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MAPPINGS OF THREE-DIMENSIONAL CR MANIFOLDS AND THEIR HOLOMORPHIC EXTENSION

M. S. BAOUENDI, S. R. BELL, AND LINDA PREISS ROTHSCHILD

CONTENTS

I. Introduction

§1. Introduction and main results. A smooth manifold M is called a *CR* (*Cauchy–Riemann*) *manifold* if there is a subbundle $\mathscr{V}$ (called the *CR bundle*) of $\mathbf{C}TM$, the complexified tangent bundle of M, satisfying $[\mathscr{V}, \mathscr{V}] \subset \mathscr{V}$ and $\mathscr{V} \cap \overline{\mathscr{V}} = \{0\}$. If M and M' are CR manifolds with CR bundles $\mathscr{V}$ and $\mathscr{V}'$, a smooth mapping H: $M \to M'$ is called CR if for every $p \in M$,

$$H'(\theta) \in \mathscr{V}'_{H(p)} \tag{1.1}$$

for all $\theta \in \mathscr{V}_p$, the fiber of $\mathscr{V}$ at p. Here H': $\mathbf{C}TM \to \mathbf{C}TM'$ is the differential map of H. If M and M' are three-dimensional, then necessarily, $dim_{\mathbf{C}}\mathscr{V} = dim_{\mathbf{C}}\mathscr{V}' = 1$.

Locally, near $p_0 \in M$ and $p_0' = H(p_0) \in M'$, there exist smooth nonvanishing vector fields L and L', sections of $\mathscr{V}$ and $\mathscr{V}'$, respectively. Condition (1.1) can then be written

$$H'(L_p) = \lambda(p) L'_{H(p)} \tag{1.2}$$

for some smooth function λ defined on M near p_0.

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By a *string* on M we shall mean a differential operator of the form

$$S = A_1 \dots A_\ell, \tag{1.3}$$

where each $A_i = L$ or $\bar{L}$, and $|S| = \ell$ is the length of the string S.

The *multiplicity* of H at p_0 is defined as the smallest integer $k + 1$ for which there exists a string S, $|S| = k$, with

$$S\lambda(p_0) \neq 0. \tag{1.4}$$

(Multiplicity 1 will mean $\lambda(p_0) \neq 0$.) It is clear that the multiplicity is independent of the choice of L and L'.

Following Kohn [15], the three-dimensional CR manifold M is of type m at p_0 ($m \geqslant 2$) if the shortest commutator of L and $\bar{L}$ which is not a linear combination of L and $\bar{L}$ at p_0, is of length m. (If no such m exists, then M is not of finite type at p_0.) We write m' for the type of M' at p_0'.

In this paper we shall always assume that $dim_{\mathbf{R}} M = dim_{\mathbf{R}} M' = 3$. Our first result connects the types m and m' with the multiplicity of H at p_0.

THEOREM 1. *Let H be a CR mapping from M into M', with $H(p_0) = p_0'$, and $k + 1$ its multiplicity at p_0, $0 \leqslant k < \infty$. If M' is of finite type m' at p_0', then the following hold*:

(*i*) *H is a local diffeomorphism if and only if $k = 0$.*

(*ii*) *M is of finite type $m = (k + 1)m'$ at p_0.*

(*iii*) *If S is a string for which $S\lambda(p_0) \neq 0$, $|S| = k$, then $S = L^k$. In particular, $L^k\lambda(p_0) \neq 0$.*

(*iv*) *If $k > 0$, then $dim_{\mathbf{C}} H'(\mathbf{C}T_{p_0}M) = 1$ and*

$$H'(\mathbf{C}T_{p_0}M) \not\subset \mathscr{V}'_{p_0'} \oplus \bar{\mathscr{V}}'_{p_0'}.$$

Theorem 1 is proved in part II, §§2–5.

Remark. If (iv) is added as a hypothesis of Theorem 1, then the proofs of (ii) and (iii) may be considerably simplified (see §4 for similar arguments). In the case where M and M' are embedded pseudoconvex hypersurfaces in $\mathbf{C}^2$ and H extends as a holomorphic mapping from one side of M to one side of M', (iv) always holds (see §9, Theorem 9). Also, in this case (i) follows from Derridj [9].

We say that the mapping H is *flat* at p_0 if all partial derivatives of H of any order vanish at p_0.

The following result is a corollary of Theorem 1 and deals with the case where both M and M' are assumed to be of finite type. It is proved in §6.

THEOREM 2. *Let H be a CR mapping from M to M' and assume that M and M' are of finite type at p_0 and $p_0' = H(p_0)$, respectively. The following conditions are*

equivalent:
(*i*) *H is of finite multiplicity at* p_0.
(*ii*) $H'_{p_0} \neq 0$.
(*iii*) *H is not flat at* p_0.

Examples given in §6 show that the conclusions of Theorems 1 and 2 are no longer valid if the finite type conditions are dropped in the assumptions.

In part III we assume that M and M' are real-analytic and we prove, under the assumptions of Theorem 1, that H is real-analytic in a neighborhood of p_0. This extends the classical result of Lewy [16] and Pinčuk [19] for M, M' strictly pseudoconvex and H a diffeomorphism, and also the more general diffeomorphic case proved by Baouendi–Jacobowitz–Treves [2].

Our result is the following:

THEOREM 3. *If H is a smooth CR mapping from M into M', where M and M' are real-analytic CR manifolds, and if H is of finite multiplicity at* p_0, *and M' of finite type at* $p_0' = H(p_0)$, *then H is real-analytic in a neighborhood of* p_0.

If M and M' are real-analytic and embedded in $\mathbf{C}^2$, then under the assumptions of Theorem 3, H extends as a proper holomorphic map in a neighborhood of p_0 in $\mathbf{C}^2$, valued in $\mathbf{C}^2$. We show in §9 (Theorem 8) that, if H is of multiplicity κ at p_0, and if $\kappa \geqslant 2$, then branching occurs at p_0 and the mapping is κ to 1 near p_0.

In part IV we consider the case of pseudoconvex manifolds, and we give some applications of Theorems 1, 2, and 3 to questions involving globally defined CR mappings and extension of proper holomorphic mappings between bounded domains with real-analytic boundaries. We state now some of these results.

THEOREM 4. *Let H be a smooth CR self-mapping of a real-analytic compact CR manifold M. Assume M is of finite type, and H of finite multiplicity at each point of M. Then H is a local analytic diffeomorphism.*

For domains in $\mathbf{C}^2$ we have the following results.

THEOREM 5. *Let D and D' be two open bounded sets of* $\mathbf{C}^2$ *with real-analytic boundaries. Let* $F\colon D \to D'$ *be a proper holomorphic mapping with* $F \in C^\infty(\overline{D})$. *If F is nowhere flat on* ∂D, *then F extends holomorphically to a neighborhood of* $\overline{D}$. *More precisely, there exist* D_1 *and* D_1', *two open bounded neighborhoods of* $\overline{D}$ *and* $\overline{D}'$, *respectively, such that F extends as a holomorphic proper mapping from* D_1 *into* D_1'.

The following generalizes a result of Bedford–Bell [5].

THEOREM 6. *Let D be an open bounded set in* $\mathbf{C}^2$ *with real-analytic boundary and F a proper holomorphic self-mapping of D. If* $F \in C^\infty(\overline{D})$ *and F is nowhere flat on* ∂D, *then F extends as a biholomorphism from an open neighborhood of* $\overline{D}$ *onto another.*

Using the result of Bell–Catlin [8] and Diederich–Fornaess [11], we obtain

THEOREM 7. *Let D and D' be two bounded pseudoconvex domains in $\mathbf{C}^2$ with real-analytic boundaries. If F is a proper holomorphic mapping from D into D', then F extends as a proper holomorphic mapping from a neighborhood of $\overline{D}$ to a neighborhood of $\overline{D}'$.*

This theorem generalizes and improves results of Bell [7] and of Bedford–Bell [6] in which much stronger additional hypotheses are placed on the domains. It is also more general than a result announced recently by S. Maingot [17].

The results of our paper were announced in [1].

II. Proof of Theorems 1 and 2

§2. Strings, commutators, and rearrangements. We give here some preliminary definitions and lemmas needed for the proof of Theorem 1.

For any positive integer p we introduce a linear ordering on the set of all strings of length p as follows. Any string S can be written

$$S = \overline{L}^{q_r} L^{p_r} \dots L^{p_2} \overline{L}^{q_1} L^{p_1}, \tag{2.1}$$

with $p_i > 0$, $2 \leqslant i \leqslant r$, $q_i > 0$, $1 \leqslant i \leqslant r - 1$, and $\Sigma(q_i + p_i) = p = |S|$. If $\tilde{S}$ is another such string, $|\tilde{S}| = p$, with associated integers $\tilde{p}_i, \tilde{q}_i, 1 \leqslant i \leqslant \tilde{r}$, we write $S > \tilde{S}$ if $p_i = \tilde{p}_i$ and $q_i = \tilde{q}_i$ for $i < r_0$ and either $p_{r_0} > \tilde{p}_{r_0}$ or $p_{r_0} = \tilde{p}_{r_0}$ and $q_{r_0} > \tilde{q}_{r_0}$.

If S is a string as in (1.3) and B a vector field on M, we shall write

$$(ad\,S)B = [A_1, [A_2[\dots[A_\ell, B]\dots]]],$$

with the usual notation for commutators $[A, B] = AB - BA$. Any commutator of L and $\overline{L}$ of length $p + 2$ is a linear combination of commutators of the form $(ad\,S)[L, \overline{L}]$, for some strings S with $|S| = p$.

(2.2) LEMMA. *If A and B are commutators (of the L and $\overline{L}$) of lengths n_1 and n_2, respectively, with $n_i \geqslant 2$, $i = 1, 2$, then near p_0*

$$[A, B] = \sum_{j=1}^{\max(n_1+1,\, n_2+1)} \gamma_j C_j, \qquad \gamma_j \in C^\infty, \tag{2.3}$$

where C_j is a commutator of length j.

Proof. In local coordinates we write

$$A = \alpha_1 L + \alpha_2 \overline{L} + \alpha_3 \frac{\partial}{\partial s} \tag{2.4}$$

$$B = \beta_1 L + \beta_2 \overline{L} + \beta_3 \frac{\partial}{\partial s}, \tag{2.5}$$

with L, $\bar{L}$, $\partial/\partial s$ linearly independent at p_0. Hence $\alpha_3(\partial/\partial s)$ and $\beta_3(\partial/\partial s)$ are linear combinations of commutators of lengths $\leqslant n_1$ and n_2, respectively. Since

$$[A, B] = \left[\alpha_3 \frac{\partial}{\partial s}, \beta_3 \frac{\partial}{\partial s}\right] \tag{2.6}$$

modulo a linear combination of commutators of lengths $\leqslant \max(n_1 + 1, n_2 + 1)$, the result follows.

We remark that Lemma (2.2) holds because a CR structure of codimension 1 is semirigid in the sense of [3]. Note that $\max(n_1 + 1, n_2 + 1) \leqslant n_1 + n_2 - 1$.

If S and $\tilde{S}$ are strings of length p containing the same number of L's, then we say $\tilde{S}$ is a *rearrangement* of S.

(2.7) LEMMA. *If $\tilde{S}$ is a rearrangement of S, with $|S| = |\tilde{S}| = p$, then $ad\,S[L, \bar{L}] - ad\,\tilde{S}[L, \bar{L}]$ is a linear combination of commutators of lengths $\leqslant p + 1$.*

Proof. $S - \tilde{S}$ is a linear combination of terms of the form $S_1[L, \bar{L}]S_2$ with $|S_1| + |S_2| = p - 2$. Since

$$ad(S_1[L, \bar{L}]S_2)[L, \bar{L}] = ad\,S_1 ad[L, \bar{L}](ad\,S_2[L, \bar{L}]),$$

the result follows from Lemma (2.2).

We write $[j : q]$ for any commutator of the form $ad\,S[L, \bar{L}]$, where $|S| = q - 2$ and S contains exactly j $\bar{L}$'s. By Lemma (2.7), $[j : q]$ is, up to a linear combination of commutators of length $< q$, of the form

$$(ad\,\bar{L})^j (ad\,L)^{q-2-j}[L, \bar{L}]. \tag{2.8}$$

If the type of M at p_0 is $m < \infty$, let j_0 be minimal so that $[j_0 : m]$ is not a linear combination of L and $\bar{L}$, at p_0, and let j_0' be the corresponding number associated to M' at p_0', with type m'. The number $k + 1$ denotes the multiplicity of H at p_0 throughout the rest of the paper (see (1.4)).

(2.9) LEMMA. *If $2 \leqslant q < m$ or if $q > m$, then for any string S with $|S| = k - q \geqslant 0$ and any $j < q$,*

$$[j : q]S\lambda(p_0) = 0. \tag{2.9}$$

Proof. The lemma is clear for $m = \infty$. For $q < m < \infty$,

$$[j : q] = aL + b\bar{L} + c[j_0 : m]$$

with $c(p_0) = 0$. Hence $[j : q]S\lambda(p_0) = 0$. For $q > m$, $[j : q]$ is a linear combination of commutators of lengths $< q$, so that $[j : q]S\lambda(p_0) = 0$ in this case also.

(2.10) LEMMA. *Assume* $m \leqslant k$, k *as in* (1.4). *Suppose that for all strings* S, $|S| = k - m$, *and all* p, $0 \leqslant p \leqslant m - 1$,

$$[p:m]S\lambda(p_0) = 0. \tag{2.11}$$

Then for any strings $T_1, T_2, |T_i| = k$, *for which* T_2 *is a rearrangement of* T_1,

$$T_1\lambda(p_0) = T_2\lambda(p_0). \tag{2.12}$$

Conversely, (2.12) *implies* (2.11).

Proof. By commuting terms in T_1 and T_2, we may check that $T_1\lambda(p_0) - T_2\lambda(p_0)$ is a linear combination of terms of the form $S_1[j:q]S_2\lambda(p_0)$, where $|S_1| + |S_2| + q = k$. By further transpositions, it can be shown that these are linear combinations of terms of the form (2.11). The converse is obtained by expanding (2.11).

We shall refer to (2.12) as the *rearrangement hypothesis*.

The following will be needed in §5.

(2.13) LEMMA. *If* $j < j_0$ *and* $2 \leqslant q \leqslant m$, *then*

$$[j:q] = aL + b\bar{L} + c[j_0:m], \qquad a, b, c \in C^\infty, \tag{2.14}$$

where $c(p_0) = 0$, *and for any string* S, $|S| = m - q$, *such that* S *contains exactly* $(j_0 - j)$ $\bar{L}$*'s*,

$$Sc(p_0) = 1. \tag{2.15}$$

Furthermore, if $|\tilde{S}| = m - q$ *and* $\tilde{S}$ *contains fewer than* $(j_0 - j)$ $\bar{L}$*'s, then*

$$\tilde{S}c(p_0) = 0. \tag{2.16}$$

Also, (2.16) *holds if* $|\tilde{S}| < m - q$.

Proof. By the definition of m and j_0 and by using Lemma (2.7),

$$ad\,L^{m-q-p}ad\,\bar{L}^p[j:q] = [j_0:m] + \mathcal{O}([m-1]),$$

where $\mathcal{O}([\ell])$ denotes a linear combination of commutators of lengths $\leqslant \ell$, if $p = j_0 - j$. If $p < j_0 - j$,

$$ad\,L^{m-q-p}ad\,\bar{L}^p[j:q] = \alpha[j_0:m] + \mathcal{O}([m-1]) \tag{2.17}$$

with $\alpha(p_0) = 0$. If $|\tilde{S}| < m - q$, then

$$ad\,\tilde{S}[j:q] = \mathcal{O}([m-1]),$$

so that (2.16) and (2.15) follow by inductive arguments with Lemma (2.7).

§3. **Outline of the proof of Theorem 1.** The main step in proving Theorem 1 is to show

$$H'(\mathbb{C}T_{p_0}M) \not\subset \mathscr{V}'_{p_0'} \oplus \bar{\mathscr{V}}'_{p_0'}.$$

For this, we proceed by first assuming that the rearrangement hypothesis (2.12) holds. In fact, we eventually show in general not only that (2.12) is satisfied but also the much stronger statement ((iii) of Theorem 1) that if T is a string of length k, then $T\lambda(p_0) = 0$ unless $T = L^k$.

Assuming (2.12), we explicitly find a vector field Y on M for which $H'(Y_{p_0}) \neq 0 \bmod L', \bar{L}'$. In fact, Y is a commutator of length $(k+1)m'$ of the $L, \bar{L}$. To find Y we use the formula (3.2) below, which may be iterated to calculate $H'(C_{p_0})$, C any commutator. (It can be seen without much difficulty that $H'(C_{p_0}) = 0 \bmod L', \bar{L}'$ for any commutator C of length less than $(k+1)m'$.) The most important idea in constructing Y as above is to use the ordering on strings introduced in §2. This allows us to avoid cancellations in the coefficient of $[j_0' : m']'$.

In the second half of the proof (given in §5) we assume the rearrangement hypothesis fails. As is shown in Proposition (3.4) below, this implies that $H'_{p_0} \equiv 0$ and $m \leqslant k$. We show that under these conditions the rearrangement hypothesis does hold, contradicting the assumption.

It will be essential to calculate $H'(C)$, where C is a commutator of L and $\bar{L}$. The following is obtained by a modification of a standard argument (see, e.g., [20], p. 95.)

(3.1) PROPOSITION. *Suppose A and B are vector fields on a manifold M, and $H: M \to M'$ a smooth mapping from M into a manifold M' defined locally near $p_0 \in M$. Suppose there exist smooth functions $\alpha_j(u)$, $\beta_j(u)$ on M, and smooth vector fields A_j' and B_j' on M', $1 \leqslant j \leqslant r$, such that*

$$H'(A_u) = \sum \alpha_j(u) A'_{jH(u)} \quad \text{and} \quad H'(B_u) = \sum \beta_j(u) B'_{jH(u)}.$$

Then

$$H'([A,B]_u) = \sum_j (A\beta_j)(u) B'_{jH(u)} - \sum_j (B\alpha_j)(u) A'_{jH(u)} + \sum_{p,q} \alpha_p(u)\beta_q(u)\left[A_p', B_q'\right]_{H(u)}. \tag{3.2}$$

We shall apply Proposition (3.1) to calculate $H'([A, B])$ when $A = L$ or $\bar{L}$ and B is a commutator. In particular, we obtain

$$H'([L, \bar{L}]) = (L\bar{\lambda})\bar{L}' - (\bar{L}\lambda)L' + \lambda\bar{\lambda}[L', \bar{L}']. \tag{3.3}$$

(3.4) PROPOSITION. *Let M, M', H be as in Theorem* 1, *and assume $m' < \infty$, $k \geqslant 1$. For the following conditions,*
(*i*) $H'_{p_0} \neq 0$,
(*ii*) $m \geqslant k+1$,
(*iii*) *the rearrangement hypothesis* (2.12) *holds,*
we have the implications (*i*) *implies* (*ii*), *and* (*ii*) *implies* (*iii*).

Proof. For $[p:q]$ with $q < k+1$, it is clear from repeated application of (3.2) that $H'([p:q])(p_0) = 0$. This proves (i) implies (ii).

For (ii) implies (iii) observe that if $|T_i| = k$, $i = 1, 2$, then $T_1 - T_2$ is a linear combination of terms of the form $[p:q]S$ with $q \leqslant m-1$. Hence $(T_1 - T_2)\lambda(p_0) = 0$ by Lemma (2.9).

§4. Proof of Theorem 1 under the rearrangement hypothesis. We shall prove Theorem 1 under the additional hypothesis (2.12) of rearrangement. By Proposition (3.4), this will prove Theorem 1 if $H'_{p_0} \neq 0$ or if $m \geqslant k+1$.

First, for $\lambda(p_0) \neq 0$, we may assume $\lambda \equiv 1$. We shall show that H'_{p_0} is surjective, which will prove H is a local diffeomorphism. Let j'_0 be as above. By iterating (3.2) we obtain

$$H'\big([j'_0 : m']_{p_0}\big) = [j'_0 : m']'_{p'_0}, \tag{4.1}$$

which implies that, when $k = 0$, H'_{p_0} is an isomorphism from $T_{p_0}M$ onto $T_{p'_0}M'$.

For $k \geqslant 1$ we need more information on the calculations obtained by applying (3.2). Suppose C' is a commutator of L', $\overline{L}'$, and suppose S_j, T_ℓ, $1 \leqslant j \leqslant p$, $1 \leqslant \ell \leqslant q$, are strings of L, $\overline{L}$ of lengths $\leqslant k$; then

$$A' = (S_1\lambda)(S_2\lambda)\dots(S_p\lambda)(T_1\overline{\lambda})(T_2\overline{\lambda})\dots(T_q\overline{\lambda})C' \tag{4.2}$$

defines a tangent vector at every point $H(u)$, with $u \in M$, where we have used the convention that the functions in the product are evaluated at u and C' at $H(u)$. We define, for $B = L$ or $\overline{L}$, an operator $\pi(B)$ acting on expressions of the form (4.2) by

$$\begin{aligned}\pi(B)A' = {} & B\big((S_1\lambda)\dots(S_p\lambda)(T_1\overline{\lambda})\dots(T_q\overline{\lambda})\big)C' \\ & + \kappa(S_1\lambda)\dots(S_p\lambda)(T_1\overline{\lambda})\dots(T_q\overline{\lambda})[B', C'],\end{aligned} \tag{4.3}$$

where $B' = L'$, $\kappa = \lambda$ if $B = L$, and $B' = \overline{L}'$, $\kappa = \overline{\lambda}$ if $B = \overline{L}$. By expanding the first term on the right in (4.3), we see that $\pi(B)A'$ is a sum of terms of the same form as (4.2). By iteration and linearity we may extend the operator π by

$$\pi(S) = \pi(B_1)\pi(B_2)\dots\pi(B_r), \tag{4.4}$$

for $S = B_1 B_2 \dots B_r$.

The following is the key step in the proof of Theorem 1.

(4.5) LEMMA. *Assume that the rearrangement hypothesis* (2.12) *holds. Let* j_0' *be as in* §2. *Then there is a string* $S_0 = L^{p_2}\bar{L}^{q_1}L^{p_1}$ *with* $p_1 + q_1 + p_2 = m'(k+1) - 2$ *such that the expansion of*

$$\pi(S_0)(L\bar{\lambda}\bar{L}' - \bar{L}\lambda L' + (\lambda\bar{\lambda})[L', \bar{L}']) \tag{4.6}$$

contains exactly one nonzero term at p_0', *mod* $L', \bar{L}'$.

Proof. From the definitions of k, j_0' and $\pi(S)$ it is clear that there exist strings S, $|S| = m'(k+1) - 2$, for which the expansion of (4.6) (prior to any cancellation) contains at least one nonzero term at p_0', mod L', $\bar{L}'$. Let S_0 be the maximum string in that set, in the linear ordering defined in §2. We claim that the expansion (4.6) then has exactly one nonzero term, which is a multiple of

$$(\bar{L}^{k-\ell_1}L^{\ell_1}\lambda)^{m'-j_0'-1}(\bar{L}^{k-\ell_2}L^{\ell_2}\bar{\lambda})(\bar{L}^{\ell_1}L^{k-\ell_1}\bar{\lambda})^{j_0'}[j_0' : m']', \tag{4.7}$$

where ℓ_1 is the largest integer for which $\bar{L}^{k-\ell_1}L^{\ell_1}\lambda(p_0) \neq 0$, and ℓ_2 is the largest integer for which $L^{k-\ell_2}\bar{L}^{\ell_2}\lambda(p_0) \neq 0$. (We note that ℓ_1 and ℓ_2 exist by the rearrangement hypothesis and the fact that k is finite.)

For this, note that if S_0 is maximal, it must be of the form $L^{p_2}\bar{L}^{q_1}L^{p_1}$, and $p_1 = (m' - j_0' - 2)(\ell_1 + 1) + (\ell_1 + \ell_2)$. Indeed, the expansion of $\pi(L^{p_1})(\lambda\bar{\lambda}[L', \bar{L}'])$ contains a term of the form

$$(L^{\ell_1}\lambda)^{m'-j_0'-2}(L^{\ell_1}\lambda)(L^{\ell_2}\bar{\lambda})(ad\,L')^{m'-j_0'-2}[L', \bar{L}']), \tag{4.8}$$

which leads to a term of the form (4.7) after applying $\pi(L^{p_2})\pi(\bar{L}^{q_1})$ for some p_2, q_1. On the other hand, $\pi(L^{p_1+1})(\lambda\bar{\lambda}[L', \bar{L}'])$ cannot lead to a nonzero term by the maximality of ℓ_1, ℓ_2 and the minimality of j_0'.

We claim that (4.8) is the only term in the expansion of

$$\pi(L^{p_1})((L\bar{\lambda})\bar{L}' - (\bar{L}\lambda)L' + \lambda\bar{\lambda}[L', \bar{L}']) \tag{4.9}$$

which can lead to a nonzero term after applying $\pi(L^{p_2}\bar{L}^{q_1})$ for some p_2, q_1 with $p_2 + q_1 = m'(k+1) - p_1 - 2$. Indeed, any other term would have a factor of either $L^{\ell_1'}\lambda$, $\ell_1' > \ell_1$ or $L^{\ell_2'}\bar{\lambda}$, $\ell_2' > \ell_2$ or $(ad\,L')^n[L', \bar{L}']$ with $n > m' - j_0' - 2$. By the definitions of ℓ_1 and ℓ_2, $\bar{L}^{k-\ell_1'}L^{\ell_1'}\lambda$ and $\bar{L}^{k-\ell_2'}L^{\ell_2'}\bar{\lambda}$ both vanish at p_0. Also, if $n > m' - j_0' - 2$, then $(ad\,\bar{L}')^{m'-n-2}(ad\,L')^n[L', \bar{L}']$ is a linear combination of $L', \bar{L}'$ at p_0, by minimality of j_0'. Therefore, it is impossible to find another term in the expansion of $\pi(L^{p_1})(\lambda\bar{\lambda}[L', \bar{L}'])$ leading to a nonzero term.

To complete the proof of Lemma (4.5), we must also examine the terms in (4.6) arising from $\pi(L^{p_1})(L\bar{\lambda}\bar{L}' - \bar{L}\lambda L')$. Similar arguments to those used above show that this also gives a positive multiple of (4.8), proving Lemma (4.5).

We may now complete the proof of Theorem 1 under the rearrangement hypothesis. Indeed, since

$$S_1[j:q]S_2\lambda(p_0) = 0 \tag{4.10}$$

for all strings S_1, S_2 with $|S_1| + q + |S_2| \leqslant k$, by Lemma (2.10), it follows from successive applications of (3.2) that for any string S, $|S| \leqslant m'(k+1) - 2$,

(4.11)

$$H'((adS)[L,\bar{L}])(p_0) = \pi(S)(L\bar{\lambda}\bar{L}' - \bar{L}\lambda L' + \lambda\bar{\lambda}[L',\bar{L}']), \operatorname{mod}(L',\bar{L}').$$

Hence by Lemma (4.5),

$$\dim H'(\mathbf{C}T_{p_0}M) = 1 \operatorname{mod} L', \bar{L}'. \tag{4.12}$$

To complete the proof of Theorem 1, note that since $H'(adS[L,\bar{L}])(p_0) = 0 \operatorname{mod} L', \bar{L}'$ for $|S| < m'(k+1) - 2$, it follows from (4.12) and Lemma (4.5) that $m = m'(k+1)$. If S is any string, $|S| = k$, we need to show

$$S\lambda(p_0) = 0 \quad \text{unless } S = L^k. \tag{4.13}$$

For this, we may assume, by the rearrangement hypothesis, that $S = T\bar{L}$, $|T| = k - 1$. Then, since $k + 1 < m$,

$$H'(adT[L,\bar{L}])(p_0) = 0,$$

we obtain (4.13) by calculating the left-hand side of (3.2) and checking the coefficient of L', using Lemma (2.10).

§5. Proof of Theorem 1 without the rearrangement hypothesis. By Proposition (3.4), if the rearrangement hypothesis does not hold, then

$$H'_{p_0} = 0 \quad \text{and} \quad m \leqslant k.$$

We shall assume these throughout this section and arrive at the contradiction that the rearrangement hypothesis does hold. For this, we use the fact that for any commutator $Z = adT[L,\bar{L}]$, the coefficients of L', $\bar{L}'$, and $[j_0':m']'$ in $H'(Z_{p_0})$ all vanish. In Lemma (5.1) below we apply this to the coefficient of L' to prove that vanishing of terms of the form $S\bar{L}^p\lambda(p_0)$, $|S| = k - p$, implies vanishing of terms of the form $[q:m]T\lambda(p_0)$, $|T| = k - m$, $q \geqslant p - 1$. In Lemma (5.6) we use the vanishing of the coefficient of $[j_0':m']'$ to show that $T\bar{L}^{m-j_0-1}\lambda(p_0) = 0$, $|T| = k - (m - j_0 - 1)$. Finally, we apply Lemmas (5.1) and (5.6) together with a simple calculation to show that $S\bar{L}\lambda(p_0) = 0$ for all S with $|S| = k - 1$. This, together with Lemma (5.1) and (2.10), will prove that $S\lambda(p_0) = 0$ for all strings

S with $|S| = k$, $S \neq L^k$, therefore contradicting the assumption that the rearrangement hypothesis fails.

A key computation is the following.

(5.1) LEMMA. *Suppose that for some p, $k \geqslant p > 0$, we have*

$$S\bar{L}^p\lambda(p_0) = 0 \tag{5.2}$$

for all strings S with $|S| = k - p$. Then for all q, $m - 1 \geqslant q \geqslant p - 1$ and all strings T with $|T| = k - m$,

$$[q:m]T\lambda(p_0) = 0. \tag{5.3}$$

Proof. We prove this lemma by induction on q. It holds for $q = m - 1$, since $[m-1:m] = 0$. Assume (5.3) for q replaced by $q+1, q+2, \ldots, m-2$; we shall prove it for q.

Assume that T, $|T| = k - m$, has q_0 $\bar{L}$'s. We first prove it for

$$T_0 = \bar{L}^{q_0}L^{k-m-q_0}.$$

Since $H'_{p_0} = 0$, the coefficient α of L' in $H'(ad\,T_0 ad\,L^{m-(q+1)}ad\,\bar{L}^q[L,\bar{L}])(p_0)$ is 0. Since

$$\begin{aligned}\alpha = & -T_0L^{m-(q+1)}\bar{L}^{q+1}\lambda(p_0) - \bar{L}^{q_0}L^{k-q_0-(q+2)}(ad\,\bar{L})^q[L,\bar{L}]\lambda(p_0)\\ & -\bar{L}^{q_0}L^{k-q_0-(q+2)-1}ad\,L(ad\,\bar{L})^q[L,\bar{L}]\lambda(p_0) - \cdots\\ & -\bar{L}^{q_0}(ad\,L)^{k-(q+2)-q_0}(ad\,\bar{L})^q[L,\bar{L}]\lambda(p_0)\end{aligned} \tag{5.5}$$

with the first term on the right in (5.5) equal to 0 by (5.2), the sum of the remaining terms is 0. We now shift the commutators to the left in each of the remaining terms, which gives a sum of terms with negative coefficients, each of which is of the form $[r:N]T\lambda(p_0)$. If $N < m$, this term vanishes, since $[r:N](p_0) = 0 \bmod L, \bar{L}$ for $m > N$. If $N > m$ this term also vanishes, since then $[r:N]T$ can be written as a linear combination of strings of lengths $< k$. If $[r:m]T\lambda(p_0)$ appears, then necessarily $r \geqslant q$, since $(\mathrm{ad}\,\bar{L})^q$ appears in each term. If $r > q$ this term vanishes by the inductive hypothesis. If $r = q$ it is easy to see that $T = T_0$. Hence, all the nonzero terms are negative multiples of $[q:m]T_0\lambda(p_0)$. This proves (5.3) for $T = T_0$.

Proceeding by induction on the position of the rightmost $\bar{L}$ in T, we may assume that (5.3) holds for all T for which the rightmost $\bar{L}$ is in a position $\leqslant r-1$ units from the left, $r \leqslant k - m$. Let T be a string for which the rightmost $\bar{L}$ is in the rth position. By calculating the coefficient of L' in $H'(ad\,T ad\,L^{m-(q+1)}(ad\,\bar{L})^q[L,\bar{L}])(p_0)$ and using arguments as above, we may prove (5.3) for this T also. This proves Lemma (5.1).

We need to prove that the hypothesis (5.2) of Lemma (5.1) holds for all $p > 0$, since then rearrangement holds, by Lemma (2.10). The following technical result is needed.

(5.6) LEMMA. *If $m \leqslant k$, then for all strings T, $|T| = k - (m - j_0 - 1)$,*

$$T\bar{L}^{m-j_0-1}\lambda(p_0) = 0. \tag{5.7}$$

We shall now complete the proof of the rearrangement hypothesis, assuming Lemma (5.6), and then prove Lemma (5.6).

By the definition of j_0 (after taking complex conjugates) we may write

$$[L, \bar{L}] = \alpha[m - j_0 - 2 : m] + uL + v\bar{L},$$

where $S_1\alpha(p_0) = 0$ for all S_1 with $|S_1| < m - 2$ by Lemma (2.13). We identify the coefficients of L' after applying H' to both sides of the above identity, using (3.3) and (3.2). We obtain

$$- \bar{L}\lambda = -\alpha L^{j_0}\bar{L}^{m-j_0-1}\lambda - \alpha \sum_{q=1}^{j_0} L^{j_0-q}(ad\,L)^{q-1} \times (ad\,\bar{L})^{m-j_0-2}[L, \bar{L}]\lambda + \mathcal{O}, \tag{5.8}$$

where $S\mathcal{O}(p_0) = 0$ for $|S| \leqslant k - 1$. Indeed, any term coming from the third term on the right in (3.2) will have a coefficient annihilated by any S, $|S| \leqslant k - 1$, since $m \leqslant k$.

Now we shall apply an arbitrary string S, $|S| = k - 1$, to (5.8) and show that the right-hand side vanishes at p_0. Since $S_1\alpha = 0$ for $|S_1| < m - 2$, we need only show

$$S_2L^{j_0}\bar{L}^{m-j_0-1}\lambda(p_0) = 0, \tag{5.9}$$

and

$$S_2L^{j_0-q}(ad\,L)^{q-1}(ad\,\bar{L})^{m-j_0-2}[L, \bar{L}]\lambda(p_0) = 0, \tag{5.10}$$

for all S_2, $|S_2| \leqslant k - m + 1$. First, (5.9) is immediate from Lemma (5.6). For (5.10) we first pull all the commutators to the left-hand side of the string and then apply both Lemmas (5.1) and (5.6).

Hence we have shown that for all S, $|S| = k - 1$,

$$S\bar{L}\lambda(p_0) = 0. \tag{5.11}$$

But now the rearrangement hypothesis follows from Lemma (5.1) (with $p = 1$)

and then Lemma (2.10). (Note that in fact we have proved that $T\lambda(p_0) = 0$, $|T| = k$, $T \neq L^k$.)

Proof of Lemma (5.6). Because of the technical nature of this proof, we will divide it into several parts.

Claim 1. $H'((ad\,L)^{m-j_0-1}[L, \overline{L}]) = \beta_1[j_0' : m']' \bmod L', \overline{L}'$, where $S\beta_1(p_0) = 0$ for $|S| \leqslant m'(k+1) - (m - j_0 + 1)$.

To prove this, we first note that by the definition of j_0, if S_1 is any string with $|S_1| = j_0 - 1$, then

$$ad\,S_1(ad\,L)^{m-j_0-1}[L, \overline{L}](p_0) = 0 \bmod L, \overline{L} \tag{5.12}$$

since the commutator on the left in (5.12) has at most j_0 $\overline{L}$'s, while $[j_0 : m]$ has $(j_0 + 1)$ $\overline{L}$'s and j_0 is minimal. By a slight variation of Lemma (2.13) we may write

$$(ad\,L)^{m-j_0-1}[L, \overline{L}] = \alpha L + \beta\overline{L} + \gamma[j_0 : m], \tag{5.13}$$

where $S_1\gamma(p_0) = 0$ for all strings S_1 with $|S_1| \leqslant j_0 - 1$. Now by (5.13)

$$H'\big((ad\,L)^{m-j_0-1}[L, \overline{L}]\big) = \gamma H'([j_0 : m]) \bmod L', \overline{L}'. \tag{5.14}$$

Next, we calculate $H'([j_0 : m])$ by using (3.2). We may assume, by Lemma (2.7), that $[j_0 : m] = (ad\,\overline{L})^{j_0}(ad\,L)^{m-j_0-2}[L, \overline{L}]$. Again by Lemma (2.13), if S' is any string in $L', \overline{L}'$, then

$$ad\,S'[L', \overline{L}'] = \beta_2[j_0' : m']' \bmod L', \overline{L}', \tag{5.15}$$

where $T'\beta_2(p_0') = 0$ if $|T'| + |S'| < m' - 2$. Since $L(f \circ H)(p_0) = \lambda(L'f) \circ H(p_0)$, for any smooth function f on M', from (5.15) we have

$$T(\beta_2 \circ H)(p_0) = 0 \tag{5.16}$$

for any string T with $|T| < (m' - 2 - |S'|)(k+1)$. (Note that another string of length k is needed for every λ or $\overline{\lambda}$ brought out.)

Now, by iterating (3.2) and using (5.15) and (5.16), we obtain by examining the form of each term,

$$H'([j_0 : m]) = \beta_3[j_0' : m']' \bmod (L', \overline{L}'), \tag{5.17}$$

where $S\beta_3(p_0) = 0$ for all strings S with $|S| < m'(k+1) - m$. Now Claim 1 follows immediately from (5.14) and (5.17).

Claim 2. For $0 \leqslant J \leqslant m - j_0 - 1$,

$$(5.18) \qquad (ad\, L')^J [L', \bar{L}'] = \rho_J [j_0' : m']', \operatorname{mod}(L', \bar{L}'),$$

such that

$$(5.19) \qquad \rho_J = L'^J c + \mathcal{O}_J,$$

where c is a smooth function on M' satisfying

$$(5.20) \qquad S'c(p_0') = 1, \quad \text{where } S' = \bar{L}'^{j_0'} L'^{m'-j_0'-2}$$

$$(5.21) \qquad T'c(p_0') = 0 \quad \text{if } |T'| < m' - 2$$

$$(5.22) \qquad U'\mathcal{O}_J(p_0') = 0 \quad \text{if } |U'| \leqslant m' - 2 - J.$$

Furthermore, $S' = \bar{L}'^{j_0'} L'^{m'-j_0'-2}$ is maximal, in the ordering of §2, satisfying (5.20).

To prove Claim 2, we begin with Lemma (2.13), which gives the existence of c satisfying (5.20) and (5.21), as well as the maximality of S', and such that (5.18) holds with $J = 0$. For $J = 1$ we note that

$$(5.23) \qquad (ad\, L')(c[j' : m']') = (L'c)[j' : m']' + c[L', [j' : m']'],$$

so that $\mathcal{O}$ contains c as a factor. Iterating this, we see that (5.22) follows from (5.21), which proves Claim 2.

Claim 3.

$$(5.24) \qquad H'\left((ad\, L)^{m-j_0-1}[L, \bar{L}]\right)$$

$$= \sum_{K=0}^{m-j_0-1} \alpha_K \left(L'^{m-j_0-K-1} c \circ H + \mathcal{O}_{m-j_0-K-1} \circ H\right)$$

$$\times [j_0' : m']'_{H(u)}, \operatorname{mod} L', \bar{L}',$$

where each α_K is a linear combination, with positive coefficients, of terms occurring in the expansion of $L^K(\lambda^{m-j_0-K}\bar{\lambda})$. Furthermore, there is a string S such that

$$(5.25) \qquad S(L'^{m-j_0-K-1} c \circ H)(p_0) \neq 0 \quad \text{and}$$

$$|S| = (m' - 2 - (m - j_0 - K - 1))(k + 1).$$

If S is maximal in the ordering of §2 and satisfying (5.25), then

$$S = S_1 L^{n_1}, \qquad n_1 = m' - j_0' - 2 - (m - j_0 - K - 1). \tag{5.26}$$

To prove Claim 3, first note that (5.24) follows by iterating (3.2) and then substituting for $(ad\,L')^J[L', \overline{L}']$ by using (5.18) and (5.19). The existence of S in (5.25) follows from (5.20), again using $H'(L) = \lambda L'$.

Claim 4. If (5.7) does not hold, then there exists a string S,

(5.27)
$$\begin{aligned}|S| &= (m - j_0 - K + 1)k - K + (m' - 2 - (m - j_0 - K - 1))(k + 1)\\ &= m'(k + 1) - (m - j_0 + 1),\end{aligned}$$

such that

$$\sum_{K=0}^{m-j_0-1} S\Big(\alpha_K\Big(L'^{m-j_0-K-1}c + \mathcal{O}_{m-j_0-K-1}\Big)\circ H\Big)(p_0) \neq 0. \tag{5.28}$$

The proof of Claim 4 is similar to the maximality argument used to prove Lemma (4.5). First, we assert that for each K there is a string S_K satisfying (5.27) such that $S_K(\alpha_K(L'^{m-j_0-K-1}c \circ H))(p_0) \neq 0$. For this, we use the negation of (5.7) and a maximality argument to show that there exists a string U_K, $|U_K| = (m - j_0 - K + 1)k - K$ for which $U_K \alpha_K(p_0) \neq 0$. (In fact, U_K exists even if $\overline{L}$ were replaced by L in (5.7), and in many cases no assumption is necessary.) Next, we can find a string V_K such that (5.25) holds with S replaced by V_K. Then the existence of S_K is proved by the maximality argument. Now, by (5.26) one can check that if S_K is maximal, then the terms in the expansions of the $S_K\alpha_K(L'^{m-j_0-K-1}c \circ H)$, which are nonvanishing at p_0, are all positive multiples of the same monomial. Finally, (5.22) shows that the $\mathcal{O}_J$ terms can be neglected, which completes the proof of Claim 4.

We may now finish the proof of Lemma (5.6). By Claims 1 and 3,

$$S\left(\sum_{K=0}^{m-j_0-1} \alpha_K\Big(L'^{m-j_0-K-1}c \circ H + \mathcal{O}_{m-j_0-K-1} \circ H\Big)\right) = 0 \tag{5.29}$$

for all strings S with

$$|S| \leqslant m'(k + 1) - (m - j_0 + 1), \tag{5.30}$$

which contradicts (5.27) and (5.28) of Claim 4. Hence Lemma (5.6) is proved.

The proof of Theorem 1 is now complete.

§6. Proof of Theorem 2. Examples. The fact that (i) implies (ii) is an immediate corollary of Theorem 1. Indeed, since M' is of finite type at p_0 and H' of finite multiplicity, we have $\dim \operatorname{Im} H'_{p_0} \geq 1$ by Theorem 1. The fact that (ii) implies (iii) is trivial.

It remains to prove that not (i) implies not (iii). Suppose that H is not of finite multiplicity at p_0. Since M is of finite type at p_0, this implies that the function λ given by (1.2) is flat at p_0. Indeed, any partial derivative of λ at p_0 is given by a linear combination of commutators of L and $\bar{L}$ applied to λ, and hence by a linear combination of strings of L and $\bar{L}$ applied to λ. We shall show that H is flat at p_0, contradicting (iii).

Let u_j, u'_j, $1 \leq j \leq 3$, be local real coordinates on M and M', respectively, around p_0 and p'_0 vanishing at these points. Also, let H_j, $1 \leq j \leq 3$, be the components of H written in these coordinates. By (1.2) we have for, $1 \leq j \leq 3$,

$$(LH_j)(u) = \lambda(u)(L'u'_j)(H(u)), \tag{6.1}$$

$$(\bar{L}H_j)(u) = \overline{\lambda(u)}\,(\bar{L}'u'_j)(H(u)). \tag{6.2}$$

Applying any string of L and $\bar{L}$ to (6.1) and (6.2) and using the flatness of λ at p_0, we conclude that each component H_j is flat at p_0, which completes the proof of Theorem 2.

Examples show that the conclusions of Theorems 1 and 2 are no longer valid if the finite type conditions are dropped in the assumptions.

(6.3) *Example.* Let $z = x + iy$ and $w = s + it$ be the variables in $\mathbf{C}^2$. Let M be the hypersurface of $\mathbf{C}^2$ defined by $t = |z|^2$ and M' the one defined by $t = 0$. Let H be the restriction to M of the holomorphic map from $\mathbf{C}^2$ into $\mathbf{C}^2$ defined by $F(z, w) = (z, 0)$. It is clear that H is a CR mapping from M into M' of multiplicity 1 at the origin, and that $H'_0 \neq 0$, however, H is not a local diffeomorphism. Note here that M is of type 2 at 0, whereas M' is not of finite type at 0.

(6.4) *Example.* We use the same notation as in Example (6.3). Here M is defined by $t = 0$, M' by $t = |z|^2$, and H is the restriction to M of the holomorphic mapping $F(z, w) = (0, w)$. It is easily checked that H is not of finite multiplicity at 0 (in fact, here the function λ in (1.2) is identically 0), and $H'_0 \neq 0$. Hence (ii), (iii), but not (i), hold in Theorem 2.

III. Proof of Theorem 3

§7. Embedding in $\mathbf{C}^2$ and local coordinates. The proof of Theorem 3 uses the general approach of [2]. Since M and M' are real-analytic, they can be considered as embedded in $\mathbf{C}^2$, where the variables are denoted by z and w. We assume $p_0 = H(p_0) = 0$, and write

$$z = x + iy, \qquad w = s + it.$$

As in [2], after holomorphic changes of coordinates, we may assume that M and M' are respectively given by

$$t = \varphi(z, \bar{z}, s), \tag{7.1}$$

$$t = \psi(z, \bar{z}, s), \tag{7.2}$$

with φ and ψ real-analytic, and real-valued, defined in a neighborhood of 0 in $\mathbb{R}^3$, and satisfying

$$\varphi(z, 0, s) = \psi(z, 0, s) = 0. \tag{7.3}$$

The mapping H is then locally given by a pair of smooth CR functions (f, g) defined on M and satisfying

$$\frac{g - \bar{g}}{2i} = \psi\left(f, \bar{f}, \frac{g + \bar{g}}{2}\right). \tag{7.4}$$

Since M is parametrized by

$$(x, y, s) \mapsto (z, s + i\varphi(z, \bar{z}, s)),$$

we may always think of f and g as functions of z, $\bar{z}$, s defined in a neighborhood of 0 in $\mathbb{R}^3$ and satisfying

$$Lf = Lg = 0 \tag{7.5}$$

with

$$L = \frac{\partial}{\partial \bar{z}} - i\frac{\varphi_{\bar{z}}}{1 + i\varphi_s}\frac{\partial}{\partial s}. \tag{7.6}$$

It is clear that the function λ in (1.2) is given by

$$\lambda(z, \bar{z}, s) = (L\bar{f})(z, \bar{z}, s),$$

and, by Theorem 1, we have

$$L^j\bar{f}(0) = 0, 0 \leqslant j \leqslant k, L^{k+1}\bar{f}(0) \neq 0. \tag{7.7}$$

Since, by Theorem 1, M is of finite type at the origin, it follows from Baouendi–Treves [4] that f and g extend holomorphically to at least one side of M. Therefore we may assume that for $|z| < r$ the functions $s \mapsto f(z, \bar{z}, s)$ and

$s \mapsto g(z, \bar{z}, s)$ extend holomorphically to a rectangle of the form

$$\{s + it, |s| < r, 0 < t < r\} \tag{7.8}$$

uniformly in z. By the arguments in [2], in order to prove Theorem 3, it suffices to show that these functions extend also holomorphically for $|s| < r$ and $-r < t < 0$.

By the implicit function theorem we obtain from (7.4) and (7.3)

$$\bar{g} = Q(f, \bar{f}, g), \tag{7.9}$$

where $Q(z, \zeta, w)$ is holomorphic near the origin in $\mathbb{C}^3$, and, by (7.3), satisfies

$$Q(z, 0, w) = Q(0, \zeta, w) = w. \tag{7.10}$$

Since M' is of type m' at 0, there exist positive integers p, q such that

$$p + q = m', \qquad Q_{z^i\zeta^j}(0) = 0, 0 \leqslant i + j < m', \tag{7.11}$$

and

$$Q_{z^p\zeta^q}(0) \neq 0; \tag{7.12}$$

we assume that q is minimal in (7.11), (7.12).

The following expansion, valid for z, s in a neighborhood of 0 in $\mathbb{R}^3$, and any small complex number λ, will be useful in the proof of Theorem 3

$$Q(f, \lambda, g)(z, \bar{z}, s) = \sum_{\alpha=0}^{\infty} \frac{(\lambda - \bar{f})^\alpha}{\alpha!} Q_{\zeta^\alpha}(f, \bar{f}, g)(z, \bar{z}, s). \tag{7.13}$$

§8. Technical lemmas. End of the proof of Theorem 3. Before giving the proof of Theorem 3, we need to state and prove a few technical lemmas.

(8.1) LEMMA. *If $r > 0$ is sufficiently small, then for every $z_0 \in \mathbb{C}$, $|z_0| < r$, and every $\alpha \in \mathbb{Z}_+$, there exist two holomorphic functions u_α, v_α defined in the rectangle*

$$R = \{s + it \in \mathbb{C}, |s| < r, -r < t < 0\}, \tag{8.2}$$

and C^∞ in its closure $\bar{R}$, such that

$$h_\alpha(s) = Q_{\zeta^\alpha}(f, \bar{f}, g)(z_0, \bar{z}_0, s) = \frac{u_\alpha(s)}{v_\alpha(s)}, \ |s| < r. \tag{8.3}$$

Proof of Lemma (8.1). Since the function $s \mapsto g(z_0, \bar{z}_0, s)$ extends holomorphically to the rectangle defined by (7.8), its complex conjugate $s \mapsto \bar{g}(z_0, \bar{z}_0, s)$ extends holomorphically to R defined by (8.2). Therefore the claim is true for $\alpha = 0$ by using (7.9) and taking $v_0 \equiv 1$, $u_0 = \bar{g}$.

We shall prove the lemma by induction on α. By repeatedly applying L, given by (7.6), to (7.9), we obtain for every α, $n \in \mathbf{Z}_+$:

$$(8.4) \quad L^{n\alpha}\bar{g} = Q_{\zeta^\alpha}(f, \bar{f}, g)(L^n\bar{f})^\alpha + \sum_{\substack{p_1 + \cdots + p_\beta = n\alpha \\ \beta \leqslant n\alpha}} a_{\beta p_1 \dots p_\beta} Q_{\zeta^\beta}(f, \bar{f}, g)(L^{p_1}\bar{f}) \dots (L^{p_\beta}\bar{f}),$$

where $a_{\beta p_1 \dots p_\ell} \in \mathbf{Z}_+$, and $a_{\alpha n \dots n} = 0$.

Let z_0 be fixed, $|z_0| < r$, and n the smallest positive integer such that the function $s \mapsto (L^n\bar{f})(z_0, \bar{z}_0, s)$ does not vanish identically. If r is small enough, we have $1 \leqslant n \leqslant k + 1$, where k is given in (7.7).

Suppose we want to prove the lemma for a given $\alpha \in \mathbf{Z}_+$. We can use (8.4) and divide by $(L^n\bar{f})^\alpha$. The terms in the Σ of the right-hand side of (8.4) can be handled as follows. If $\beta < \alpha$ then we use the induction hypothesis, i.e., (8.3) with α replaced by β. If $\beta \geqslant \alpha$ in (8.4), then, by the "pigeon hole principle," necessarily one of the p_j, $1 \leqslant j \leqslant \beta$, is strictly less than n; therefore, by the choice of n,

$$(L^{p_j}\bar{f})(z_0, \bar{z}_0, s) \equiv 0, \; |s| < r$$

for some p_j.

The proof of the lemma is now complete by observing that $L^{n\alpha}\bar{g}$ and $L^n\bar{f}$ extend holomorphically in s to the rectangle R defined by (8.2), provided r is sufficiently small, independently of n, α and z_0.

(8.5) LEMMA. *Let k, p, q be as defined in* (7.7) *and* (7.12). *The following identity holds in a neighborhood of $z = 0$, $s = 0$:*

$$(8.6) \qquad f^p + \sum_{j=0}^{p-1} a_j(L^\alpha\bar{f}, L^\beta\bar{g}) f^j \equiv 0,$$

where the functions a_j are holomorphic with respect to their arguments $L^\alpha\bar{f}$, $L^\beta\bar{g}$, $0 \leqslant \alpha$, $\beta \leqslant (k + 1)q$, near $L^\alpha\bar{f}(0)$, $L^\beta\bar{g}(0)$, and

$$(8.7) \qquad a_j(L^\alpha\bar{f}, L^\beta\bar{g})\big|_{z=0, s=0} = 0.$$

Proof of Lemma (8.5). The starting point in this proof is identity (8.4) with $n = k + 1$ and $\alpha = q$. Next, from (7.9) we have

$$(8.8) \qquad g = \overline{Q}(\bar{f}, f, \bar{g}).$$

Substituting (8.8) in (8.4) (with $\alpha = q$, $n = k + 1$), we obtain:

$$(8.9) \qquad L^{(k+1)q}\bar{g} = Q_{\zeta^q}(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g}))(L^{k+1}\bar{f})^q + \sum_{\substack{p_1 + \cdots + p_\beta = (k+1)q \\ \beta \leqslant (k+1)q}} a_{\beta p_1 \ldots p_\beta} \times Q_{\zeta^\beta}(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g}))(L^{p_1}\bar{f}) \ldots (L^{p_\beta}\bar{f}).$$

Now we think of f, $L^\alpha \bar{f}$, $L^\beta \bar{g}$, $0 \leqslant \alpha$, $\beta \leqslant (k+1)q$ as independent variables. Making use of (7.10), (7.11), and (7.12), we can apply the Weierstrass preparation theorem to equation (8.9), with respect to the variable f, which yields the desired conclusion (8.6).

As in Lemma (8.1), it is convenient to introduce the following notation:

$$(8.10) \qquad h_\alpha(z, \bar{z}, s) = Q_{\zeta^\alpha}(f, \bar{f}, g)(z, \bar{z}, s) = Q_{\zeta^\alpha}(f, \bar{f}, \bar{Q}(\bar{f}, f, \bar{g}))(z, \bar{z}, s).$$

We can now state:

(8.11) LEMMA. *There is a neighborhood V of $z = 0$, $s = 0$ such that the following identity holds in V for all $\alpha \in \mathbb{Z}_+$,*

$$(8.12) \qquad h_\alpha^p + \sum_{j=0}^{p-1} b_{\alpha j}(L^\gamma \bar{f}, L^\delta \bar{g}) h_\alpha^j \equiv 0,$$

where the functions $b_{\alpha j}(u_\gamma, v_\delta)$, $0 \leqslant \gamma$, $\delta \leqslant (k+1)q$, are holomorphic for $|u_\gamma - L^\gamma \bar{f}(0)| < \eta$, $|v_\delta - L^\delta \bar{g}(0)| < \eta$, and satisfying

$$(8.13) \qquad |b_{\alpha j}(u_\gamma, v_\delta)| \leqslant [C^{\alpha+1}\alpha!]^{p-j},$$

with C and $\eta > 0$ independent of α and j.

Proof of Lemma (8.11). For z and s sufficiently small let $\rho_j(z, \bar{z}, s)$, $1 \leqslant j \leqslant p$, be the roots of the polynomial in f defined by (8.6). Using the definition of h_α given by (8.10), we have the following identity in a neighborhood of $z = s = 0$:

$$(8.14) \qquad \prod_{j=1}^{p} \left(h_\alpha - Q_{\zeta^\alpha}(\rho_j, \bar{f}, \bar{Q}(\bar{f}, \rho_j, \bar{g}))\right) = 0.$$

Since the left-hand side of (8.14) is a symmetric function of the roots ρ_j, $1 \leqslant j \leqslant p$, the identity (8.12) follows by expanding the product (8.14), using the classical Newton's theorem for symmetric functions and the expression of the coefficients a_j in (8.6). The estimates (8.13) follow easily from (8.14) and the estimates

$$|Q_{\zeta^\alpha}(z, \zeta, w)| \leqslant C^{\alpha+1}\alpha!,$$

uniformly in z, ζ, w sufficiently small.

We need to state a lemma in one complex variable.

(8.15) LEMMA. *Let r be a positive number and R the rectangle in the complex plane defined by (8.2). Let u, v be two functions defined in $\overline{R}$ and satisfying*
(*i*) $u, v \in C^\infty(\overline{R})$ *and u, v are holomorphic in R,*
(*ii*) $h(s) = u(s)/v(s) \in C^\infty([-r, r])$,
(*iii*) *there exist a positive integer p, and for $0 \leqslant j \leqslant p-1$, functions $a_j \in C^\infty(\overline{R})$, holomorphic in R, such that*

$$(h(s))^p + \sum_{j=0}^{p-1} a_j(s)(h(s))^j = 0, \forall s \in [-r, r]. \tag{8.16}$$

Then h extends holomorphically to R as $u(w)/v(w)$, $w = s + it$, and $u/v \in C^\infty(R \cup (-r, r))$.
In addition, if for some $C \geqslant 0$,

$$\sup_R |a_j(w)| \leqslant C^{p-j}, \qquad 0 \leqslant j \leqslant p-1, \tag{8.17}$$

then

$$\sup_R \left|\frac{u(w)}{v(w)}\right| \leqslant pC. \tag{8.18}$$

Proof of Lemma (8.5). We may assume $v \not\equiv 0$, for otherwise $u \equiv 0$ and the conclusion of the lemma is trivial. Let $s_0 \in (-r, r)$, such that $v(s_0) \neq 0$ and $\tilde{h}(w)$ be the meromorphic function in R defined by $u(w)/v(w)$. Since (8.16) holds in a real neighborhood of s_0, by analytic continuation we have

$$(\tilde{h}(w))^p + \sum_{j=0}^{p-1} a_j(w)(\tilde{h}(w))^j = 0, \tag{8.19}$$

for $w \in R$, except possibly at the poles of $\tilde{h}$. Since the coefficients a_j are bounded in R, (8.19) implies that $\tilde{h}$ is also bounded in R. We conclude that $\tilde{h}$ is a holomorphic bounded function in R satisfying (8.19). Therefore, since its boundary value $h(s)$ on $(-r, r)$ is C^∞, we reach the first conclusion of the lemma. Now the desired estimate (8.18) easily follows from (8.17) and (8.19).

Remark. It should be noted that the first conclusion of the lemma is no longer valid if we drop either (i) or (iii) as shown by the following examples.

(8.20) *Example.* For $w = s + it$ let

$$f(w) = \exp(-w^{-1/4}), \tag{8.21}$$

with the determination of the argument of w, $-3\pi/2 < Arg\, w < \pi/2$. Now set $h(s) = \sqrt{s} f(s)$ for $s \geqslant 0$, and $h(s) = i\sqrt{-s} f(s)$ for $s \leqslant 0$. Clearly we have $h \in C^\infty(\mathbb{R})$, and $h(s)^2 - s(f(s))^2 = 0$. The function $s(f(s))^2$ extends holomorphically to the negative half-plane defined by $Im\, w < 0$, whereas h has no such extension.

(8.22) *Example.* With the notation of Example (8.20), let

$$u(w) = (f(w))^2 \exp(iw^{-1}),$$

$$v(w) = f(w) \exp(2iw^{-1}),$$

$$h(s) = \frac{u(s)}{v(s)}, \qquad s \in \mathbb{R}.$$

The reader can easily check that u, v satisfy (i) and (ii) of Lemma (8.15) (with $r = \infty$), whereas h is not the boundary value, near the origin, of a bounded holomorphic function in the negative half-plane.

End of the proof of Theorem 3. By using Lemmas (8.1), (8.11), and (8.15), we see that there exists $r > 0$ such that for $|z| < r$, $|\lambda| < r$, the right-hand side of (7.13) extends holomorphically as a function of s to the rectangle R defined by (8.2). Since the left-hand side of (7.13) extends holomorphically to the rectangle defined by (7.8), we conclude that the function

$$s \mapsto Q(f, \lambda, g)(z, \bar{z}, s)$$

is real-analytic in s for $|s| < r$, uniformly in z, λ, $|z| < r$, $|\lambda| < r$. Therefore the same conclusion holds for g by taking $\lambda = 0$ and using (7.10). Now, by taking $\lambda \neq 0$ small, and by (7.12), we may apply the Weierstrass preparation theorem to (7.13) with respect to f (regarded as an independent variable) to conclude that f satisfies a monic polynomial with real-analytic coefficients in s uniformly in z. The real analyticity of f now follows by applying Lemma 6.1 of [2]. This completes the proof of Theorem 3.

IV. Applications

§9. Proofs of Theorems 5–7 and related results. Before we begin to prove Theorems 5–7, we will state and prove a theorem which, for a CR mapping, justifies the use of the term *finite multiplicity*.

THEOREM 8. *Let M and M' be two real-analytic connected CR manifolds of finite type at p_0 and p_0', respectively, and $H: M \to M'$ a real-analytic CR mapping with $H(p_0) = p_0'$.*

(*i*) *If H is not of finite multiplicity at p_0, then H is constant, i.e., $H(M) = \{p_0'\}$.*

(*ii*) *If H is of finite multiplicity κ at p_0, then p_0' is an interior point of $H(M)$. More precisely, for every neighborhood U of p_0 in M, there exists an open neighborhood V of p_0' in M' such that $V \subset H(U)$. In addition, there is a finite number of real-analytic curves $\gamma_1, \dots, \gamma_r$ contained in V such that, for every point $p' \neq p_0'$ in V which does not lie on one of these curves, there exist exactly κ points $p_1, \dots, p_\kappa$ in U satisfying $H(p_j) = p'$ with H of multiplicity 1 at each p_j.*

Proof of Theorem 8. Statement (i) follows immediately from Theorem 2 and the analyticity of H. Indeed, if H is not of finite multiplicity at p_0, then H must be flat at p_0. Thus H is constant. The proof of statement (ii) is somewhat more involved. As in the proof of Theorem 3, we may consider the map H to be a real-analytic map between manifolds M and M' which are real-analytic hypersurfaces in $\mathbf{C}^2$. The real analyticity of H implies that there exist functions $F(z, w)$ and $G(z, w)$ which are holomorphic in a neighborhood of p_0 such that the mapping (F, G) restricted to M agrees with $H = (f, g)$, where z, w, f, and g are the special coordinates used in the proof of Theorem 3 (see §7). In particular, we assume that $p_0 = p_0' = 0$. These coordinates were chosen so that, among other things, the function $g(z, \bar{z}, s)$ is such that $(\partial^j g/\partial z^j)(0) = 0$ for all j. Indeed, if we apply L^j to Equation (7.9) and use (7.5) and (7.10), together with the fact that $L^j = \partial^j/\partial \bar{z}^j$ at 0, we see that $(\partial^j g/\partial z^j)(0) = 0$. Hence, we conclude that

$$\frac{\partial^j G}{\partial z^j}(0,0) = 0 \quad \text{for all } j. \tag{9.1}$$

It follows easily from part (iv) of Theorem 1 and the choice of the coordinates that $(\partial g/\partial s)(0) \neq 0$. Hence, we deduce that

$$\frac{\partial G}{\partial w}(0,0) \neq 0. \tag{9.2}$$

Furthermore, it follows from (7.7) (here $k + 1 = \kappa$) that

$$\frac{\partial^\kappa F}{\partial z^\kappa}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial^j F}{\partial z^j}(0,0) = 0 \quad \text{for all } j < \kappa. \tag{9.3}$$

Now (9.2) implies that we may solve the equation $G(z, w) = w'$ for w, and in so doing obtain a change of coordinates in which the G component of our mapping (F, G) takes the form $G(z, w) = w$. Furthermore, (9.1) and (9.2) yield that, in these new coordinates, the function F continues to satisfy the conditions in (9.3). Thus, by the Weierstrass preparation theorem, after factoring out a nonzero holomorphic function of (z, w, z'), we may write the equation $F(z, w) - z' = 0$ in the form

$$z^{\kappa} + \sum_{j<\kappa} a_j(w, z')z^j = 0 \tag{9.4}$$

where the $a_j(w, z')$ are holomorphic and $a_j(0,0) = 0$ for all j. Now, since (9.3) implies that the equation $F(z,0) - z' = 0$ is satisfied by precisely κ distinct values of z when $|z'| \neq 0$ is small, it follows that the discriminant of the Weierstrass polynomial in (9.4) is 0 for (w, z') in a one-dimensional complex-analytic subvariety of $\mathbf{C}^2$. Since $w = w'$, we conclude that there is a one-dimensional complex-analytic subvariety Λ of $\mathbf{C}^2$ passing through the origin such that a point p' near 0 which is not in Λ has exactly κ preimages under (F, G). Now, as a real-analytic set, $\Lambda \cap M'$ can be written as a locally finite union of real-analytic manifolds (see Hironaka [14], Theorem (4.8)). Furthermore, the manifolds which compose $\Lambda \cap M'$ must be one- or zero-dimensional real-analytic subsets of M' because, being of finite type, M' cannot contain a complex variety (see Diederich–Fornaess [10]). By shrinking the neighborhood of the origin under consideration, the zero-dimensional parts, if any, can be assumed to be at most the origin. This proves the existence of the curves $\gamma_1, \ldots, \gamma_r$. To finish the proof, we must see that if $p' \in M'$ near p_0', then the preimages of p' under (F, G) must lie in M. Indeed, condition (9.2) implies that the equation

$$\operatorname{Im} G(z, w) - \psi\Big(F(z, w), \overline{F(z, w)}, \operatorname{Re} G(z, w)\Big) = 0$$

defines a hypersurface in $\mathbf{C}^2$, which clearly coincides with M near the origin. Finally, note that if V is a small enough neighborhood of p_0', then, for points in $M \setminus H^{-1}(\Lambda \cap M' \cap V)$ near p_0, the extension of H to $\mathbf{C}^2$ has a local holomorphic inverse, and hence has nonvanishing Jacobian. Thus H is of multiplicity 1 at these points. This completes the proof of Theorem 8.

Proof of Theorem 4. We follow closely the argument used in [5]. If d is a positive integer, let $S(d)$ denote the set of points in M which are of type d or greater. Note that the set $S(d)$, if it is not empty, is a closed real-analytic subset of M which can be written as a finite disjoint union of connected components according to Theorem (4.8) of [14]. Let V be equal to the set of points in M at which the multiplicity of H is greater than 1. In order to derive a contradiction, we assume that V is not empty. Define $d_{\max}$ to be the type of the point of highest type in V. Pick a point p_0 in $V \cap S(d_{\max})$. Theorem 8 implies that H is an open mapping, and therefore, that H is a map of M onto itself. Thus, we may choose a point p_1 in $H^{-1}(p_0)$, and inductively define a sequence of points p_i such that $p_i \in H^{-1}(p_{i-1})$. We now claim that all the points p_i must be of type $d_{\max}$. Indeed, if p_i is of type τ_i in M and if H is of multiplicity κ_i at p_i, then, by Theorem 1, we have that $\tau_i = \kappa_i \tau_{i+1}$. Now if $\kappa_1 > 1$, then p_1 is a point in V whose type is strictly greater than $d_{\max}$. This violates the definition of $d_{\max}$. Thus,

$\kappa_1 = 1$. Iteration of this argument reveals that $\kappa_i = 1$ for all i, and therefore that $\tau_i = d_{\max}$ for all i. Let $S(d_{\max}) = C_1 \cup C_2 \cup \cdots \cup C_N$ be a decomposition of $S(d_{\max})$ as a disjoint union of closed connected real-analytic sets. We need the following lemma.

(9.5) LEMMA. *If H is of multiplicity* 1 *at $z \in C_k$, and $H(z) \in C_j$, then $H(C_k) \subset C_j$, and H is of multiplicity* 1 *at each point of C_k.*

Accepting this lemma for the moment, notice that $p_i \in S(d_{\max})$ for all i. Therefore, there must be a component C_j which contains infinitely many of the points p_i. Suppose that p_i and p_{i+L} both belong to C_j. Now Lemma (9.5) implies that the iterated map H^L maps C_j into itself and is a local diffeomorphism in a neighborhood of C_j (by Theorem 1, (i)). Furthermore, the iterated map $H^{(i+1)L}$ maps C_j into itself and is also locally diffeomorphic there. But the multiplicity of this map is greater than 1 at p_i. (This is because the mapping H^{i+n} branches at p_i whenever $n \geqslant 1$ due to the choice of p_0.) Hence $H^{(i+1)L}$ must branch at p_i. This is a contradiction. Thus V must be empty. The theorem will be proved provided that we verify the lemma.

Proof of Lemma (9.5). Define a set U to be equal to the set of points z in C_k such that $H(z) \in C_j$ and such that the multiplicity of H at z is equal to 1. Note that by assumption $U \neq \varnothing$. First, we shall show that U is open in C_k. Indeed, if z_0 is in U, then there is a neighborhood W of z_0 such that H is a diffeomorphism of W onto $H(W)$ and, of course, the multiplicity of H at points in W is 1. Hence, the points in W of type $\geqslant d_{\max}$ get mapped by H to points of the same type. Since the sets C_i are closed, and since H is continuous, we may shrink W if necessary, so that $W \cap C_i = \phi$ if $i \neq k$ and so that $C_i \cap H(W \cap C_k) = \phi$ if $i \neq j$. Since we know that H maps $W \cap C_k$ into $S(d_{\max})$, we conclude that $H(W \cap C_k)$ is contained in C_j. Therefore, U is open in C_k. Next, we show that U is closed. Suppose U is not closed. Then there exists a point p in the closure of U minus U. This point p must be in C_k because C_k is closed, and $H(p)$ must be in C_j because H is continuous. Furthermore, the multiplicity κ of H at p must be greater than 1 because p is not in U. Thus, p must be a point of type $\geqslant \kappa d_{\max}$, at which branching occurs. This violates the definition of $d_{\max}$. Therefore, U is both open and closed in C_k, and we conclude that $U = C_k$. Thus, H maps C_k into C_j and the multiplicity of H on C_k is 1.

Proof of Theorem 5. The fact that the boundary of a bounded domain in $\mathbb{C}^2$ with real-analytic boundary is a CR manifold of finite type was proved by Diederich and Fornaess in [10]. Therefore we can apply Theorem 2 to $H = F|_{\partial D}$ and conclude that H is of finite multiplicity at each point of ∂D. Now Theorem 3 implies that F extends holomorphically to a neighborhood of $\overline{D}$. By using Theorem 1, (iv), and the argument of the proof of Theorem 8, it follows that if ρ' is a defining function for D' (i.e., $D' = \{\rho' < 0\}$ and $d\rho' \neq 0$ on $\partial D'$), then $\rho' \circ F$ is a defining function for D. Choosing $\varepsilon > 0$ small enough, we may set $D_1' = \{\rho' < \varepsilon\}$ and $D_1 = F^{-1}(D_1')$. This completes the proof of Theorem 5.

Proof of Theorem 6. Suppose that F is a proper self-map of D that satisfies the hypotheses of Theorem 6. The boundary of D is a CR manifold of finite type and the restriction of F to the boundary is a CR self-mapping. According to a result of Pinčuk [18], it will follow that F is biholomorphic if we show that the Jacobian of F does not vanish on the boundary of D, i.e., that the multiplicity of F as a CR-mapping of the boundary of D into itself is everywhere equal to 1. But this is guaranteed by Theorems 2 and 5.

Proof of Theorem 7. It is proved in [8] and [11] that F extends to be in $C^\infty(\overline{D})$ and that the Jacobian determinant of F cannot vanish to infinite order at any boundary point of D. Thus, F is not flat at any boundary point and Theorem 7 follows from Theorem 5.

We wish to state one last theorem. To do so, we must first make some definitions. A function ρ is called a defining function for a hypersurface M near a point $z_0 \in M$ if there is a neighborhood U of z_0 such that $d\rho \neq 0$ on U and $M \cap U = \{z : \rho(z) = 0\}$.

Suppose that M and M' are C^∞ pseudoconvex hypersurfaces in $\mathbf{C}^2$ of finite type (with defining functions ρ and ρ') which both contain the origin, and suppose that $H: M \to M'$ is a C^∞ CR-mapping with $H(0) = 0$. It is known that H extends to be holomorphic on the pseudoconvex side of M; we shall denote this extension by H, too. Therefore, by shrinking U, we may suppose that H is holomorphic on $\Omega = \{z \in U: \rho(z) < 0\}$ and that H maps Ω into the neighborhood U' of the origin on which ρ' is defined. Furthermore, we know that $H \in C^\infty(\Omega \cup M)$. Let Ω' denote the set $\{z \in U' : \rho'(z) < 0\}$. Finally we can state the theorem.

THEOREM 9. *Under the above conditions, if H maps Ω into Ω', then H is not flat at the origin. In fact, $d(r \circ H) \neq 0$ near the origin where r is any defining function for M'. Hence, if, in addition, M and M' are real-analytic, then H extends to be holomorphic in a neighborhood of the origin.*

Proof of Theorem 9. We shall modify an argument of Fornaess, which he used originally in the biholomorphic case [13]. It is shown in [12] that, by shrinking U', we may suppose that ρ' is such that $-(-\rho')^{2/3}$ is strictly plurisubharmonic on Ω'. Now $R = -(-\rho' \circ H)^{2/3}$ is a plurisubharmonic (and hence subharmonic) function on Ω which is continuous up to M and which assumes its maximum value of 0 on M. Now if B is a ball which is contained in Ω such that $\partial B \cap M = \{0\}$, we may apply the Poisson integral formula associated to B to the superharmonic, nonnegative function $-R(z)$ to obtain

$$-R(z) \geqslant \int_{\partial B} P(z,\zeta)[-R(\zeta)]\, d\sigma_\zeta \tag{9.5}$$

where $z \in B$ and $P(z,\zeta)$ is the Poisson kernel associated to B. We now restrict $z \in B$ to lie along the inward-pointing normal to ∂B at the origin (which is also a normal to M). As in the proof of the classical Hopf lemma, we may apply the inequality $P(z, \zeta) \geqslant K' \operatorname{dist}(z, \partial B)$, where $K' > 0$ is independent of $\zeta \in \partial B$, to (9.5) to obtain the estimate

$$-R(z) \geqslant K' \operatorname{dist}(z, M) \int_{\partial B} - R(\zeta)\, d\sigma_\zeta$$

$$\geqslant K \operatorname{dist}(z, M),$$

where $K > 0$ is independent of the point z on the normal to M at 0. Thus, $\rho'(H(z)) \leqslant -K \operatorname{dist}(z, M)^{3/2}$ for z on the inward-pointing normal to M at 0. This forces us to conclude that $d(\rho' \circ H)(0) \neq 0$ and, in fact, that $d(r \circ H)(0) \neq 0$ for any defining function r for M'. In the coordinates of §7, this means that $\partial g/\partial s(0) \neq 0$, and therefore that H is not flat at the origin. The extendibility of H now follows from Theorems 2 and 3.

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BAOUENDI AND BELL: DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

ROTHSCHILD: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92093