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NONEXISTENCE OF OPTIMAL L^2 ESTIMATES FOR THE BOUNDARY LAPLACIAN OPERATOR ON CERTAIN WEAKLY PSEUDOCONVEX DOMAINS

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1. Introduction. In this paper it is shown that the boundary Laplacian operator on some "reasonable" weakly pseudoconvex domains does not satisfy optimal L^2 estimates (see (1.2) below) in contrast to the strictly pseudoconvex case (Folland-Stein [3]). However, a closely related example in a class of operators defined by Grušin (Section 5) does satisfy these optimal estimates. The techniques used here are those developed by Helffer-Nourrigat [8].

A differential operator L is called hypoelliptic if $Lu = f$, with f smooth in an open set U implies u also smooth in U. Now suppose that $\{X_1, X_2, \ldots, X_n\}$ is a set of smooth, real vector fields on a manifold M. We are interested in conditions which guarantee that an operator of the form

$$L = \sum_{k \le d} a_{i_1 i_2 \cdots i_k}(x) X_{i_1} X_{i_2} \cdots X_{i_k} \tag{1.1}$$

with $a_{i_1 i_2 \cdots i_k}(x)$ smooth, complex-valued functions, is hypoelliptic. Hörmander [9], introduced the following condition on the X_i:

(H) $\{X_1, X_2 \ldots, X_n\}$ together with their commutators of length $\le r$, r a fixed integer, span the tangent space at $x_0 \in M$.

If $\{X_1, \ldots, X_n\}$ satisfies (H), then, following Helffer-Nourrigat [7] we say that L is <u>maximally hypoelliptic</u> if for every $s \le d$, there exists $C > 0$ and a neighborhood ω of x_0 such that

$$\|X_{i_1} X_{i_2} \ldots X_{i_s} u\|^2 \le C(\|Lu\|^2 + \|u\|^2). \tag{1.2}$$

For all $u \in C_0^\infty(\omega)$, where $\| \ \|$ denotes L^2 norm.

We exhibit here (Sections 4 and 5) two examples of operators of the form (1.2), one arising from several complex variables and the other in a class considered by Grušin [5]. Although the operators are formally similar, it will be shown that one is maximally hypoelliptic and the other is not.

2. <u>Homogeneous left invariant differential operators on nilpotent groups.</u>

We review briefly some results of [6], [7], and [8], and establish notation. Let $\mathfrak{g}$ be a graded nilpotent Lie algebra, i. e.

$$\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^2 + \ldots + \mathfrak{g}^r$$

a linear direct sum with $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, where $\mathfrak{g}^s = (0)$, $s > r$

Then $\mathfrak{g}$ has a family $\delta_s, s > 0$ of dilations which are automorphisms,

$$\delta_s Y_j = s^j Y_j, \quad Y_j \in \mathfrak{g}^j.$$

A left invariant differential operator L on G, the simply connected nilpotent group corresponding to $\mathfrak{g}$, is <u>homogeneous</u> of degree d if it can be written

$$L = \sum_{i_1 + i_2 + \ldots + i_k = d} a_{i_1 i_2 \ldots i_k} Y_{i_1} Y_{i_2} \ldots Y_{i_k}, \tag{2.1}$$

$Y_{i_j} \in \mathfrak{g}^{i_j}$. Generalizing slightly the definition given in Section 1, we call L <u>maximally hypoelliptic</u> if for every homogeneous left invariant differential operator A of degree d on G there is a constant $C > 0$ such that

$$\|Au\| \leq C\|Lu\| \tag{2.2}$$

for all $u \in C_0^\infty(G)$. A representation-theoretic criterion for hypoellipticity of L was conjectured by Rockland [13] and proved in general by Helffer-Nourrigat [6]. In this context it had been known (Folland [1]) that hypoellipticity is equivalent to maximal hypoellipticity.

(2.3) Theorem (Helffer-Nourrigat [6]). *Let L be a homogeneous left invariant differential operator on G. Then L is hypoelliptic if and only if for every irreducible unitary representation π of G on a Hilbert space $\mathcal{H}$, $\pi(L)$ is injective in the space of C^∞ vectors of $\mathcal{H}$.*

Suppose now that $X_1, X_2, \ldots X_n$ are arbitrary C^∞ vector fields satisfying (H). By the methods of Rothschild-Stein [16] one may lift the X_j to vector fields on a higher dimensional space so that an operator L of the form (1.1) is lifted to an operator $\tilde{L}$. $\tilde{L}$ has the property that it may be approximated by a homogeneous left invariant differential operator D on a nilpotent group in such a way that if D is maximally hypoelliptic, so are L and $\tilde{L}$. (See [7] for complete definitions and details.) However, it may happen that L is maximally hypoelliptic although $\tilde{L}$ is not. Under more restrictive conditions on the vector fields it is possible to associate a group at each point and to obtain necessary and sufficient conditions for maximal hypoellipticity by using the group criterion (see Rothschild [15] for details.) The condition on the $X_1, X_2, \ldots X_n$, first introduced by Métivier [14], is that the dimension of the space spanned by the commutators of length

$\leq j$ at x should be constant for all x in a neighborhood of x_0 and all $j \leq r$. Unfortunately there are many interesting examples of vector fields satisfying (H), such as $\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\}$, which do not satisfy the above condition.

3. Quasi homogeneous vector fields. Let $\mathcal{P}(\mathbb{R}^N)$ be the Lie algebra of all global vector fields on $\mathbb{R}^N$ with polynomial coefficients and suppose δ_s, $s > 0$ is a family of dilations on $\mathbb{R}^N$. The dilations δ_r induce an automorphism of $\mathcal{P}$ by

$$\delta_s X(\varphi) = [X(\varphi \circ \delta_s)] \circ \delta_{s^{-1}},$$

$X \in \mathcal{P}$, $\varphi \in C^\infty(\mathbb{R}^N)$. X is (quasi-) homogeneous of degree λ if $\delta_s X = s^\lambda X$.

Now let $\{X_1, X_2, \ldots, X_n\} \in \mathcal{P}$ satisfy

(3.1 i) X_j is homogeneous of degree 1, $1 \leq j \leq n$, and

(3.1 ii) The Lie algebra $\mathfrak{g}$ generated by the $\{X_i\}$ is finite dimensional.

If (3.1) is satisfied, then

$$\mathfrak{g} = \sum_{i=1}^{r} \mathfrak{g}^i$$

where

$$\mathfrak{g}^i = \{X \in \mathfrak{g} : X \text{ is (quasi)-homogeneous of degree } i\},$$

and $\delta_s | \mathfrak{g}^i = s^i$ is an automorphism.

The condition (3.1) was first defined by Folland [2] and later studied more extensively by Helffer-Nourrigat [8]. Folland proved that sets of vector fields satisfying (3.1) may be characterized as follows. Let $\mathfrak{h}$ be the (graded) subalgebra of all vector fields in $\mathfrak{g}$ vanishing at 0. Let G, H be the corresponding simply connected Lie groups for $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then

$\{X_1, X_2, \ldots, X_n\}$ may be identified with the restrictions to the homogeneous space $H\backslash G$ of a basis $\{Y_1, Y_2, \ldots, Y_n\}$ of $\mathfrak{g}^1 \subset \mathfrak{g}$.

Now suppose L is of the form

$$L = \sum a_{i_1 i_2 \cdots i_d} X_{i_1} X_{i_2} \cdots X_{i_d}$$

with $a_{i_1 i_2 \cdots i_d} \in \mathbb{C}$ and $X_{i_j} \in \{X_1, X_2 \ldots, X_n\}$ satisfying (3.1). Helffer and Nourrigat [8] have given a necessary condition that L be maximally hypoelliptic, which they have shown sufficient for $r \leq 3$. To state this condition we need some facts about representations of nilpotent Lie groups. Given $\xi \in \mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, the Kirillov correspondence [10] determines an irreducible unitary representation π_ξ. π_ξ is equivalent to $\pi_{\xi'}$ if and only if ξ and ξ' are in the same orbit under the action of Ad^*G, the co-adjoint representation of G on $\mathfrak{g}^*$. Given a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $\mathrm{sp}(\pi_{(0,H)})$ be the set of all irreducible unitary representations of G corresponding to the closure (in the Euclidean sense) of $\{\xi \in \mathfrak{g}^*: \xi = \mathrm{Ad}^* g \cdot \xi', \ \xi'$ vanishes on $\mathfrak{h}$, for some $g \in G\}$.

(3.2) <u>Theorem</u> (Helffer-Nourrigat [8], [17]). <u>Let</u> L <u>be as above and let</u> $\tilde{L} = \sum a_{i_1 i_2 \cdots i_d} Y_{i_1} Y_{i_2} \cdots Y_{i_d}$ <u>be the corresponding left invariant differential operator on</u> G. <u>Then if</u> L <u>is maximally hypoelliptic</u> $\pi(\tilde{L})$ <u>has trivial kernel in the space of</u> C^∞ <u>vectors (= rapidly decreasing functions) for every irreducible unitary representation</u> $\pi \in \mathrm{sp}(\pi_{(0,H)})$. <u>The converse is known to hold for</u> $r \leq 3$.

It is conjectured by Helffer-Nourrigat [17] that the condition of the above theorem is also sufficient for any r.

4. <u>The boundary Laplacian operator for the domain</u> $\{z \in \mathbb{C}^{k+1}:$

$$\operatorname{Im} z_{k+1} > |z_1|^4 + |z_2|^4 \ldots + |z_k|^4\} .$$

The boundary of the strictly pseudoconvex domain $\{z \in \mathbb{C}^{k+1}:$ $\operatorname{Im} z_{k+1} > |z_1|^2 + |z_2|^2 + \ldots + |z_k|^2\}$ may be identified with the Heisenberg group H^k (see Folland-Stein [3]). As in [3] we introduce the coordinates (z,t), $z \in \mathbb{C}^k$, $t \in R$, $(z,t) \longleftrightarrow (z_1, \ldots, z_{k+1})$, where $t = \operatorname{Re} z_{k+1}$. The space of holomorphic vector fields is then spanned by $Z_j = U_j + \sqrt{-1}V_j$, where

$$U_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad V_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} .$$

The boundary Laplacian, $\square_b$ with respect to the metric which makes $\{U_i, V_j, T, j = 1, 2, \ldots, k\}$ an orthonormal basis is then a vector-valued, diagonalizable operator, each of whose components is of the form

$$\sum_{i=1}^{k} U_i^2 + V_i^2 + \sum_{j=1}^{k} \varepsilon_j [U_j, V_j] \tag{4.1}$$

where each ε_j is either $+1$ or -1. In particular, the ε_j all have the same sign if and only if $q = 0$ or $q = k$. If $0 < q < k$, each operator (4.1) is hypoelliptic and in fact maximally hypoelliptic with U_j, V_j, $j = 1, 2, \ldots k$ as the chosen directions. By making use of this model case, Folland and Stein were able to prove local optimal estimates in various function spaces for $\square_b$ on arbitrary strictly pseudoconvex domains. From these optimal estimates for the $\bar{\partial}$ operator have been obtained by Greiner-Stein [4].

One may ask whether there is an analogous theory for some weakly pseudoconvex domains. In particular, consider the domain

$$\mathcal{S} = \{z \in \mathbb{C}^{k+1} : \operatorname{Im} z_{k+1} = |z_1|^4 + |z_2|^4 + \ldots + |z_k|^4\} .$$

with respect to the coordinates (z, t) the holomorphic vector fields have a basis U_j', V_j', where

$$U_j' = \frac{\partial}{\partial x_j} + 2y_j(x_j^2 + y_j^2)\frac{\partial}{\partial t} \qquad V_j' = \frac{\partial}{\partial y_j} - 2x_j(x_j^2 + y_k^2)\frac{\partial}{\partial t}.$$

Except on the line $x_j = y_j = 0$, $j = 1, 2, \ldots, n$, the vector fields U_j', V_j' and their commutators $[U_j', V_j']$ span the tangent space. On the exceptional line, the commutators of length four are needed to span the tangent space.

The associated Lie algebra is of step four with $2k$ generators, $A_1, A_2, \ldots, A_k$, $B_1, B_2, \ldots, B_k$. We may divide out by an ideal so that a basis for the nonzero commutators is given by

Length 1: $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$

Length 2: $[A_j, B_j]$ $j = 1, 2, \ldots, k$

Length 3: $[A_j, [A_j, B_j]]$, $[B_j, [A_j, B_j]]$, $j = 1, 2, \ldots, k$,

Length 4: $[[A_j, [A_j, B_j]]] = [B_j, [B_j, [A_j, B_j]]]$, $j = 1, 2, \ldots, k$.

We denote the above basis by $\mathfrak{B}$. On the line $x_j = y_j = 0$ $j = 1, 2, \ldots, k$ the associated homogeneous space is $H\backslash G$, where the Lie algebra $\mathfrak{h}$ of H is spanned by A_j, B_j, $\mathrm{ad}(A_j)^2 B_j$ and $(\mathrm{ad}\, A_j)(\mathrm{ad}\, B_j)A_j$. By the same computation as in the strictly pseudo-convex case (see [3]), the scalar components of the boundary Laplacian $\square_b$ may be identified with the restrictions of

$$\sum_{j=1}^{k} A_j^2 + B_j^2 + \sqrt{-1}\sum_{j=1}^{k} \varepsilon_j [A_j, B_j]$$

to $H\backslash G$ with each $\varepsilon = \pm 1$. Again, if $\square_b$ is acting on (p, q) forms the ε_j are all $+1$ or -1 if and only if $q = 0$ or $q = k$. For $0 < q < k$, the scalar components are

$$\sum_{j=1}^{k} U_j'^2 + V_j'^2 + \sum_{j=1}^{k} \varepsilon_j [U_j', V_j']$$

where not all the ε_j have the same sign. One may ask whether any of these components is maximally hypoelliptic.

(4.2) Proposition. $\sum_{j=1}^{n} U_j'^2 + V_j'^2$ is maximally hypoelliptic.

Proof. This is just a special case of [16, Theorem 15].

Thus the question of maximal hypoellipticity depends on the lower order terms. By applying the necessary condition of Helffer-Nourrigat (Theorem 3.2 above), we shall see that maximal hypoellipticity does not hold in this case. This does not imply that the related optimal estimates for $\bar{\partial}$ do not hold (see Greiner-Stein [4]) but only that they cannot be obtained from boundary estimates.

(4.3) Theorem. $\Box_b$ on $\partial \mathcal{S}'$ is not maximally hypoelliptic on $(0,q)$ forms for any q. In particular, no scalar component of $\Box_b$ is maximally hypoelliptic.

Proof. We shall show that there is a linear function $\xi_0 \in Sp(0,H)$ such that $\pi_{\xi_0}(\sum A_j^2 + B_j^2 + \sqrt{-1}\sum \varepsilon_j [A_j, B_j])$ has a non-trivial kernel. Indeed, if $\xi_0([A_1, B_1]) = \varepsilon_1$, and ξ_0 vanishes on the other elements of $\mathcal{B}$, then

$$(4.4) \qquad \pi_{\xi_0}(\sum A_j^2 + B_j^2 + \sqrt{-1}\sum \varepsilon_j [A_j, B_j]) = (\frac{\partial^2}{\partial u_1^2} - u_1^2) + 1,$$

which annihilates the Hermite function $e^{-u_1^2/2}$. Thus it suffices to show that $\pi_{\xi_0} \in sp(\pi_{(0,H)})$. Let π_{ξ_n} be defined by

$$\xi_n[A_1, [A_1, [A_1, B_1]]] = \frac{\varepsilon_1}{n},$$

ξ_n vanishes on the elements of length $1, 2, 3$ of the basis $\mathcal{B}$. Now put, for $r_n \in \mathbb{R}$,

$$\xi_n' = Ad^* \operatorname{Exp} r_n B_1 \cdot \xi_n .$$

Then

$$\xi_n'(A_1) = \xi_n(\mathrm{Exp}(-r_n \operatorname{ad} B_1)\cdot A_1) =$$
$$\xi_n(A_1) - r_n[B_1, A_1] + \frac{r_n^2}{2}[B_1,[B_1,A_1]] + \frac{r_n^3}{3!}[B_1,[B_1,[B_1,A_1]]])$$
$$= \frac{+r_n^3}{3!}\,\frac{1}{n}$$

$$\xi_n'(B_1) = \xi_n(B_1) = 0$$

$$\xi_n'([A_1, B_1]) = \xi_n([A_1,B_1]) - r_n[B_1,[A_1,B_1]] + \frac{r_n^2}{2!}[B_1,[B_1,[A_1,B_1]]]$$
$$= \frac{r_n^2\,\varepsilon_1}{2n}$$

$$\xi_n'[A_1[A_1,B_1]] = 0$$

$$\xi_n'([B_1,[A_1,B_1]]) = \frac{-r_n\,\varepsilon_1}{n}$$

$$\xi_n^1[A_1,[A_1,[A_1,B_1]]] = \frac{\varepsilon_1}{n},$$

$$\xi_n'(X) = 0 \quad \text{for all other} \quad X \in \mathcal{B}.$$

Now let $s_n \in \mathbb{R}$ and put

$$\xi_n'' = \mathrm{Ad}^* \mathrm{Exp}\, s_n[A,[A,B]] \circ \xi_n .$$

Then one can check that

$$\xi_n''(A_1) = \xi_n'(A_1) - s_n\,\xi_n'([A_1,[A_1,[A_1,B_1]]]) = \left(\frac{+r_n^3}{3!\,n} - s_n\right)\varepsilon_1, \tag{4.5}$$

and

$$\xi_n''(X) = \xi_n'(X) \qquad X \in \mathcal{B},\ X \neq A_1 .$$

Finally, put

$$r_n = \sqrt{2n}, \quad s_n = (r_n^3/3!\,n). \tag{4.6}$$

Then

$$\xi_n''[A_1, B_1] = \varepsilon_1$$

$$\xi_n''([B_1,[A_1,B_1]]) = \frac{\sqrt{2}\,\varepsilon_1}{\sqrt{n}}$$

$$\xi_n''([A_1,[A_1,[A_1,B_1]]]) = \varepsilon_1/n$$

and $\xi_n''(X) = 0$, all other $X \in \mathcal{B}$. Hence $\xi_n \to \xi_0$ as $n \to \infty$.

To complete the proof of Theorem 4.3 we calculate π_{ξ_0}. A maximal isotropic subalgebra for ξ_0 is spanned by

$$\{B_1, A_j, B_j, [A_j, B_j],\ \operatorname{ad} A_j^2\, B_j,\ \operatorname{ad} B_j^2\, A_j,\ \operatorname{ad} A_j^3\, B_j,\ j \geq 2\}.$$

Then

$$\pi_{\xi_0}(A_1) = \frac{\varepsilon_j \partial}{\partial u}, \quad \pi_{\xi_0}(B_1) = \sqrt{-1}\, u_j \cdot \varepsilon_j I$$

and $\pi_{\xi_0}(A_j) = \pi_{\xi_0}(B_j) = 0$, $j \geq 2$, which proves (4.4).

(4.7) <u>Remark</u>. It is unknown whether the scalar components of $\square_b$ on $\partial\mathcal{L}'$ are non-maximally hypoelliptic. By Kohn's recent work [11] it is known that $\bar{\partial}$ is subelliptic, however.

5. <u>A related maximally hypoelliptic operator</u>. The difficulty of resolving the question of whether an operator like $\square_b$ is maximally hypoelliptic may be illustrated by a related example which may be analyzed by the method of Grušin [5]. Let

$$U_j'' = \frac{d}{dx_j}, \quad V_j'' = x_j^3/3\ \frac{\partial}{\partial t},$$

$j = 1, 2, \ldots, k$, and put

$$L = \sum_{j=1}^{k} (U_j''^2 + V''^2) + \sqrt{-1}\sum \varepsilon_j [U_j'', V_j'']$$

(5.1)

$$= \sum \frac{\partial^2}{\partial x_j^2} + x_j^6 \frac{\partial^2}{\partial t^2} + \sqrt{-1} \sum \varepsilon_j x_j^2 \frac{\partial}{\partial t}$$

where each $\varepsilon_j = +1$ or -1, (5.2) Theorem. *Let L be defined by (5.1). L is maximally hypoelliptic if and only if L is hypoelliptic if and only if the ε_j do not all have the same sign.*

Proof. L is homogeneous of degree 2 if x_i is given weight 1, $i = 1, 2, \ldots, k$ and t is given weight 4. Furthermore, L is elliptic away from $x_1 = x_2 = 0$. Hence the criterion of Grušin [5] may be applied to conclude that L is hypoelliptic if and only if

(5.3) $\quad L^\wedge(\tau)$ is injective on $\mathcal{S}(\mathbb{R}^k)$ for $\tau = \pm 1$,

where $L^\wedge(\tau)$ denotes the partial Fourier transform i.e.

$$L^\wedge(\tau) = \sum_{i=1}^{k} \left(\frac{\partial^2}{\partial x_i^2} - x_i^6 \tau^2\right) - 3\varepsilon_i x_i^2 \tau .$$

To check that (5.3) holds if not all the ε_j have the same sign, it suffices to show that if $\varphi \in \mathcal{S}(\mathbb{R}^k)$

$$(L^\wedge \varphi, \varphi) \neq 0 .$$

Suppose first that $\tau = 1$. Then

$$L^\wedge(1) = \sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2} - x_j^6 - 3\varepsilon_j x_j^2 = \sum_{j=1}^{k} L_j^\wedge .$$

Hence

$$(L^\wedge \varphi, \varphi) = \sum_{j=1}^{k} -\left\|\frac{\partial \varphi}{\partial x_j}\right\|^2 - \|x_j^3 \varphi\|^2 - 3\varepsilon_j \|x_j \varphi\|^2 . \tag{5.4}$$

Now put $D_j = \frac{\partial}{\partial x_j} + x_j^3$. By the inequality

$$\left\|\frac{\partial \varphi}{\partial x_j}\right\|^2 + \|x_j^3 \varphi\|^2 \geq 3\|x_j \varphi\|^2 , \tag{5.5}$$

which is obtained by expanding $(D_j D_j^\tau \varphi, \varphi) \geq 0$ and $(D_j^\tau D_j \varphi, \varphi) \geq 0$, we see that $(L^\wedge \varphi, \varphi)$ is non-positive. Furthermore, if at least one of the ε_j is positive, $(L_j^\wedge \varphi, \varphi) < 0$, and therefore

$$(L^\wedge \varphi, \varphi) < 0 . \tag{5.6}$$

If $\tau = -1$, a similar argument shows that $(\hat{L}\varphi, \varphi) \neq 0$ holds if at least one of the ε_j is negative. Hence, by Grušin's criterion [5, Theorem 1.2], L is hypoelliptic. Conversely if all the ε_j have the same sign, then $\varphi = e^{-x_1^4/4} e^{-x_2^4/4} \cdots e^{-x_k^4/4}$ is in the kernel of either $\hat{L}(1)$ or $\hat{L}(-1)$, as is easily checked,

Now suppose that the ε_j have different signs; we want to check that L is maximally hypoelliptic. For this it suffices to show that for all $u \in C_0^\infty(\mathbb{R}^{2k+1})$

$$\|Au\|^2 \leq C(\|Lu\|^2 + \|u\|^2) \tag{5.7}$$

for $A = \frac{\partial^j}{\partial x_\ell}$, $(x_\ell^3 \frac{\partial}{\partial t})^j$, $\frac{\partial}{\partial t}$, $\ell = 1, 2, \ldots, k$, and $j = 1, 2$. Suppose first that one can prove

$$\|Au\| \leq C'\|Lu\| \tag{5.8}$$

for all A as above with $j = 2$. Then one easily obtains (5.7) for $j = 1$. For instance,

$$\left\|\frac{\partial u}{\partial x_\ell}\right\|^2 = \left(\frac{\partial^2}{\partial x_\ell^2} u, u\right) |= \left(\frac{1}{M}\frac{\partial^2 u}{\partial x_\ell^2} U, Mu\right)| \tag{5.9}$$
$$\leq \frac{1}{M}\left\|\frac{\partial^2 u}{\partial x_\ell^2}\right\|^2 + M\|u\|^2$$

for any constant M. Hence (5.8) and (5.9) yield (5.7) for $A = \frac{\partial}{\partial x_\ell}$. To prove (5.8) by homogenity of A and L it suffices to show

$$\|\hat{A}(\tau)\varphi\|^2 \leq C_\tau\|\hat{L}(\tau)\varphi\|^2 \tag{5.10}$$

for $\tau = 1, -1$.

It may be possible to obtain (5.10) directly from the estimates in Grušin [5], but I shall give an independent proof here based on the ideas of Helffer-Nourrigat [7] and [8]. (In [8, section 3.4] it is implied that hypoelliptic Grušin-type operators like L are maximally hypoelliptic, but a proof is not given there.) To prove (5.10) one first proves

(5.11) $$\|A^\wedge u\|^2 \le C(\|L^\wedge u\|^2 + \|u\|^2_{H^1}),$$

where

$$\|u\|^2_{H^1} = \sum \left\|\frac{\partial u}{\partial x_i}\right\|^2 + \|x_i^3 u\|^2 + \|u\|^2 .$$

For this, note that

(5.12) $$\|L_1^\wedge u\|^2 \le C_1(\|L^\wedge u\|^2 + \|u\|^2_{H^1}),$$

where

$$L_1^\wedge = \sum \frac{\partial^2}{\partial x_i} - x_i^6 .$$

Next we shall show that

(5.13) $$\|A^\wedge u\|^2 \le C_2\|L_1^\wedge u\|^2 .$$

For this, let $Y_{i,1}, Y_{i,2}$ be generators of a free nilpotent algebra $\mathfrak{g}_i$ of step 4. Then $\tilde{L}_1 = \sum_i Y_{i1}^2 + Y_{i2}^2$ is hypoelliptic, and by Folland [1, theorem 6.1] satisfies the estimate

$$\|B\varphi\|^2 \le C_2\|\tilde{L}_1\varphi\|^2$$

if $B = Y_1^2, Y_2^2$, or Y_1Y_2. Hence by Helffer-Nourrigat [6, Proposition 2.1] if π is any irreducible unitary representation of G on $L^2(S)$,

(5.14) $$\|\pi(B)v\|^2 \le C_2'\|\pi(\tilde{L}_1)v\|^2$$

for all $v \in C_0^\infty(S)$. Now if π is taken to be the irreducible representation given by $Y_{i,1} \to \frac{\partial}{\partial x_i}$ $Y_{i,2} \to \sqrt{-1}\, x_i^3$ (5.13) follows from (5.14). It is obvious that (5.11) follows from (5.12) and (5.13). Now if $N > 0$ is arbitrary,

$$\left\|\frac{\partial u}{\partial x_j}\right\|^2 = \left|\left(\frac{1}{N}\frac{\partial^2}{\partial x_j^2}u, Nu\right)\right| \le \frac{1}{N^2}\left\|\frac{\partial^2}{\partial x_j}u\right\|^2 + N^2\|u\|^2 .$$

Similarly,

$$\|x_j^3 u\|^2 \le \frac{1}{N^2}\|x_j^6 u\|^2 + N^2\|u\|^2 .$$

Hence

$$(5.15) \qquad \|u\|^2_{H^1} \le C\left(\frac{1}{N^2}\left(\sum \left\|\frac{\partial^2}{\partial x_j^2}\right\|^2 + \|x_j^6 u\|^2\right) + N^2\|u\|\right).$$

Now if (5.15) is combined with (5.13) for N sufficiently large we obtain

$$(5.16) \qquad \|\hat{A} u\|^2 \le C^1(\|\hat{L} u\|^2 + \|u\|^2) .$$

Finally, since $\hat{L}$ is injective on the space of Schwartz functions, the inequality (5.10) will follow from the following abstract lemma of Peetre.

(5.17) Theorem [12, Lemma 5.1]. Let E, F, G be reflexive Banach spac with $E \subset F$ compact injection and C a continuous linear operator from E to G. Then the following are equivalent:

(i) The image of C in G is closed and the kernel of C is finite dimensional.

(ii) There exists a constant C such that

$$\|u\|_E \le C\{\|Cu\|_G + \|u\|_F\} \qquad \forall u \in E.$$

Furthermore, if $\ker C = (0)$, (i) and (ii) are equivalent to

(iii) There exists a constant C' such that

$$\|u\|_E \le C'\|Cu\|_G .$$

Now the proof of (5.13) is complete from which (5.10) and Theorem 5.2 follow.

Remark. An argument similar to that above proves maximal hypoellipticity for a large class of Grušin-type operators. Note that by the criterion of Rothschild-Stein [16, Theorem 2], L is not the restriction to $H\backslash G$ of any homogeneous hypoelliptic left invariant differential operator on G.

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