NONEXISTENCE OF OPTIMAL $L^2$ ESTIMATES FOR THE BOUNDARY LAPIACIAN OPERATOR ON CERTAIN WEAKLY PSEUDOCONVEX DOMAINS

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1. Introduction. In this paper it is shown that the boundary Laplacian operator on some "reasonable" weakly pseudoconvex domains does not satisfy optimal $L^2$ estimates (see (1.2) below) in contrast to the strictly pseudoconvex case (Folland-Stein [3]). However, a closely related example in a class of operators defined by Grušin (Section 5) does satisfy these optimal estimates. The techniques used here are those developed by Helffer-Nourrigat [8].

A differential operator $L$ is called hypoelliptic if $Lu = f$, with $f$ smooth in an open set $U$ implies $u$ also smooth in $U$. Now suppose that $(X_1, X_2, \ldots, X_n)$ is a set of smooth, real vector fields on a manifold $M$. We are interested in conditions which guarantee that an operator of the form

$$L = \sum_{k \leq d} a_{i_1i_2\ldots i_k}(x) X_{i_1}X_{i_2}\ldots X_{i_k}$$

with $a_{i_1i_2\ldots i_k}(x)$ smooth, complex-valued functions, is hypoelliptic. Hörmander [9], introduced the following condition on the $X_i$:

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\( \{X_1, X_2, \ldots, X_n\} \) together with their commutators of length \( \leq r, \ r \) a fixed integer, span the tangent space at \( x_0 \in M. \) If \( \{X_1, \ldots, X_n\} \) satisfies (H), then, following Helffer-Nourrigat [7] we say that \( L \) is \textit{maximally hypoelliptic} if for every \( s \leq d, \) there exists \( C > 0 \) and a neighborhood \( \omega \) of \( x_0 \) such that
\[
\|X_{i_1}X_{i_2} \ldots X_{i_s}u\|^2 \leq C(\|Lu\|^2 + \|u\|^2).
\]
For all \( u \in C^\infty_0(\omega), \) where \( \| \| \) denotes \( L^2 \) norm.

We exhibit here (Sections 4 and 5) two examples of operators of the form (1.2), one arising from several complex variables and the other in a class considered by Grušin [5]. Although the operators are formally similar, it will be shown that one is maximally hypoelliptic and the other is not.

2. \textbf{Homogeneous left invariant differential operators on nilpotent groups.}\n
We review briefly some results of [6], [7], and [8], and establish notation. Let \( g \) be a graded nilpotent Lie algebra, i.e.
\[
g = g^1 + g^2 + \ldots + g^r
\]
a linear direct sum with \([g^i, g^j] \subset g^{i+j}, \) where \( g^s = (0), \ s > r.\) Then \( g \) has a family \( \delta_s, s > 0, \) of dilations which are automorphisms,
\[
\delta_s Y_j = s^i Y_j, \ Y_j \in g^j.
\]
A left invariant differential operator \( L \) on \( G, \) the simply connected nilpotent group corresponding to \( g, \) is \textbf{homogeneous} of degree \( d \) if it can be written
\[
L = \sum_{i_1 + i_2 + \ldots + i_k = d} a_{i_1 i_2 \ldots i_k} Y_{i_1} Y_{i_2} \ldots Y_{i_k},
\]
\( Y_{i_j} \in g^j. \) Generalizing slightly the definition given in Section 1, we call \( L \) \textbf{maximally hypoelliptic} if for every homogeneous left
invariant differential operator \( A \) of degree \( d \) on \( G \) there is a constant \( C > 0 \) such that

\[
\| Au \| \leq C \| Lu \|
\]

for all \( u \in C_0^\infty(G) \). A representation-theoretic criterion for hypoellipticity of \( L \) was conjectured by Rockland [13] and proved in general by Helffer-Nourrigat [6]. In this context it had been known (Folland [1]) that hypoellipticity is equivalent to maximal hypoellipticity.

(2.3) **Theorem** (Helffer-Nourrigat [6]). Let \( L \) be a homogeneous left invariant differential operator on \( G \). Then \( L \) is hypoelliptic if and only if for every irreducible unitary representation \( \pi \) of \( G \) on a Hilbert space \( \mathcal{H} \), \( \pi(L) \) is injective in the space of \( C^\infty \) vectors of \( \mathcal{H} \).

Suppose now that \( X_1, X_2, \ldots, X_n \) are arbitrary \( C^\infty \) vector fields satisfying (H). By the methods of Rothschild-Stein [16] one may lift the \( X_j \) to vector fields on a higher dimensional space so that an operator \( L \) of the form \((1, 1)\) is lifted to an operator \( \tilde{L} \). \( \tilde{L} \) has the property that it may be approximated by a homogeneous left invariant differential operator \( D \) on a nilpotent group in such a way that if \( D \) is maximally hypoelliptic, so are \( L \) and \( \tilde{L} \). (See [7] for complete definitions and details.) However, it may happen that \( L \) is maximally hypoelliptic although \( \tilde{L} \) is not. Under more restrictive conditions on the vector fields it is possible to associate a group at each point and to obtain necessary and sufficient conditions for maximal hypoellipticity by using the group criterion (see Rothschild [15] for details.) The condition on the \( X_1, X_2, \ldots, X_n \), first introduced by Métivier [14], is that the dimension of the space spanned by the commutators of length
\( \leq j \) at \( x \) should be constant for all \( x \) in a neighborhood of \( x_0 \) and all \( j \leq r \). Unfortunately there are many interesting examples of vector fields satisfying \((H)\), such as \( \{ \frac{\partial}{\partial x}, x^{\frac{\partial}{\partial y}} \} \), which do not satisfy the above condition.

3. **Quasi homogeneous vector fields.** Let \( \mathcal{P}(\mathbb{R}^N) \) be the Lie algebra of all global vector fields on \( \mathbb{R}^N \) with polynomial coefficients and suppose \( \delta_s, s > 0 \) is a family of dilations on \( \mathbb{R}^N \). The dilations \( \delta_s \) induce an automorphism of \( \mathcal{P} \) by

\[ \delta_s X(\varphi) = [X(\varphi \circ \delta_s)] \circ \delta_{s^{-1}}, \]

\( X \in \mathcal{P}, \varphi \in C^\infty(\mathbb{R}^N) \). \( X \) is \textit{(quasi-)} homogeneous of degree \( \lambda \) if

\[ \delta_s X = s^{\lambda} X. \]

Now let \( \{X_1, X_2, \ldots, X_n\} \in \mathcal{P} \) satisfy

- (3.1 i) \( X_j \) is homogeneous of degree \( 1 \), \( 1 \leq j \leq n \), and
- (3.1 ii) The Lie algebra \( \mathfrak{g} \) generated by the \( \{X_i\} \) is finite dimensional.

If (3.1) is satisfied, then

\[ \mathfrak{g} = \sum_{i=1}^{r} \mathfrak{g}^i \]

where

\[ \mathfrak{g}^i = \{ X \in \mathfrak{g} : X \text{ is (quasi)-homogeneous of degree } i \}, \]

and \( \delta_s \mathfrak{g}^i = s^i \mathfrak{g}^i \) is an automorphism.

The condition (3.1) was first defined by Folland [2] and later studied more extensively by Helffer-Nourrigat [8]. Folland proved that sets of vector fields satisfying (3.1) may be characterized as follows. Let \( \mathfrak{h} \) be the (graded) subalgebra of all vector fields in \( \mathfrak{g} \) vanishing at \( 0 \). Let \( G, H \) be the corresponding simply connected Lie groups for \( \mathfrak{g} \) and \( \mathfrak{h} \) respectively. Then
\( \{X_1, X_2, \ldots, X_n\} \) may be identified with the restrictions to the homogeneous space \( H \backslash G \) of a basis \( \{Y_1, Y_2, \ldots, Y_n\} \) of \( \mathfrak{g} \subseteq \mathfrak{g} \).

Now suppose \( L \) is of the form

\[
L = \sum_{i_1, i_2, \ldots, i_d} a_{i_1, i_2, \ldots, i_d} X_{i_1} X_{i_2} \ldots X_{i_d}
\]

with \( a_{i_1, i_2, \ldots, i_d} \in \mathbb{C} \) and \( X_{i_j} \in \{X_1, X_2, \ldots, X_n\} \) satisfying (3.1). Helffer and Nourrigat [8] have given a necessary condition that \( L \) be maximally hypoelliptic, which they have shown sufficient for \( r \leq 3 \). To state this condition we need some facts about representations of nilpotent Lie groups. Given \( \xi \in \mathfrak{g}^* \), the linear dual of \( \mathfrak{g} \), the Kirillov correspondence [10] determines an irreducible unitary representation \( \pi_\xi \). \( \pi_\xi \) is equivalent to \( \pi_{\xi'} \) if and only if \( \xi \) and \( \xi' \) are in the same orbit under the action of \( \text{Ad}^* G \), the co-adjoint representation of \( G \) on \( \mathfrak{g}^* \). Given a sub-algebra \( \mathfrak{h} \) of \( \mathfrak{g} \), let \( \text{sp}(\pi_{(0, H)}) \) be the set of all irreducible unitary representations of \( G \) corresponding to the closure (in the Euclidean sense) of \( \{\xi \in \mathfrak{g}^* : \xi = \text{Ad}^* g \cdot \xi', \xi' \text{ vanishes on } \mathfrak{h}, \text{ for some } g \in G\} \).

(3.2) **Theorem (Helffer-Nourrigat [8], [17]).** Let \( L \) be as above and let \( \tilde{L} = \sum_{i_1, i_2, \ldots, i_d} a_{i_1, i_2, \ldots, i_d} Y_{i_1} Y_{i_2} \ldots Y_{i_d} \) be the corresponding left invariant differential operator on \( G \). Then if \( L \) is maximally hypoelliptic \( \pi(\tilde{L}) \) has trivial kernel in the space of \( C^\infty \) vectors (= rapidly decreasing functions) for every irreducible unitary representation \( \pi \in \text{sp}(\pi_{(0, H)}) \). The converse is known to hold for \( r \leq 3 \).

It is conjectured by Helffer-Nourrigat [17] that the condition of the above theorem is also sufficient for any \( r \).

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4. The boundary Laplacian operator for the domain \( \{ z \in \mathbb{C}^{k+1} : \) 
\[ \text{Im} \, z_{k+1} > |z_1|^4 + |z_2|^4 + \ldots + |z_k|^4 \} \).

The boundary of the strictly pseudoconvex domain \( \{ z \in \mathbb{C}^{k+1} : \) \[ \text{Im} \, z_{k+1} > |z_1|^2 + |z_2|^2 + \ldots + |z_k|^2 \} \) may be identified with the Heisenberg group \( H^k \) (see Folland-Stein [3]). As in [3] we introduce the coordinates \((z, t), \, z \in \mathbb{C}^k, \, t \in \mathbb{R}, \, (z, t) \leftrightarrow (z_1, \ldots, z_{k+1}), \) where \( t = \text{Re} \, z_{k+1} \). The space of holomorphic vector fields is then spanned by \( Z_j = U_j + \sqrt{-1} V_j, \) where
\[ U_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad V_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}. \]

The boundary Laplacian, \( \square_b \) with respect to the metric which makes \( \{ U_1, V_j, T, \, j = 1, 2, \ldots, k \} \) an orthonormal basis is then a vector-valued, diagonalizable operator, each of whose components is of the form
\[ (4.1) \quad \sum_{i=1}^{k} U_i^2 + \sum_{j=1}^{k} V_j^2 + \sum_{j=1}^{k} \epsilon_j [U_j, V_j] \]
where each \( \epsilon_j \) is either +1 or -1. In particular, the \( \epsilon_j \) all have the same sign if and only if \( q = 0 \) or \( q = k \). If \( 0 < q < k \), each operator (4.1) is hypoelliptic and in fact maximally hypoelliptic with \( U_j, V_j, \, j = 1, 2, \ldots, k \) as the chosen directions. By making use of this model case, Folland and Stein were able to prove local optimal estimates in various function spaces for \( \square_b \) on arbitrary strictly pseudoconvex domains. From these optimal estimates for the \( \square \) operator have been obtained by Greiner-Stein [4].

One may ask whether there is an analogous theory for some weakly pseudoconvex domains. In particular, consider the domain
\[ \mathcal{D}' = \{ z \in \mathbb{C}^{k+1} : \text{Im} \, z_{k+1} = |z_1|^4 + |z_2|^4 + \ldots + |z_k|^4 \}. \]
with respect to the coordinates \((z, t)\) the holomorphic vector fields have a basis \(U'_j, V'_j\), where
\[
U'_j = \frac{\partial}{\partial x_j} + 2y_j(x_j^2 + y_j^2) \frac{\partial}{\partial t}, \quad V'_j = \frac{\partial}{\partial y_j} - 2x_j(x_j^2 + y_j^2) \frac{\partial}{\partial t}.
\]

Except on the line \(x_j = y_j = 0\), \(j = 1, 2, \ldots, n\), the vector fields \(U'_j, V'_j\) and their commutators \([U'_j, V'_j]\) span the tangent space. On the exceptional line, the commutators of length four are needed to span the tangent space.

The associated Lie algebra is of step four with \(2k\) generators, \(A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k\). We may divide out by an ideal so that a basis for the nonzero commutators is given by

- **Length 1:** \([A_j, B_j]\), \(j = 1, 2, \ldots, k\)
- **Length 2:** \([A_j, [A_j, B_j]]\), \([B_j, [A_j, B_j]]\), \(j = 1, 2, \ldots, k\)
- **Length 3:** \([A_j, [A_j, B_j]]\) = \([B_j, [B_j, [A_j, B_j]]]\), \(j = 1, 2, \ldots, k\).

We denote the above basis by \(\mathcal{B}\). On the line \(x_j = y_j = 0\) \(j = 1, 2, \ldots, k\) the associated homogeneous space is \(H \setminus G\), where the Lie algebra \(\mathfrak{h}\) of \(H\) is spanned by \(A_j, B_j, \text{ad}(A_j)^2 B_j\) and \((\text{ad} A_j)(\text{ad} B_j) A_j\). By the same computation as in the strictly pseudo-convex case (see [3]), the scalar components of the boundary Laplacian \(\Box_b\) may be identified with the restrictions of
\[
\sum_{j=1}^k A_j^2 + B_j^2 + \sum_{j=1}^k \epsilon_j [A_j, B_j]
\]
to \(H \setminus G\) with each \(\epsilon = \pm 1\). Again, if \(\Box_b\) is acting on \((p, q)\) forms the \(\epsilon_j\) are all \(+1\) or \(-1\) if and only if \(q = 0\) or \(q = k\). For \(0 < q < k\), the scalar components are
\[
\sum_{j=1}^k U'_j^2 + V'_j^2 + \sum_{j=1}^k \epsilon_j [U'_j, V'_j]
\]
where not all the $\varepsilon_j$ have the same sign. One may ask whether any of these components is maximally hypoelliptic.

(4.2) **Proposition.** \( \sum_{j=1}^{n} U_j^2 + V_j^2 \) is maximally hypoelliptic.

**Proof.** This is just a special case of [16, Theorem 15].

Thus the question of maximal hypoellipticity depends on the lower order terms. By applying the necessary condition of Helffer-Nourrigat (Theorem 3.2 above), we shall see that maximal hypoellipticity does not hold in this case. This does not imply that the related optimal estimates for $\mathfrak{D}$ do not hold (see Greiner-Stein [4]) but only that they cannot be obtained from boundary estimates.

(4.3) **Theorem.** $\Box_b$ on $\mathfrak{D}'$ is not maximally hypoelliptic on $(0, q)$ forms for any $q$. In particular, no scalar component of $\Box_b$ is maximally hypoelliptic.

**Proof.** We shall show that there is a linear function $\xi_0 \in \text{Sp}(0, H)$ such that $\pi \left( \sum A_j^2 + B_j^2 + \sqrt{-1} \sum \varepsilon_j [A_j, B_j] \right)$ has a non-trivial kernel. Indeed, if $\xi_0 \left( [A_j, B_j] \right) = \varepsilon_j$ and $\xi_0$ vanishes on the other elements of $\mathfrak{a}$, then

\[ \pi \left( \sum A_j^2 + B_j^2 + \sqrt{-1} \sum \varepsilon_j [A_j, B_j] \right) = \left( \frac{\partial^2}{\partial u_1^2} - u_1^2 \right) + 1, \]

which annihilates the Hermite function $e^{-u_1^2/2}$. Thus it suffices to show that $\pi \xi_0 \in \text{sp}(\pi(0, H))$. Let $\xi_n$ be defined by

$\xi_n[A_j, [A_j, [A_j, B_j]]] = \frac{\varepsilon_j}{n}$,

$\xi_n$ vanishes on the elements of length $1, 2, 3$ of the basis $\mathfrak{a}$. Now put, for $r_n \in \mathbb{R}$,

$\xi_n^i = \text{Ad}^* \text{Exp} r_n B_1 \cdot \xi_n$. 
Then
\[
\xi_n'(A_1) = \xi_n(\text{Exp}(-\tau_n \text{ad } B_1) \cdot A_1) = \\
\xi_n(A_1) - r_n [B_1, A_1] + \frac{r_n^2}{2} [B_1, [B_1, A_1]] \\
+ \frac{r_n^3}{3!} [B_1, [B_1, [B_1, A_1]]] \\
= \frac{r_n^3}{3!} \frac{1}{n}
\]
\[
\xi_n'(B_1) = \xi_n(B_1) = 0
\]
\[
\xi_n'([A_1, B_1]) = \xi_n([A_1, B_1]) - r_n [B_1, [A_1, B_1]] + \frac{r_n^2}{2} [B_1, [B_1, [A_1, B_1]]] \\
= \frac{r_n^2}{2n} \varepsilon_1
\]
\[
\xi_n'[A_1, [A_1, B_1]] = 0
\]
\[
\xi_n'([B_1, [A_1, B_1]]) = \frac{-r_n \varepsilon_1}{n}
\]
\[
\xi_n'[A_1, [A_1, B_1], [A_1, B_1]] = \frac{\varepsilon_1}{n},
\]
\[
\xi_n'(X) = 0 \text{ for all other } X \in \mathcal{B}.
\]

Now let \( s_n \in \mathbb{R} \) and put
\[
\xi_n'' = \text{Ad}^* \text{Exp } s_n [A, [A, B]] \circ \xi_n.
\]

Then one can check that
\[
(4.5) \quad \xi_n''(A_1) = \xi_n'(A_1) - s_n \xi_n'([A_1, [A_1, [A_1, B_1]]]) \\
= \left( \frac{r_n^3}{3!} - s_n \right) \varepsilon_1,
\]
and
\[
\xi_n''(X) = \xi_n'(X) \quad X \in \mathcal{B}, \ X \neq A_1.
\]

Finally, put
\[
(4.6) \quad r_n = \sqrt{2n}, \quad s_n = \left( \frac{r_n^3}{3! n} \right).
\]

Then
\[
\xi_n''[A_1, B_1] = \varepsilon_1
\]
\[ \xi_n'' ([B_1, [A_1', B_1]]) = \frac{\sqrt{2} \varepsilon_1}{\sqrt{n}} \]
\[ \xi_n'' ([A_1', [A_1', [A_1', B_1]]]) = \varepsilon_1/n \]
and \[ \xi_n''(X) = 0, \] all other \( X \in \mathfrak{g} \). Hence \( \xi_n \to \xi_0 \) as \( n \to \infty \).

To complete the proof of Theorem 4.3 we calculate \( \pi_{\xi_0} \). A maximal isotropic subalgebra for \( \xi_0 \) is spanned by
\[ \{ B_1, A_j, B_j, [A_j, B_j], \ \text{ad} \ A_j^2 B_j, \ \text{ad} \ B_j^2 A_j, \ \text{ad} \ A_j^3 B_j, \ j \geq 2 \}. \]

Then
\[ \pi_{\xi_0} (A_1) = \frac{\varepsilon_j}{\partial u}, \ \pi_{\xi_0} (B_1) = \sqrt{-1} \ u_j \cdot \varepsilon_j I \]
and \( \pi_{\xi_0} (A_j) = \pi_{\xi_0} (B_j) = 0, \ j \geq 2, \) which proves (4.4).

(4.7) **Remark.** It is unknown whether the scalar components of \( \Box_\mathcal{B} \) on \( \partial \mathcal{S} \) are non-maximally hypoelliptic. By Kohn's recent work [11] it is known that \( \overline{\mathcal{S}} \) is subelliptic, however.

5. **A related maximally hypoelliptic operator.** The difficulty of resolving the question of whether an operator like \( \Box_\mathcal{B} \) is maximally hypoelliptic may be illustrated by a related example which may be analyzed by the method of Grusin [5]. Let
\[ U_j'' = \frac{d}{dx_j}, \ V_j'' = x_j^3/3 \ \frac{\partial}{\partial t}, \]
\[ j = 1, 2, \ldots, k, \] and put
\[ L = \sum_{j=1}^k (U_j''^2 + V_j''^2) + \sqrt{-1} \sum \varepsilon_j [U_j'', V_j''] \]
\[ = \sum \frac{\partial^2}{\partial x_j^2} + x_j \frac{\partial^2}{\partial t^2} + \sqrt{-1} \sum \varepsilon_j x_j^2 \frac{\partial}{\partial t} \]
where each \( \varepsilon_j = +1 \) or \(-1\).
(5.2) Theorem. Let \( L \) be defined by (5.1). \( L \) is maximally hypoelliptic if and only if \( L \) is hypoelliptic if and only if the \( \varepsilon_j \) do not all have the same sign.

Proof. \( L \) is homogeneous of degree 2 if \( x_i \) is given weight 1, \( i = 1, 2, \ldots, k \) and \( t \) is given weight 4. Furthermore, \( L \) is elliptic away from \( x_1 = x_2 = 0 \). Hence the criterion of Grushin [5] may be applied to conclude that \( L \) is hypoelliptic if and only if

\[
(5.3)\quad \hat{L}(\tau) \text{ is injective on } \mathcal{S}(\mathbb{R}^k) \text{ for } \tau = \pm 1,
\]

where \( \hat{L}(\tau) \) denotes the partial Fourier transform i.e.

\[
\hat{L}(\tau) = \sum_{i=1}^{k} \left( \frac{\partial^2}{\partial x_i^2} - x_i^6 \tau^2 \right) - 3 \varepsilon_i x_i^2 \tau.
\]

To check that (5.3) holds if not all the \( \varepsilon_j \) have the same sign, it suffices to show that if \( \varphi \in \mathcal{S}(\mathbb{R}^k) \)

\[
(L\hat{\varphi}, \varphi) \neq 0.
\]

Suppose first that \( \tau = 1 \). Then

\[
\hat{L}(1) = \sum_{j=1}^{k} \left( \frac{\partial^2}{\partial x_j^2} - x_j^6 \right) - 3 \varepsilon_j x_j^2 = \sum_{j=1}^{k} L_j.
\]

Hence

\[
(5.4)\quad (L\hat{\varphi}, \varphi) = \sum_{j=1}^{k} \left( \left\| \frac{\partial \varphi}{\partial x_j} \right\|^2 - \left\| x_j^3 \varphi \right\|^2 - 3 \varepsilon_j \left\| x_j \varphi \right\|^2 \right).
\]

Now put \( D_j = \frac{\partial}{\partial x_j} + x_j \). By the inequality

\[
(5.5)\quad \left\| \frac{\partial \varphi}{\partial x_j} \right\|^2 + \left\| x_j^3 \varphi \right\|^2 \geq 3 \left\| x_j \varphi \right\|^2,
\]

which is obtained by expanding \((D_j D_j^T \varphi, \varphi) \geq 0\) and \((D_j^T D_j \varphi, \varphi) \geq 0\), we see that \((L\hat{\varphi}, \varphi)\) is non-positive.

Furthermore, if at least one of the \( \varepsilon_j \) is positive, \((L_j \varphi, \varphi) < 0\), and therefore

\[
(5.6)\quad (L\hat{\varphi}, \varphi) < 0.
\]
If $\tau = -1$, a similar argument shows that $(L^\wedge \varphi, \varphi) \neq 0$ holds if at least one of the $\varepsilon_j$ is negative. Hence, by Grusin's criterion [5, Theorem 1.2], $L$ is hypoelliptic. Conversely, if all the $\varepsilon_j$ have the same sign, then $\varphi = e^{-x_1^2/4} e^{-x_2^2/4} \ldots e^{-x_k^2/4}$ is in the kernel of either $L^\wedge(1)$ or $L^\wedge(-1)$, as is easily checked.

Now suppose that the $\varepsilon_j$ have different signs; we want to check that $L$ is maximally hypoelliptic. For this it suffices to show that for all $u \in C_0^\omega(\mathbb{R}^{2k+1})$

$$(5.7) \quad \|Au\|^2 \leq C(\|Lu\|^2 + \|u\|^2)$$

for $A = \frac{\partial^j}{\partial x_k^j}$, $(x_1^2 \frac{\partial}{\partial t})^j$, $\frac{\partial}{\partial t}$, $\ell = 1, 2, \ldots, k$, and $j = 1, 2$. Suppose first that one can prove

$$(5.8) \quad \|Au\| \leq C' \|Lu\|$$

for all $A$ as above with $j = 2$. Then one easily obtains (5.7) for $j = 1$. For instance,

$$(5.9) \quad \left\| \frac{\partial u}{\partial x_k^j} \right\|^2 = \langle \frac{\partial^2}{\partial x_k^2} u, u \rangle = \left\| \frac{1}{M} \frac{\partial^2}{\partial x_k^2} u, Mu \right\| \leq \frac{1}{M} \left\| \frac{\partial^2 u}{\partial x_k^2} \right\|^2 + M\|u\|^2$$

for any constant $M$. Hence (5.8) and (5.9) yield (5.7) for $A = \frac{\partial}{\partial x_k^j}$. To prove (5.8) by homogeneity of $A$ and $L$ it suffices to show

$$(5.10) \quad \|A^\wedge(\tau)\varphi\|^2 \leq C_\tau \|L^\wedge(\tau)\varphi\|^2$$

for $\tau = 1, -1$.

It may be possible to obtain (5.10) directly from the estimates in Grusin [5], but I shall give an independent proof here based on the ideas of Helffer-Nourrigat [7] and [8]. (In [8, section 3.4] it is implied that hypoelliptic Grusin-type operators like $L$ are maximally
hypoelliptic, but a proof is not given there. ) To prove (5.10) one
first proves

\[ \| A^* u \| \leq C ( \| L^* u \| + \| u \|_{H^1} ); \]

where

\[ \| u \|_{H^1} = \sum \| \frac{\partial u}{\partial x_i} \| + \| x_1^2 u \| + \| u \|. \]

For this, note that

\[ \| L^* u \| \leq C_1 ( \| L^* u \| + \| u \|_{H^1} ), \]

where

\[ L^* = \sum \frac{\partial^2}{\partial x_i^2} - x_1^6. \]

Next we shall show that

\[ \| A^* u \| \leq C_2 \| L^* u \|. \]

For this, let \( Y_{i,1}, Y_{i,2} \) be generators of a free nilpotent algebra
\( \mathfrak{g}_1 \) of step 4. Then \( Y^* = \sum Y_{i,1}^2 + Y_{i,2}^2 \) is hypoelliptic, and
by Folland [1, theorem 6.1] satisfies the estimate

\[ \| B \phi \| \leq C_2 \| Y^* \phi \|^2 \]

if \( B = Y_{i,1}, Y_{i,2}, \) or \( Y_{i,1} Y_{i,2}. \) Hence by Helffer-Nourrigat [6,
Proposition 2.1] if \( \pi \) is any irreducible unitary representation of \( \mathbb{G} \)
on \( L^2(S), \)

\[ \| \pi(B) v \| \leq C_2 \| \pi(Y^*) v \|^2 \]

for all \( v \in C_0^\infty(S). \) Now if \( \pi \) is taken to be the irreducible
representation given by \( Y_{i,1} \mapsto \frac{\partial}{\partial x_i}, Y_{i,2} \mapsto \sqrt{-1} x_1^3 \)
(5.13) follows from (5.14). It is obvious that (5.11) follows from (5.12)
and (5.13). Now if \( N > 0 \) is arbitrary,

\[ \| \frac{\partial u}{\partial x_j} \|^2 = \left| \left( \frac{1}{N} \frac{\partial^2}{\partial x_j^2} u, Nu \right) \right| \leq \frac{1}{N^2} \| \frac{\partial^2}{\partial x_j^2} u \|^2 + N^2 \| u \|^2. \]
Similarly,
\[ \| x_j^3 u \|^2 \leq \frac{1}{N^2} \| x_j^6 u \|^2 + N^2 \| u \|^2 . \]

Hence
\[ (5.15) \quad \| u \|^2_{H^1} \leq C \left( \frac{1}{N^2} \left( \sum \| \frac{\partial^2}{\partial x_j^2} u \|^2 + \| x_j^6 u \|^2 \right) + N^2 \| u \| \right) . \]

Now if (5.15) is combined with (5.13) for \( N \) sufficiently large we obtain
\[ (5.16) \quad \| A^\wedge u \|^2 \leq C (\| L^\wedge u \|^2 + \| u \|^2) . \]

Finally, since \( L^\wedge \) is injective on the space of Schwartz functions, the inequality (5.10) will follow from the following abstract lemma of Peetre.

(5.17) Theorem [12, Lemma 5.1]. Let \( E, F, G \) be reflexive Banach spaces with \( E \subset F \) compact injection and \( C \) a continuous linear operator from \( E \) to \( G \). Then the following are equivalent:

(i) The image of \( C \) in \( G \) is closed and the kernel of \( C \) is finite dimensional.

(ii) There exists a constant \( C \) such that
\[ \| u \|_E \leq C \{ \| Cu \|_G + \| u \|_F \} \quad \forall u \in E . \]

Furthermore, if \( \ker C = \{0\} \), (i) and (ii) are equivalent to

(iii) There exists a constant \( C' \) such that
\[ \| u \|_E \leq C' \| Cu \|_G . \]

Now the proof of (5.13) is complete from which (5.10) and Theorem 5.2 follow.

Remark. An argument similar to that above proves maximal hypoellipticity for a large class of Grusin-type operators. Note that
NONEXISTENCE OF OPTIMAL L^2 ESTIMATES

by the criterion of Rothschild-Stein [16, Theorem 2], L is not the restriction to \( H \setminus G \) of any homogeneous hypoelliptic left invariant differential operator on \( G \).

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