type

In this last section we shall define the so-called Segre mappings and give an explicit example to show how these mappings can be used to prove convergence of formal mappings sending a generic submanifold into another. The closely related Segre varieties (see below) have featured prominently in the study of mappings such as in the work of Webster [W77] on algebraicity. Segre mappings are also a crucial tool in proving results concerning finite determination and approximation.

For simplicity we shall take $M = M' \subset \mathbb{C}^N$ and $p = p' = 0$ in this section, where M is a real analytic generic submanifold of codimension d given near 0 by the equation $\rho(Z, \bar{Z}) = 0$. Here $\rho = (\rho_1, \dots, \rho_d)$, is a real analytic local defining

function of M near 0 with $\partial_Z \rho_1(0), \dots, \partial_Z \rho_d(0)$ linearly independent. We consider \bar{Z} as an independent variable so that $\rho_j(Z, \zeta)$ is a convergent power series in the $2N$ indeterminates (Z, ζ) . We shall denote the ring of such convergent power series with complex coefficients by $\mathbb{C}\{Z, \zeta\}$. The *generic rank*, $\text{Rk } F$, of a formal mapping $F: (\mathbb{C}_x^k, 0) \rightarrow (\mathbb{C}_y^p, 0)$ is defined as the rank of the Jacobian matrix $\partial F/\partial x$ regarded as a \mathbb{K}_x -linear mapping $\mathbb{K}_x^k \rightarrow \mathbb{K}_x^p$, where \mathbb{K}_x denotes the field of fractions of $\mathbb{C}\{\{x\}\}$. Hence $\text{Rk } F$ is the largest integer s such that there is an $s \times s$ minor of the matrix $\partial F/\partial x$ which is not 0 as a formal power series in x .

Let $\gamma(\zeta, t)$, where $\zeta = (\zeta_1, \dots, \zeta_N)$, $t = (t_1, \dots, t_n)$, and $n = N - d$, be a holomorphic mapping $(\mathbb{C}^N \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}^N, 0)$ such that

$$(5.1) \quad \rho(\gamma(\zeta, t), \zeta) = 0, \quad \text{rk } \frac{\partial \gamma}{\partial t}(0, 0) = n.$$

The existence of such $\gamma(\zeta, t)$ is a consequence of the implicit function theorem and the fact that $\partial_Z \rho_1, \dots, \partial_Z \rho_d$ are linearly independent at 0 . We shall call a holomorphic mapping $\gamma(\zeta, t)$ satisfying (5.1) a *Segre variety mapping* for the germ of M at 0 . The mapping $t \mapsto \gamma(\zeta, t)$, for t near $0 \in \mathbb{C}^n$, parametrizes the *Segre variety* of M at $\bar{\zeta}$.

We define a sequence of holomorphic mappings $v^j: (\mathbb{C}^{nj}, 0) \rightarrow (\mathbb{C}^N, 0)$, called the *iterated Segre mappings* of M at 0 (relative to γ), inductively as follows:

$$(5.2) \quad \begin{aligned} v^1(t^1) &:= \gamma(0, t^1), \\ v^{j+1}(t^1, \dots, t^{j+1}) &:= \gamma(\bar{v}^j(t^1, \dots, t^j), t^{j+1}). \end{aligned}$$

Here for $u(x) \in \mathbb{C}\{x\}$ we denote by $\bar{u}(x)$ the series obtained by replacing the coefficients in u by their complex conjugates. The relevance of the iterated Segre mappings is given by the following theorem.

THEOREM 5.1. (Baouendi, Ebenfelt, & Rothschild [BER03]) *Let $M \subset \mathbb{C}^N$ be as above. Then*

$$(5.3) \quad \rho(v^{k+1}(t^1, \dots, t^{k+1}), \bar{v}^k(t^1, \dots, t^k)) = 0, \quad k = 1, 2, \dots,$$

$$(5.4) \quad \text{Rk } v^1 \leq \text{Rk } v^2 \leq \dots \leq \text{Rk } v^j \leq \text{Rk } v^{j+1} \leq N, \quad j = 1, 2, \dots,$$

and M is of finite type at 0 if and only if $\text{Rk } v^{d+1} = N$. Moreover, for any j , $\text{Rk } v^j$ is independent of the choice of the Segre variety mapping γ .

We shall illustrate here how the Segre mappings can be used to prove convergence and finite determination in a very simple case. We let M be the Lewy hypersurface in \mathbb{C}^2 , i.e.

$$(5.5) \quad M = \{Z = (z, w) \in \mathbb{C}^2 : \text{Im } w = |z|^2\},$$

and let $\rho(Z, \zeta) = w - \tau - 2iz\chi$, where $\zeta = (\chi, \tau) \in \mathbb{C}^2$. Then $\gamma = (\gamma_1, \gamma_2): (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^2, 0)$ is uniquely defined by choosing $\gamma_1(\zeta, t) = t$, so that by (5.1) and (5.2),

$$(5.6) \quad \gamma_2(\zeta, t) = \tau + 2it\chi \quad \text{and} \quad v^1(t^1) = (t^1, 0),$$

Applying (5.2) for the next iteration, we obtain

$$(5.7) \quad v^2(t^1, t^2) = \gamma(\bar{v}^1(t^1), t^2) = (t^2, 2it^1 t^2), \quad \text{with } \text{Rk } v^2 = 2.$$

By Theorem 5.1 we recover the well-known fact, which is in this case easier to prove directly from the definition, that the Lewy hypersurface is of finite type at 0. For our purposes, however, it will be necessary to go one further iteration to obtain

$$(5.8) \quad v^3(t^1, t^2, t^3) = \gamma(\bar{v}^2(t^1, t^2), t^3) = (t^3, -2it^1t^2 + 2it^2t^3).$$

Now suppose that $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is a formal invertible mapping sending M into itself. Considering \bar{Z} as an independent variable ζ it follows that

$$(5.9) \quad \rho(F(Z), \bar{F}(\zeta)) = a(Z, \zeta)\rho(Z, \zeta),$$

for some $a(Z, \zeta) \in \mathbb{C}[[Z, \zeta]]$. Applying (5.3) of Theorem 5.1 with $k = 2$ we obtain

$$(5.10) \quad \rho(F(v^3(t^1, t^2, t^3)), \bar{F}(\bar{v}^2(t^1, t^2))) = 0.$$

Writing $F(z, w) = (f(z, w), g(z, w))$ and using our choice of ρ , we hence have the identity

$$(5.11) \quad g(t^3, 2i(t^1t^2 - t^2t^3)) - \bar{g}(t^2, 2it^1t^2) = 2if(t^3, 2i(t^1t^2 - t^2t^3))\bar{f}(t^2, 2it^1t^2)$$

Setting $t^1 = t^2 = 0$ in (5.11), and noting that $f(0) = g(0) = 0$, we obtain $g(z, 0) = 0$. Since F is assumed to be invertible, it follows that $f_z(0) \neq 0$. Hence after differentiating (5.11) in t^3 we may solve for $\bar{f}(t^2, 2it^1t^2)$ as a quotient of formal series

$$(5.12) \quad \bar{f}(t^2, 2it^1t^2) = \frac{(g_z - 2it^2g_w)(t^3, 2i(t^1t^2 - t^2t^3))}{(f_z - 2it^2f_w)(t^3, 2i(t^1t^2 - t^2t^3))}.$$

Now set $t^3 = t^1$ in (5.12) to obtain

$$(5.13) \quad \bar{f}(v^2(t^1, t^2)) = \bar{f}(t^2, 2it^1t^2) = \frac{g_z(t^1, 0) - 2it^2g_w(t^1, 0)}{f_z(t^1, 0) - 2it^2f_w(t^1, 0)} =: R(t^1, t^2).$$

To prove the convergence of the formal series F we shall make use of the the following result (see e.g. [EH77] or [BM88]).

LEMMA 5.2. *Let $S : (\mathbb{C}^p, 0) \rightarrow \mathbb{C}$ be a formal series and $h : (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^p, 0)$ a germ of a holomorphic mapping with $\text{Rk } h = p$. Then if $S \circ h$ is convergent, so is S .*

I claim that the convergence of F will follow from that of $R(t^1, t^2)$. For this, assume first that $R(t^1, t^2)$ is convergent. Since $\text{Rk } v^2 = 2$, it follows from Lemma 5.2 that $\bar{f}(\chi, \tau)$ (and hence f) is also convergent. Setting $t^1 = t^3$ in (5.11), and making use of the fact that $g(t^3, 0) = 0$, it follows from the convergence of f and again Lemma 5.2 that $\bar{g}(\chi, \tau)$ is also convergent. This yields the claim that F is convergent.

It remains to show the convergence of $R(t^1, t^2)$ given by (5.13). By the formula for R , it will suffice to show that

$$(5.14) \quad t \mapsto \partial^\alpha h(t, 0) \text{ is convergent for } |\alpha| = 1, h = f \text{ or } g.$$

Putting $t^1 = 0$ in (5.13) we obtain

$$(5.15) \quad \bar{f}(t^2, 0) = \frac{g_z(0) - 2it^2g_w(0)}{f_z(0) - 2it^2g_w(0)} = R(0, t^2),$$

where we note that $R(0, t^2)$ is a rational (and hence convergent) function of t^2 whose coefficients depend only on the first derivatives of F at 0. This shows that

$\bar{f}(t^2, 0)$ is convergent. Differentiating (5.15) we have

$$(5.16) \quad \bar{f}_x(t^2, 0) = \frac{d}{dt^2} R(0, t^2).$$

Since $g(z, 0) = 0$, we have $g_z(t, 0) = 0$. This proves the convergence of $f_z(t, 0)$ and $g_z(t, 0)$.

To complete the proof of the convergence of F , it suffices to show that $t \mapsto \bar{f}_\tau(t, 0)$ and $t \mapsto \bar{g}_\tau(t, 0)$ are convergent. For the first, differentiate both sides of (5.13) with respect to t^1 and set $t^1 = 0$. In fact this shows that $\bar{f}_\tau(t, 0)$ is a rational function whose coefficients depend only on the first two derivatives of f and g at 0. To prove that $t \mapsto g_w(t, 0)$ is convergent, set $t^3 = t^1$ in (5.11) to obtain

$$(5.17) \quad -\bar{g}(t^2, 2it^1t^2) = 2if(t^1, 0)\bar{f}(t^2, 2it^1t^2).$$

The desired convergence then follows from that of $\bar{f}(t^1, 0)$ by differentiating (5.17) with respect to t^1 and setting $t^1 = 0$.

REMARK 5.1. In the above computation one can in fact obtain an explicit formula for any local biholomorphism mapping M to itself, and it can be readily seen that all such are given by rational mappings. Furthermore, it follows that any such mapping is determined by its derivatives of order two at 0.

Although the above calculation is for a very special case, an argument along the same lines can be used to show that if $M \subset \mathbb{C}^N$ is a real analytic generic submanifold which is finitely nondegenerate and of finite type at p , then any invertible formal mapping $F : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^N, p)$ sending M into itself is convergent. (This is a special case of Theorem 3.2 stated above.) In fact, the proofs of most of the results stated in this survey rely on Theorem 5.1 and also on a crucial use of Artin's approximation theorem.

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DEPARTMENT OF MATHEMATICS, 0112, UNIVERSITY OF CALIFORNIA AT SAN DIEGO, LA JOLLA, CA 92093-0112, USA

E-mail address: lrothschild@ucsd.edu