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Minimality and the Extension of Functions from Generic Manifolds

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1. Introduction. Let M be a smooth submanifold of $\mathbf{C}^p$ of codimension l given locally near $m_0 \in M$ by $\rho_j(Z, \overline{Z}) = 0$, $1 \le j \le l$, with ρ_j smooth, real-valued, and $d\rho_1, \dots, d\rho_l$ linearly independent. If, in addition, the complex differentials $\partial\rho_1, \dots, \partial\rho_l$ are also linearly independent, then M is called *generic*. (Here $\partial\rho_j = \sum_{k=1}^{p} \frac{\partial\rho_j}{\partial Z_k} dZ_k$.) We write $\rho = (\rho_1, \dots, \rho_l)$.

We denote by $\mathbf{C}TM$ the complexified tangent space of M. If $m \in M$ we denote by $\mathscr{V}_m$ the space of antiholomorphic tangent vectors at m, i.e.,

$$\mathscr{V}_m = \left\{ \sum_{j=1}^{p} a_j \frac{\partial}{\partial \overline{Z}_j} \in \mathbf{C}T_m M,\ a_j \in \mathbf{C} \right\}. \tag{1.1}$$

If M is generic then $\dim \mathscr{V}_m = N - l$ is independent of m. We denote by $\mathscr{V}$ the associated subbundle of $\mathbf{C}TM$. We shall refer to $\mathscr{V}$ as the CR *subbundle* of M, and denote by $\mathbf{L}$ the smooth sections of $\mathscr{V}$. If h is a function or distribution defined on M, then h is called CR if h is annihilated by all the vector fields in $\mathbf{L}$. If a C^1 function f on M is the boundary value on M of a function holomorphic in some open set whose boundary contains M, then, by continuity, f must be a CR function. The converse does not always hold; e.g., take $M = \mathbf{R} \subset \mathbf{C}$. We address here the question of finding geometric conditions on a generic manifold M which guarantee that every CR function extends holomorphically to some open set in $\mathbf{C}^p$.

We shall describe a class of open sets, called wedges, into which CR functions will extend holomorphically under favorable conditions. Let M be a generic submanifold of $\mathbf{C}^p$, $m_0 \in M$, and ρ a set of defining functions for M near m_0 as above. If $\mathscr{O}$ is a small neighborhood of m_0 in $\mathbf{C}^p$ and Γ

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an open convex cone in $\mathbf{R}^l\backslash 0$, we put

$$\mathscr{W}(\mathscr{O}, \Gamma) = \{Z \in \mathscr{O} : \rho(Z, \overline{Z}) \in \Gamma\}. \tag{1.2}$$

The set defined by (1.2) is an open subset of $\mathbf{C}^p$ whose boundary contains $M \cap \mathscr{O}$; it is called a *wedge of edge* M *in the direction* Γ. Note that if M is a hypersurface, i.e. $l = 1$, then a wedge of edge M is one of the two sides of the hypersurface.

The extension problem may now be formulated precisely as follows. We shall say that a generic manifold M is *wedge extendable* at m_0 if every CR function defined in a neighborhood of m_0 in M extends holomorphically to a wedge of edge M of the form (1.2). A complete characterization of wedge extendability is known and is the subject of this paper.

We first give a brief history of this problem. The study of extendability of CR functions defined on a hypersurface began with the important work of Hans Lewy [23], in which he showed that all CR functions on a strictly pseudoconvex hypersurface extend to the convex side. Subsequent progress on extension of CR functions from hypersurfaces was made by a number of mathematicians culminating in the work of Trépeau [28], who obtained a necessary and sufficient condition: a hypersurface M satisfies the extendability condition at a point m_0 as above if and only if there is no germ of a complex analytic hypersurface contained in M and passing through m_0. In the case of a real analytic hypersurface, this result was first obtained by Baouendi and Treves [10].

For higher codimension, i.e. $l > 1$, a number of sufficient conditions for extendability were obtained by several mathematicians, in particular, Hill and Taiani [20, 21], Ajrapetyan and Henkin [1], Henkin [19], Boggess [13], Boggess and Polking [15], Boggess and Pitts [14], Baouendi, Chang, and Treves [3], Baouendi, Rothschild, and Treves [8], and the authors [4]. See also Taiani [27] for a survey of some of these results and for further references.

Recently, A. E. Tumanov [31] has given a sufficient condition for wedge extendability of CR functions defined on a smooth generic manifold of any codimension. The authors [7] have shown that Tumanov's condition is also necessary. We will now describe this condition.

A generic manifold M as above is called *minimal* at m_0 if there is no germ of a smooth submanifold N properly contained in M and passing through m_0 such that all smooth sections L of the CR bundle $\mathscr{V}$ near m_0 are tangent to N. Note that if such an N exists we must have

$$2 \dim_{\mathbf{C}} \mathscr{V} \leq \dim_{\mathbf{R}} N < \dim_{\mathbf{R}} M. \tag{1.3}$$

We can now state the characterization mentioned above.

THEOREM 1 [31, 7]. *A generic submanifold* M *in* $\mathbf{C}^p$ *is wedge extendable at* m_0 *if and only if it is minimal at* m_0.

In [31] it is proved that under the condition of minimality for every neighborhood U of m_0 in M, there is a fixed wedge $\mathscr{W}(\mathscr{O}, \Gamma)$ to which every

CR function in U extends holomorphically. Also, note that it is shown in [7] that if minimality fails then there is a smooth CR function which does not extend to any wedge. (See §3.)

2. Sufficiency of minimality. In this section we shall briefly describe some of the ideas involved in Tumanov's proof that minimality is sufficient for wedge extendability. We refer to [31] for details.

We first describe the method of analytic discs, initiated by Bishop [11]. Suppose that M is a generic manifold in $\mathbf{C}^p$ as in §1. Let D be the unit disc in the complex plane, $D = \{\zeta \in \mathbf{C},\ |\zeta| < 1\}$ and $S = \partial D$ the unit circle. By an *analytic disc* attached to M of class C^α we mean a mapping $Z: \overline{D} \to \mathbf{C}^p$ holomorphic in D, of class C^α in $\overline{D}$, and such that $Z(S) \subset M$. The method of proof is based on the following well-known result.

PROPOSITION 2.1. *If, for some fixed $\alpha \geq 0$, there is a wedge $\mathscr{W}(\mathscr{O}, \Gamma)$ of the form* (1.2) *contained in $\bigcup_Z Z(\overline{D})$, where Z ranges over all analytic discs attached to M of class C^α, then every CR function of class C^1 on M extends holomorphically to $\mathscr{W}(\mathscr{O}, \Gamma)$.*

PROOF. It follows from the approximation theorem of [9] that every C^1 CR function h on M is locally a limit in the C^1 norm of a sequence of holomorphic polynomials in $\mathbf{C}^p$. After composition of these polynomials with analytic discs attached to M and making use of the maximum principle, we conclude that this sequence converges uniformly in $\bigcup_Z Z(\overline{D})$. Its limit is the desired holomorphic extension of h.

The main point of the proof of the sufficiency is to show that the assumption of Proposition (2.1) holds if minimality is assumed.

In order to describe the construction of the analytic discs we set $p = n + l$ and introduce holomorphic coordinates $(z, w) \in \mathbf{C}^p$, $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and $w = (w_1, \dots, w_l) \in \mathbf{C}^l$, such that M is given by

$$\operatorname{Im} w = \phi(z, \bar{z}, s), \qquad s = \operatorname{Re} w, \tag{2.2}$$

where ϕ is a smooth function valued in $\mathbf{R}^l$, defined in a neighborhood of 0 in $\mathbf{R}^{2n+l}$, and satisfying $\phi(0) = 0$ and $d\phi(0) = 0$. An analytic disc of class C^α attached to M is then of the form $Z(\zeta) = (z(\zeta), w(\zeta))$, with $z: \overline{D} \to \mathbf{C}^n$ and $w: \overline{D} \to \mathbf{C}^l$ of class C^α and holomorphic in D satisfying

$$\operatorname{Im} w(\zeta) = \phi(z(\zeta), \overline{z(\zeta)}, s(\zeta)), \qquad \zeta \in S, \tag{2.3}$$

where $s(\zeta)) = \operatorname{Re} w(\zeta)$.

Let $\mathscr{E}$ be the space of all vector functions $z(\zeta)$ valued in $\mathbf{C}^n$ of class C^α such that $z(1) = 0$. For $\lambda \in \overline{D}$ define $z_\lambda(\zeta) = z(\zeta) - z(\lambda)$ for $z \in \mathscr{E}$. We look for analytic discs of the form $Z_\lambda(\zeta) = (z_\lambda(\zeta), w_\lambda(\zeta))$, where $\operatorname{Re} w_\lambda(\lambda) = 0$. It follows from (2.3) that $s_\lambda(\zeta) = \operatorname{Re} w_\lambda(\zeta)$ satisfies the Bishop equation

$$s_\lambda(\zeta) = -T_\lambda(\phi(z_\lambda, \bar{z}_\lambda, s_\lambda))(\zeta), \quad \zeta \in S, \tag{2.4}$$

where $T_\lambda = T - P_\lambda T$, with T the Hilbert transform on S and P_λ the Poisson kernel evaluated at λ. Recall that

$$
(2.5)\qquad \begin{aligned} T\phi(\sigma) &= \frac{1}{2\pi}\int_0^{2\pi} \phi(\zeta)\,\mathrm{Im}\,\frac{\zeta+\sigma}{\zeta-\sigma}\,d\theta, \\ P_\lambda\phi &= \frac{1}{2\pi}\int_0^{2\pi} \phi(\zeta)\frac{1-|\lambda|^2}{|\zeta-\lambda|^2}\,d\theta, \qquad \zeta = e^{i\theta}. \end{aligned}
$$

For $z \in \mathscr{E}$ we write $\|z\|$ for the C^α norm of z. If $\|z\|$ is sufficiently small, by contraction and the continuity of the Hilbert transform in $C^\alpha(S)$, for $0 < \alpha < 1$, (i.e. by using the implicit function theorem for Banach spaces in C^α), we conclude that equation (2.4) has a unique solution $s_\lambda(\zeta)$. Now for $\lambda \in D$ we define $F_\lambda \colon \mathscr{E} \to \mathbf{R}^l$ by

(2.6)

$$
F_\lambda(z) = \mathrm{Im}\, w_\lambda(\lambda) = \frac{1-|\lambda|^2}{2\pi}\int_0^{2\pi} \frac{\phi(z_\lambda(\zeta), \overline{z_\lambda(\zeta)}, s_\lambda(\zeta))}{|\zeta-\lambda|^2}\,d\theta, \qquad \zeta = e^{i\theta}.
$$

From here on we fix α, $\frac{1}{2} < \alpha < 1$. For $z \in \mathscr{E}$ with small norm we define $\widehat{\phi}(z)$ by

$$
(2.7)\qquad \widehat{\phi}(z) = \int_0^{2\pi} \frac{\phi(z(\zeta), \overline{z(\zeta)}, s(\zeta))}{|\zeta-\lambda|^2}\,d\theta, \qquad \zeta = e^{i\theta},
$$

where s is given by (2.4) with $\lambda = 1$. The following, which is a crucial observation in Tumanov's work, can be proved by standard estimates.

LEMMA 2.8 [31]. *If $\|z\|$ is sufficiently small, $z \in \mathscr{E}$, then for $0 < \lambda < 1$ we have*

$$
F_\lambda(z) = \frac{1-\lambda^2}{2\pi}(\widehat{\phi}(z) + O((1-\lambda^2)^{2\alpha-1})).
$$

From Lemma (2.8), Tumanov then establishes the following criterion for wedge extendability.

PROPOSITION 2.9. *If the Banach space derivative of $\widehat{\phi}$ at some point $z_0 \in \mathscr{E}$ is onto $\mathbf{R}^l$, then M is wedge extendable at 0.*

The rest of the proof of the sufficiency in Theorem 1 consists of showing that if the derivative of $\widehat{\phi}$ is not surjective at any $z \in \mathscr{E}$ with sufficiently small norm, then M is not minimal at 0. This is contained in the following statement. Let $f \colon \mathscr{E} \to M$ be given by $f(z) = (z(-1), w(-1))$, where $\mathrm{Re}\, w$ satisfies the Bishop equation (2.4), with $\lambda = 1$, and $\mathrm{Im}\, w$ is determined by (2.3).

PROPOSITION 2.10. *If the derivative of $\widehat{\phi}$ is not surjective at any $z \in \mathscr{E}$ with sufficiently small norm, then there exists $z_0 \in \mathscr{E}$ of small norm (such that $\widehat{\phi}'$ at z_0 is of maximal rank) and a neighborhood $\mathscr{U}$ of z_0 in $\mathscr{E}$, such* *that* $f(z_0) = 0$ *and* $f(\mathcal{U})$ *is a smooth proper submanifold of* M *for which all the sections of the CR bundle* $\mathcal{V}$ *are tangent.*

REMARK 2.11. Using the representation of CR distributions in [9] (see also [30]) one can also prove under the assumption of minimality, any CR distribution on M is a boundary value of a holomorphic function in a wedge of the form (1.2) with slow growth near M.

3. Necessity of minimality. Suppose that M is not minimal so that there is a proper submanifold N of M to which all the sections of the CR bundle $\mathcal{V}$ are tangent. We shall call such an N a CR submanifold of M. The proof of the necessity proceeds by first showing that there is a nonzero CR distribution τ supported on N. It is easy to see that τ cannot be the boundary value of a holomorphic function in any wedge with edge M (since such a holomorphic function would have to vanish identically). From τ it is possible to construct, for any integer k, a nonextendable CR function of class C^k.

We first give some indications of the construction of the distribution τ. A precise statement of the result is the following theorem.

THEOREM 2 [7]. *Let* M *be a generic submanifold of* $\mathbf{C}^p$ *which is not minimal at* $m_0 \in M$ *and* N *a CR submanifold of* M *containing* m_0, *with* $\dim_{\mathbf{R}} N < \dim_{\mathbf{R}} M$. *Then there exists a CR distribution* τ *defined in a neighborhood* U *of* m_0 *in* M *with* $\operatorname{supp}\tau = N \cap U$.

We assume that $m_0 = 0$ and that M is given by (2.2). We parametrize M by $(z, \bar{z}, s)$ in a neighborhood of 0 in $\mathbf{R}^{2n+l}$. With this parametrization we can choose a basis of CR vector fields L_j, $j = 1, \ldots, n$, of the form

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{1\le k\le l} \alpha_{jk}(z, \bar{z}, s)\frac{\partial}{\partial s_k}, \tag{3.1}$$

with $\alpha_{jk}(0) = 0$.

Assume that N is given by $\rho_j(z, \bar{z}, s) = 0$, $1 \le j \le l_2$, where $l_2 = l - l_1$ and the differentials $d\rho_j$ are linearly independent. Since the L_j are tangent to N, we must have $L_j\rho_k = 0$ on N, and also $\overline{L}_j\rho_k = 0$ on N, since the ρ_j are real-valued. Hence, by (3.1), $\frac{\partial\rho_j}{\partial x_k}(0) = \frac{\partial\rho_j}{\partial y_k}(0) = 0$. We conclude that rank $(\frac{\partial\rho_j}{\partial s_k}(0))_{1\le j\le l_2, 1\le k\le l} = l_2$. After a linear change of variables and using the implicit function theorem, we can assume that N is given as a subset of M by

$$s_{l_1+j} = \psi_j(z, \bar{z}, s_1, \ldots, s_{l_1}), \qquad 1 \le j \le l_2, \quad \psi_j(0) = d\psi_j(0) = 0. \tag{3.2}$$

Put $t_j = s_{l_1+j} - \psi_j(z, \bar{z}, s)$, $j = 1, \ldots, l_2$, and take the coordinates on M, $(x, y, s, t) = (x, y, s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2})$. In these coordinates the

L_j become

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{l_1} \beta_{jk} \frac{\partial}{\partial s_k} + \sum_{\substack{1\le k\le l_2 \\ 1\le r\le l_2}} \mu_{jkr} t_k \frac{\partial}{\partial t_r}, \tag{3.3}$$

where β_{jk} and μ_{jkr} are functions of (x, y, s, t). We introduce the vector fields L_j^0, obtained from L_j by setting $t = 0$:

$$L_j^0 = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{l_1} \beta_{jk}(x, y, s, 0) \frac{\partial}{\partial s_k}. \tag{3.4}$$

We look for a distribution solution τ of the system of equations $L_j \tau = 0$ of the form $\tau = V(z, \bar{z}, s)\delta(t)$, where $\delta(t) = \delta(t_1) \otimes \cdots \otimes \delta(t_{l_2})$ is the dirac measure at the origin in $\mathbf{R}^{l_2}$ and V is a smooth function nonvanishing at 0. Using the relation

$$t_k \frac{\partial}{\partial t_r} \delta(t) = -\varepsilon_{kr} \delta(t),$$

where ε_{kr} is the Kronecker symbol, we conclude that V must satisfy the equations

$$L_j^0 V - \sum_{1\le r\le l_2} \mu_{jrr}^0(z, \bar{z}, s) V = 0, \quad 1 \le j \le n. \tag{3.5}$$

Since V is nonvanishing, equation (3.5) is equivalent to

$$L_j^0(\operatorname{Log} V) = \sum_{1\le r\le l_2} \mu_{jrr}^0(z, \bar{z}, s). \tag{3.6}$$

Therefore, it suffices to show that $\sum_{1\le r\le l_2} \mu_{jrr}^0(z, \bar{z}, s)$ is in the range of L_j^0.

It should be noted that a first-order system of equations of the form (3.6), with the $\mu_{jrr}^0(z, \bar{z}, s)$ smooth functions, need not have a solution. Even in the case of a single equation solvability can fail, as was first discovered by Hans Lewy [23] in a similar context. However, in the case under consideration the right-hand side of (3.6) may not be chosen arbitrarily. By an explicit calculation it is shown in [7] that in fact (3.6) is always solvable. This proves that the singular solution τ of Theorem 2 exists.

PROPOSITION 3.7. *Under the assumptions of Theorem 2, for any $k \ge 0$ there is a CR function of class C^k defined in a neighborhood of m_0 which does not extend to any wedge with edge M near m_0.*

PROOF. We can find l vector fields D_j satisfying, for $1 \le q \le n$, $1 \le j, p \le l$,

$$[L_q, D_j] = 0, \quad [D_j, D_p] = 0, \quad D_j(s_p + i\phi_p(z, \bar{z}, s)) = \varepsilon_{jp}, \tag{3.8}$$

where the ϕ_p are the components of ϕ in (2.2). In fact D_j is of the form

$$D_j = \frac{\partial}{\partial w_j} + \sum_{1\le k\le l} a_{jk}\frac{\partial}{\partial \bar{w}_k}, \tag{3.9}$$

(See [5] for this construction.)

By the representation of distributions annihilated by a system of complex vector fields given in [10] (see also [30]), for any k there is a CR function $f \in C^k$ defined near m_0 such that $\tau = (\sum_{1\le j\le l} D_j^2)^q f$.

We reason by contradiction. Assume that f is the boundary value of a holomorphic function H in a wedge $\mathscr{W}(\mathscr{O}, \Gamma)$. It follows from (3.9) that we have $b(\frac{\partial H}{\partial w_j}) = D_j(bH)$, where b denotes boundary value. We conclude that

$$D_j f = b\left(\frac{\partial H}{\partial w_j}\right), \qquad \tau = \left(\sum_{1\le j\le l} D_j^2\right)^q f = b\left(\left(\sum_{1\le j\le l}\left(\frac{\partial^2}{\partial w_j^2}\right)\right)^q H\right).$$

It would then follow that τ is the boundary value of a holomorphic function, contradicting Theorem 2 and the remarks preceding it.

REMARK. By using the Baire Category Theorem one can also show under the assumption of Theorem 2, the existence of a smooth CR function which does not extend to any wedge. (See [7] for details.)

4. Minimality and the sections of the CR bundle $\mathscr{V}$. In this section we shall give two characterizations of the minimal CR submanifold N through a point m_0 in a generic manifold M.

THEOREM 3 [7]. *If M is a generic manifold in $\mathbf{C}^p$ and $m_0 \in M$ there is a unique germ N_0 of a CR submanifold contained in M, $m_0 \in N_0$, of minimal dimension. Also N_0 may be described as follows.*

(i) *For every sufficiently small neighborhood U of m_0 in M, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ consists of all points $m \in U'$ which can be reached from m_0 by a finite sequence of integral curves contained in U of sections* $\operatorname{Re}\mathscr{V}$.

(ii) *For every sufficiently small neighborhood U of m_0 in M, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ is the union of sets of the form $Z(S)$, where Z is a continuous analytic disc attached to M with $m_0 \in Z(S) \subset U \cap M$.*

The first characterization (i) and the uniqueness of N_0 in Theorem 3 may be restated as a local property of a set of smooth real vector fields. Let $\mathscr{X} = \{X_1, \dots, X_r\}$ be a set of real vector fields defined in a neighborhood Ω of a point m_0 in $\mathbf{R}^q$. We shall say that $\mathscr{X}$ is *minimal* at m_0 if there is no germ of a proper submanifold N through m_0 such that all the X_j are tangent to N. The following is due to Sussman [26].

THEOREM 4 [26]. *For every sufficiently small neighborhood U of m_0 in Ω, there is a neighborhood $U' \subset U$ and a smooth manifold $N \subset U'$, $m_0 \in N$, such that the X_j are all tangent to N at every point of N and for which* $\dim N$ *is minimal with this property. In addition, N consists of all points $m \in U'$ which can be reached from m_0 by a finite sequence of integral curves contained in U of vector fields in $\mathscr{X}$. In particular, $\mathscr{X}$ is minimal at m_0 if and only if every point in U' can be reached by a sequence of integral curves of the X_j contained in U.*

To give the reader an idea of how Theorem 4 is proved, we shall consider the case of two vector fields $\mathscr{X} = \{X_1, X_2\}$ in $\mathbf{R}^3$ and show that if $\mathscr{X}$ is minimal at 0 then the last conclusion of the theorem holds. Since $\mathscr{X}$ is minimal, at least one of the vector fields, say X_1, is nonvanishing at 0 (otherwise $\{0\}$ would be a submanifold to which both X_j are tangent). Let Γ be the integral curve of X_1 starting at 0, i.e. the image of $t_1 \mapsto \exp t_1 X_1 \cdot 0$. Since, by minimality, X_2 cannot be tangent to Γ in any neighborhood of 0, there exists t_1^0, as small as desired, such that X_2 is not tangent to Γ at $\exp t_1^0 X_1 \cdot 0$. Then the mapping $(t_1, t_2) \mapsto \theta(t_1, t_2) = (\exp -t_1^0 X_1)\cdot(\exp t_2 X_2)\cdot(\exp t_1 X_1)\cdot 0$ is of rank two at $(t_1^0, 0)$. By the minimality again (since the image of this map is a two dimensional manifold Σ through 0), at least one of the vector fields X_j is not tangent to this manifold in any neighborhood of 0. Therefore, there exists $(t_1^1, t_2^1) \in \mathbf{R}^2$, as close to $(t_1^0, 0)$ as desired, such that, say, X_1 is not tangent to Σ at $\theta(t_1^1, t_2^1)$. Finally, the map $(t_1, t_2, t_3) \mapsto (\exp -t_1^1 X_1)\cdot(\exp -t_2^1 X_2)\cdot(\exp t_1^0 X_1)\cdot(\exp t_3 X_1)\cdot(\exp -t_1^0 X_1)\cdot(\exp t_2 X_2)\cdot(\exp t_1 X_1)\cdot 0$ is of rank three at $(t_1^1, t_2^1, 0)$, so that the image contains a neighborhood of the origin. Then every point in that neighborhood is connected to the origin by a sequence of at most seven integral curves of the X_j.

The general case can be proved by using an induction and repeating the same argument.

For the second characterization of minimality, i.e. (ii) in Theorem 3, one uses Proposition (2.8) to show that there is a CR submanifold of M through m_0 contained in the union of sets of the form $Z(bD)$, where $Z : \overline{D} \to \mathbf{C}^{n+l}$ is continuous, holomorphic in D and satisfying $m_0 \in Z(bD) \subset U \cap M$, where U is a sufficiently small neighborhood of m_0 in M. We shall show that the image of $S = bD$ under all such holomorphic discs Z lies in the minimal submanifold N_0. Hence the proof of Theorem 3 is completed by showing the following lemma.

LEMMA 4.1 [7]. *Let M be a generic CR submanifold of $\mathbf{C}^{n+l}$ and N a CR submanifold containing m_0. Then there is a neighborhood U of m_0 in M such that for every $Z : D \to \mathbf{C}^{n+l}$ with Z holomorphic, continuous in $\overline{D}$ and satisfying $m_0 \in Z(bD) \subset U \cap M$, we have $Z(bD) \subset N$.*

The proof of Lemma (4.1) uses the local coordinates $(z, \bar{z}, s, t)$ introduced in §3 and a uniqueness argument for the solution of the Bishop equation of the form (2.4) with $\lambda = 1$.

The notion of minimality is a weaker condition than that of finite type in the sense of Kohn [22] and Bloom and Graham [12]. A generic manifold M is of *finite type* at m_0 if the Lie algebra generated by the sections of the CR bundle $\mathscr{V}$ and their complex conjugates span CTM in a neighborhood of m_0. Indeed, if m_0 is a point of finite type, and N is a CR submanifold through m_0, then since the sections of $\mathscr{V}$ are tangent to N so are all their commutators, which implies that all tangent vectors to M are also tangent to N, i.e. the germs of M and N are the same at m_0. It should be noted that if M is real analytic, then the notion of minimality coincides with that of finite type. Indeed, if M is not of finite type, then the Nagano leaf [25] passing through m_0 would be a proper real analytic CR submanifold, contradicting minimality. The following shows that if M is only smooth it can be minimal without being of finite type.

EXAMPLE 4.2. Let M be the hypersurface in $\mathbf{C}^2$ defined by

$$M = \{(z, w) \in \mathbf{C}^2 : \operatorname{Im} w = \phi(\operatorname{Im} z)\},$$

where ϕ is a smooth function on the real line satisfying $\phi(y) > 0$ for $y > 0$ and $\phi(y) \equiv 0$ for $y < 0$. Then M is minimal at the origin, since every point can be reached by a sequence of integral curves of the sections of $\operatorname{Re}\mathscr{V}$. However, M is not of finite type at 0, since the commutators of all orders of L and $\bar{L}$ vanish at that point. Here L is any section of $\mathscr{V}$.

REMARK 4.3. If $\dim_{\mathbf{R}} N_0 = 2n$, where N_0 is given by Theorem 3, and $\dim \mathscr{V} = n$ as in §2, then N_0 is a holomorphic submanifold of $\mathbf{C}^{n+l}$ of complex dimension n. In particular, if M is a hypersurface, i.e. $l = 1$, then N_0 is a complex hypersurface. In this case the condition of minimality coincides with the necessary and sufficient condition of Trépreau. Indeed, with the holomorphic coordinates (z, w) of §2, by dimension we must have that N_0 is given by an equation of the form $w - f(z, \bar{z}) = 0$, where f is a smooth function in a neighborhood of 0 in $\mathbf{R}^{2n}$. Since $w - f(z, \bar{z})$ is annihilated on N_0 by the antiholomorphic vector fields tangent to M, we conclude that $\frac{\partial f}{\partial \bar{z}_j} = 0$, i.e. f is holomorphic.

REMARK 4.4 [7]. If M is a real analytic generic manifold which is not minimal at m_0, and N a real analytic CR submanifold of M, then there is a holomorphic submanifold $\mathscr{H}$ in $\mathbf{C}^{n+l}$, such that $N = M \cap \mathscr{H}$ with $\dim_{\mathbf{R}} N = \dim_{\mathbf{C}} \mathscr{H} + n$. In particular, the minimal CR submanifold through m_0 is of this form. This is not the case if M is assumed only to be smooth rather than real analytic, as shown by the following example.

EXAMPLE 4.5. Let M be the generic submanifold of $\mathbf{C}^3$ of codimension 2 parametrized by (x, y, s, t) and given by $\{(z, w_1, w_2) : w_1 = s + i|z|^2, w_2 = t + h(x, y, s)\}$ where h is a smooth nonreal analytic function satisfying $\frac{\partial h}{\partial \bar{z}} = iz\frac{\partial h}{\partial s}$. Here N is given by $\{t = 0\}$ and $L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}$. If N were the intersection of M with a complex hypersurface, there would exist a holomorphic function $\mathscr{H}$ and holomorphic coordinates (z', w_1', w_2') such that h is the restriction of $\mathscr{H}(z', w_1', w_2')$ to N, contradicting the assumption that h is not real analytic.

5. Other results, remarks. We describe here some related results for structures more general than generic manifolds. Let Ω be an open set in $\mathbf{R}^q$. We shall say that Ω is equipped with a *hypoanalytic structure* (see [3]) if we are given locally p complex-valued smooth functions $Z_1, \dots, Z_p$ with linearly independent differentials. Therefore there are $n = q - p$ complex vector fields $L_1, \dots, L_n$ linearly independent at each point and satisfying $L_j Z_k = 0$ for all j, k. We denote by $\mathscr{V}$ the bundle spanned by these vector fields. Note that the image of Ω by the map $Z = (Z_1, \dots, Z_p)$ in $\mathbf{C}^p$ need not be a manifold. Indeed, consider the case

$$(5.1) \qquad p = n = 1, \qquad Z = x + iy^2, \qquad L = \frac{\partial}{\partial y} - 2iy\frac{\partial}{\partial x}.$$

The L in (5.1) is called the *Mizohata operator* in $\mathbf{R}^2$.

Marson [24] has defined a notion of wedge extendability for solutions of the overdetermined system $L_j h = 0$, and has extended Theorem 1 to this case. Here a point $m_0 \in \Omega$ is minimal for $\mathscr{V}$ if every point in a neighborhood of m_0 can be reached by a finite sequence of integral curves of the real and imaginary parts of the L_j, as in condition (i) of Theorem 3. His result is the following theorem.

THEOREM 5 [24]. *M is wedge extendable at $Z(m_0)$ if and only if $\mathscr{V}$ is minimal at m_0.*

In addition to its intrinsic interest, Theorem 5 has applications to extendability of holomorphic functions across singular hypersurfaces.

We shall now give some new results for CR manifolds which are not necessarily generic. Let M of codimension l in $\mathbf{C}^p$ be given locally near $m_0 \in M$ by $\rho_j(Z, \overline{Z}) = 0$, $1 \le j \le l$, with ρ_j smooth and real-valued and $d\rho_1, \dots, d\rho_l$ linearly independent. If, in addition, the complex differentials $\partial\rho_1, \dots, \partial\rho_l$ have constant rank l_1, then M is called a *CR manifold*. In this case the antiholomorphic vector fields tangent to M form a bundle $\mathscr{V}$ of dimension $n = p - l_1$. The reader can easily check that $[(l+1)/2] \le l_1 \le l$. A solution of the system $L_j h = 0$ is again called a CR function on M. We show here that after a holomorphic change in $\mathbf{C}^p$ we can put M in the form of a CR graph over a generic manifold M_1 so that Theorem 1 can be applied to obtain extendability for CR functions.

THEOREM 6. *Let M be a CR submanifold of $\mathbf{C}^p$ as above, $0 \in M$. Then there exist holomorphic coordinates around 0, a generic manifold M_1 in C^{p-l+l_1} and $l - l_1$ CR functions $\psi = (\psi_1, \dots, \psi_{l-l_1})$ on M_1, $\psi(0) = 0$,* *such that M is the graph of ψ over M_1, i.e.*

$$M = \{(m, \psi(m)) \in \mathbf{C}^p : m \in M_1\}. \tag{5.2}$$

The natural projection of M onto M_1 is a CR diffeomorphism, i.e. its differential carries $\mathscr{V}$ onto $\mathscr{V}_1$, the CR bundle for M_1. Finally, if M is real analytic, then the holomorphic coordinates can be chosen so that $\psi \equiv 0$.

PROOF. Let $\rho_1, \dots, \rho_l$ be defining functions for M in $\mathbf{C}^p$, and l_1 the rank of $\{\partial\rho_1, \dots, \partial\rho_l\}$. By linear algebra and an application of the implicit function theorem we may choose coordinates $(z_1, \dots, z_{p-l_1}, w_1, \dots, w_{2l_1-l}, v_1, \dots, v_{l-l_1})$ so that after a linear transformation the ρ_j have the form

$$\begin{aligned} \rho_j &= \operatorname{Im} w_j - \phi_j(z, \bar z, \operatorname{Re} w), \quad 1 \le j \le 2l_1 - l, \\ \rho_{j+2l_1-l} &= \operatorname{Re}(v_j - \psi_j(z, \bar z, \operatorname{Re} w)), \quad 1 \le j \le l - l_1, \\ \rho_{j+l_1} &= \operatorname{Im}(v_j - \psi_j(z, \bar z, \operatorname{Re} w)), \quad 1 \le j \le l - l_1, \end{aligned} \tag{5.3}$$

where the ϕ_j are smooth, real-valued functions, and the ψ_j are smooth complex-valued, all vanishing at the origin, together with their differentials. Let M_1 be the generic manifold in $\mathbf{C}^{p-(l-l_1)}$ defined by $\operatorname{Im} w_j - \phi_j(z, \bar z, \operatorname{Re} w) = 0$, $1 \le j \le 2l_1 - l$ and $\mathscr{V}_1$ its CR bundle. Note that $\dim_{\mathbf{C}} \mathscr{V} = \dim_{\mathbf{C}} \mathscr{V}_1 = p - l_1$. It remains to show that the ψ_j are CR functions on M_1. Let $\mathscr{L}_j$, $1 \le j \le p - l_1$, be antiholomorphic vector fields in $\mathbf{C}^{p-(l-l_1)}$ such that their restrictions to M_1 form a basis for the sections of $\mathscr{V}_1$. Since $\mathscr{L}_j \rho_k = 0$ on M_1, for $1 \le j \le p - l_1$ and $1 \le k \le 2l_1 - l$, any section L of $\mathscr{V}$ is necessarily of the form $L = \sum_{j=1}^{l-l_1} \lambda_j \frac{\partial}{\partial \bar v_j} + \sum_{j=1}^{p-l_1} \mu_j \mathscr{L}_j$. It is easy to check that $\lambda_j(0) = 0$, from which it follows that $\mathscr{L}_j \psi_k = 0$. We may choose a basis for the sections of $\mathscr{V}$ of the form

$$L_j = \sum_{k=1}^{l-l_1} \mathscr{L}_j \bar\psi_k \frac{\partial}{\partial \bar v_k} + \mathscr{L}_j, \qquad 1 \le j \le p - l_1. \tag{5.4}$$

Now assume that M is real analytic. Then the ψ_j are real analytic CR functions on M_1, which therefore extend holomorphically in $\mathbf{C}^{p-l+l_1}$. If $H_j(z, w)$ is the holomorphic extension of ψ_j, then by replacing the coordinates v_j by $v_j - H_j(z, w)$ we achieve the desired form, i.e. $\psi = 0$ in (5.2).

Conversely, note that if M is of the form (5.2), where ψ is CR on M_1, then M is CR, since it is the image of M_1 under a CR diffeomorphism. This proves Theorem 6.

The following result is proved by using Theorem 6 and then using Theorem 1.

COROLLARY. *If M is a CR manifold in $\mathbf{C}^p$ minimal at $0 \in M$ then any CR function on M is the boundary value of a holomorphic function on an* *open set of the form* $\mathscr{W}(\mathscr{O}, \Gamma)_{M_1} \times \mathbf{C}^{l-l_1}$, *where* M_1 *is as in Theorem* 6 *and* $\mathscr{W}(\mathscr{O}, \Gamma)_{M_1}$ *is a wedge of edge* M_1 *in* $\mathbf{C}^{p-l+l_1}$.

Very recently, J-M. Trépreau [29] has obtained propagation of extendability of CR functions on a generic CR manifold along minimal CR submanifolds (see Theorem 3), generalizing the results of Hanges and Treves [17] from the case of holomorphic submanifolds. He also obtains microlocal results generalizing work of Hanges and Sjöstrand [16], dealing with propagation of microlocal singularities. We refer the reader to [29] for further details.

Another question of interest is to determine when every CR function on a generic manifold M can be decomposed as a finite sum of boundary values of holomorphic functions in wedges of the form (1.2) with edge M. It is a well-known result in Fourier analysis that any distribution on $\mathbf{R}^n$ is locally a finite sum of boundary values of functions holomorphic in wedges in $\mathbf{C}^n$ with edge $\mathbf{R}^n$. This result is easily extended to the case where M is a totally real generic manifold in $\mathbf{C}^p$ (see, e.g., [3]). Andreotti and Hill [2] showed that any CR function on a hypersurface M in $\mathbf{C}^p$ is the sum of the boundary values of two holomorphic functions, defined on opposite sides of M. Henkin [18] and Ajrapetyan and Henkin [1] obtained some positive results for decomposition in higher codimension. The authors jointly with Treves [8] have shown that decomposition holds in the rigid case, i.e. when the defining function ϕ given in (2.2) is independent of s. Other positive results, including a new proof for the case of a hypersurface, are given in [4]. The first example of a nondecomposable CR function was given by Trépreau; his example is on a generic manifold of codimension two in $\mathbf{C}^3$. In [29], Trépreau has obtained more general results on nondecomposability. Connections between decomposition and extendability of holomorphic functions defined in generic wedges are discussed in [6].

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