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ON THE GEOMETRY OF ANALYTIC DISCS ATTACHED TO REAL MANIFOLDS

M. S. BAOUENDI, L. P. ROTHSCHILD
& J.-M. TREPREAU

Introduction

A generic manifold M is a real submanifold of the complex space $\mathbf{C}^N$ which is locally the generic intersection of real hypersurfaces. An analytic disc attached to M is an analytic mapping A from the unit disc in $\mathbf{C}$ into $\mathbf{C}^N$, continuous up to the boundary, mapping the unit circle S^1 into M. We shall say that A passes through a point $p_0 \in M$ if $A(1) = p_0$. Analytic discs have been extensively used by many mathematicians since the work of Lewy [14] and Bishop [6], and play a central role in questions of holomorphic extendibility and propagation of analyticity for Cauchy-Riemann (CR) functions defined on M. In this paper we study the geometry of the set of all small analytic discs attached to M through p_0 whose derivatives are Hölder continuous up to the boundary.

Using elementary Banach space techniques, including the implicit function theorem, we prove (see §2) that, for $p_0 \in M$, the set of discs defined as above forms an infinite-dimensional submanifold $\mathscr{A}_{p_0}(M)$ of the Banach space of all discs valued in $\mathbf{C}^N$. We also give a parametrization of $\mathscr{A}_{p_0}(M)$ as well as an explicit description of its tangent space at each disc in the manifold $\mathscr{A}_{p_0}(M)$. In particular, this allows us to construct families of discs near a given small disc without use of the Bishop equation [6].

For any $A \in \mathscr{A}_{p_0}(M)$, we consider (see §3) the analytic discs attached to $\Sigma(M)$, the conormal bundle of M, with base projection equal to the given disc A. Then, for $\zeta \in S^1$, we introduce the defect of A at ζ as the dimension of the subspace spanned by these discs in the fiber $\Sigma_{A(\zeta)}(M)$. In fact, this defect is independent of ζ if A is sufficiently small. For small discs the notion of defect given here coincides with that introduced by Tumanov [20], but ours is expressed in a geometric context and, in particular, is invariant and independent of the choice of coordinates.

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Our approach (see Theorem 1 in §4) yields some variations and new proofs of the main result of Tumanov [20] relating the rank of the evaluation map, $A \mapsto A(\zeta_0)$ from $\mathscr{A}_{p_0}(M)$ into M, to the defect of A. In Theorem 1(ii) we also obtain new results for the image and the rank of the derivative map $A \mapsto \partial A(e^{i\theta})/\partial\theta|_{\theta=0}$. For later applications, we extend the results of Theorem 1 to the case where M is only assumed to be a CR submanifold of $\mathbf{C}^N$, not necessarily generic. (See Theorem 1′ in §6.)

In §5, we describe the image of the mapping $A \mapsto \partial A(e^{i\theta})/\partial\theta$ for any real θ (see Theorem 2). As a corollary, we show that if the defect of a disc $A \in \mathscr{A}_{p_0}(M)$ is equal to the codimension of M in $\mathbf{C}^N$, then its derivative, $\partial A(e^{i\theta})/\partial\theta$ lies in $T^c M$, the complex bundle of M. We also show in this section (Theorem 4) that if A is of defect 0 and $\partial A(e^{i\theta})/\partial\theta|_{\theta=0}$ is in $T^c_{p_0}M$, then every CR function on M extends to a full neighborhood of p_0 in M. In §5 we also generalize the notion of minimal convexity that was introduced in [3] in the case of a hypersurface. We prove that if M is minimally convex at p_0 and A is a small disc attached to M through p_0, then $\partial A(e^{i\theta})/\partial\theta|_{\theta=0}$ does not lie in the complex bundle unless A is of maximal possible defect (Theorem 3). We conclude §5 by showing that for a homogeneous hypersurface of finite type in $\mathbf{C}^2$ all nonconstant discs are of defect 0 (Corollary (5.9)).

In §7 we give applications of the previous sections to holomorphic extension of CR functions on hypersurfaces as well as propagation of analyticity. Theorem 6 states that if $A \in A_{p_0}(M)$ and $\partial A(e^{i\theta})/\partial\theta|_{\theta=0} \in T^c_{p_0}M$, but $\partial A(e^{i\theta})/\partial\theta|_{\theta=\theta_0} \notin T^c_{A(e^{i\theta_0})}M$ for some θ_0, then any holomorphic function on one side of M near p_0 extends to the other side. In Theorem7 we prove a new propagation result, namely that the boundary of an analytic disc propagates one-sided holomorphic extendibility. This in particular implies the propagation result of Hanges and Treves [11] and its generalization (in the context of hypersurfaces) by one of the authors [19], who used microlocal analysis for its proof. It should be noted that Tumanov [21] has recently given propagation results related to those of Theorem 7. Finally, Theorem 8 gives a completely different proof of a propagation result along a minimal CR submanifold, previously obtained by one of the authors [19] by more complicated microlocal analysis arguments from [18].

As mentioned above, analytic discs and the use of the Bishop equation have appeared in many contexts. We mention here only a few: Hill and Taiani [12], Boggess and Polking [8] (see also Boggess [7]), Bedford and

Gaveau [5], and Trepreau [17]. It should also be noted that the use of the conormal bundle as a submanifold of a complex manifold appeared first in Sato-Kawai-Kashiwara [15] and, in the context of generic manifolds, in Webster [22], Lempert [13], Sjöstrand [16], [18], and others.

1. Generic manifolds and their complex and characteristic bundles

Let M be a smooth generic manifold in $\mathbf{C}^N$ of real dimension $2N-l$, given locally near p_0 by $\rho(Z)=0$, $\rho=(\rho_1,\cdots,\rho_l)$, where the ρ_j are smooth, real-valued functions vanishing at p_0 and $\partial\rho_1\wedge\cdots\wedge\partial\rho_l\neq 0$. We have used the notation $\partial f=\sum_{j=1}^{n+l}(\partial f/\partial Z_j)\,dZ_j$. Here ρ will be considered as an $l\times 1$ matrix. Note that if $\tilde\rho$ is another matrix of defining functions for M, then we have $\rho=a\tilde\rho$, where a is an invertible $l\times l$ matrix of real-valued smooth functions. For $Z\in\mathbf{C}^N$ we shall write $Z=(Z_1,\cdots,Z_N)$. We denote by $T\mathbf{C}^N$ the real tangent bundle of $\mathbf{C}^N$, and by TM the real tangent bundle of M. Also, we denote by T^cM the complex bundle of M, i.e., the real subbundle of TM invariant under J, the complex involution of $\mathbf{C}^N$. Then for $p\in M$ near p_0, we have

$$
\begin{aligned}
T_pM &= \{X\in T_p\mathbf{C}^N : \langle d\rho_j(p),X\rangle = 2\,\mathrm{Re}\langle\partial\rho_j(p),X\rangle = 0, \\
&\qquad\qquad j=1,\cdots,l\}, \qquad (1.1)\\
T_p^cM &= \{X\in T_p\mathbf{C}^N : \langle\partial\rho_j(p),X\rangle = 0,\ j=1,\cdots,l\}.
\end{aligned}
$$

Let $CT^*\mathbf{C}^N$ be the complexified cotangent bundle of $\mathbf{C}^N$. For $Z\in\mathbf{C}^N$, a covector α in $CT^*_Z\mathbf{C}^N$ can be written in the form

$$\alpha=\sum_{j=1}^{N}\lambda_j\,dZ_j+\mu_j\,d\overline{Z}_j,\qquad \lambda_j,\mu_j\in\mathbf{C}. \tag{1.2}$$

We denote by $\Lambda^{1,0}\mathbf{C}^N$ the subbundle of $CT^*\mathbf{C}^N$ consisting of covectors of the form (1.2) with $\mu_j=0$, $j=1,\cdots,N$. The bundle $\Lambda^{1,0}\mathbf{C}^N$ is then a complex manifold which may be identified with $\mathbf{C}^{2N}$ by the coordinates $(Z_1,\cdots,Z_N,\lambda_1,\cdots,\lambda_N)$.

We identify $\Lambda^{1,0}\mathbf{C}^N$ with $T^*\mathbf{C}^N$, the real cotangent bundle of $\mathbf{C}^N$, as follows. For $\theta=\sum_j c_j\,dZ_j+\overline{c}_j\,d\overline{Z}_j\in T^*_Z\mathbf{C}^N$, we associate the covector $\alpha=2i\sum_j c_j\,dZ_j\in\Lambda^{1,0}_Z\mathbf{C}^N$, so that $\langle\theta,X\rangle=\mathrm{Im}\langle\alpha,X\rangle$ for all $X\in T_Z\mathbf{C}^N$. Under this identification, the conormal bundle $\Sigma(M)$ of a submanifold M in $\mathbf{C}^n$ is then given by

$$\Sigma_p(M)=\{\alpha\in\Lambda^{1,0}_p\mathbf{C}^N : \mathrm{Im}\langle\alpha,X\rangle=0,\ X\in T_pM\}. \tag{1.3}$$

For $p \in M$, the restriction map (i.e., pullback) $r\colon \Sigma_p(M) \to T_p^*M$ given by

$$\langle r(\alpha), X\rangle = \langle \alpha, X\rangle, \qquad X \in T_pM,$$

is in general not injective. However, if M is generic, as above, then the map r defined above is injective and

$$r(\Sigma_p(M)) = (T_p^c M)^\perp \subset T_p^* M, \tag{1.4}$$

where orthogonality is taken in the sense of duality between T^*M and TM. Hence the bundle $\Sigma(M)$ can be identified with the characteristic bundle $(T^cM)^\perp$ of the CR structure on M.

If M is locally defined by $\rho = 0$ as above and p is a point in M near p_0, then $i\partial\rho_1(p), \cdots, i\partial\rho_l(p)$ form a basis of the real vector space $\Sigma_p(M)$. If, in addition M is generic, then the pullbacks to M of $i\partial\rho_1(p), \cdots, i\partial\rho_l(p)$ form a basis of the characteristic bundle of M. With the identification above we have

$$\Sigma_p(M) = (T_p^c M)^\perp = \{i\,{}^\tau t\partial\rho(p) = i\sum_j t_j\partial\rho_j,\ t \in \mathbf{R}^l\}, \tag{1.5}$$

where ${}^\tau B$ denotes the transpose of a matrix B. To summarize, we assume M is generic, so that the characteristic bundle of M can be identified with the submanifold $\Sigma(M)$ of real dimension $2N$ in the complex manifold $\Lambda^{1,0}\mathbf{C}^N$, which is of complex dimension N.

Finally, if we choose an $N \times l$ matrix C with complex coefficients such that

$$\rho_Z(p_0)C = I_{l\times l}, \tag{1.6}$$

then we can write

$$T_p^*M = \{{}^\tau\alpha\, dZ + {}^\tau\overline{\alpha}\, d\overline{Z} + i\,{}^\tau t\partial\rho(p),\ \alpha \in \mathbf{C}^N,\ {}^\tau C\alpha = 0,\ t \in \mathbf{R}^l\}, \tag{1.7}$$

where the covectors in (1.7) are identified with their pullbacks to covectors on M. Similarly, we have

$$(T_p^c M)^\perp = \{i\,{}^\tau t\partial\rho(p),\ t \in \mathbf{R}^l\}. \tag{1.8}$$

2. Analytic discs attached to generic manifolds

An *analytic disc* in $\mathbf{C}^N$ is a continuous mapping $A\colon \overline{\Delta} \to \mathbf{C}^N$ which is holomorphic in Δ, where Δ is the open unit disc in the plane and $\overline{\Delta} = \Delta \cup S^1$, where S^1 is the unit circle. We say that A is *attached to*

M through p_0 if $A(S^1) \subset M$ and $A(1) = p_0$. We shall always assume all analytic discs to be of Hölder class at least $C^{1,\alpha}(\overline{\Delta})$ for some fixed $\alpha \in (0, 1)$. We denote by $\mathscr{D}^N$ the space of all analytic discs valued in $\mathbf{C}^N$, i.e.,

$$\mathscr{D}^N = \{A: \overline{\Delta} \to \mathbf{C}^N : A \in C^{1,\alpha}(\overline{\Delta}) \cap \mathscr{H}(\Delta)\}. \tag{2.1}$$

We regard $\mathscr{D}^N$ as a real Banach space equipped with the $C^{1,\alpha}(\overline{\Delta})$ norm. For $p_0 \in \mathbf{C}^N$ we put

$$\mathscr{D}^N_{p_0} = \{A \in \mathscr{D}^N : A(1) = p_0\}. \tag{2.2}$$

We write $\langle p_0 \rangle$ for the constant disc $A(\zeta) \equiv p_0$ for all $\zeta \in \overline{\Delta}$. Note that with the above notation we have $\mathscr{D}^N_{p_0} = \langle p_0 \rangle + \mathscr{D}^N_0$. We observe that $\mathscr{D}^N_0$ is a Banach subspace of $\mathscr{D}^N$, while $\mathscr{D}^N_{p_0}$ is only and affine space. For $\varepsilon > 0$ we put

$$\mathscr{D}^N_{p_0,\varepsilon} = \{A \in \mathscr{D}^N_{p_0} : \|A - \langle p_0 \rangle\|_{1,\alpha} < \varepsilon\}. \tag{2.3}$$

Denote by $C^{1,\alpha}_*(S^1)$ the space of real-valued functions in $C^{1,\alpha}(S^1)$ vanishing at 1. If ρ is a defining function near p_0 of a generic manifold M as above, and ε sufficiently small, we introduce the map $R: \mathscr{D}^N_{p_0,\varepsilon} \to [C^{1,\alpha}_*(S^1)]^l$ defined by

$$\mathscr{D}^N_{p_0,\varepsilon} \ni A \mapsto R(A)(\cdot) = \rho(A(\cdot)) \in [C^{1,\alpha}_*(S^1)]^l. \tag{2.4}$$

With the above notation, the subset $\mathscr{A}$ of $\mathscr{D}^N_{p_0,\varepsilon}$ consisting of those analytic discs in $\mathscr{D}^N_{p_0,\varepsilon}$ which are attached to M is given by

$$\mathscr{A} = \mathscr{A}_{p_0,\varepsilon}(M) = \{A \in \mathscr{D}^N_{p_0,\varepsilon} : R(A) = 0\}. \tag{2.5}$$

The following abstract arguments will show that if ε is sufficiently small, the set $\mathscr{A}$ defined by (2.5) is actually a smooth (infinite dimensional) closed submanifold of $\mathscr{D}^N_{p_0,\varepsilon}$ and will give a description of its tangent space.

Let E and F be two real Banach spaces, $\tilde{E}$ an affine space with underlying Banach space E and $e_0 \in \tilde{E}$. For $\varepsilon > 0$, we let

$$B_\varepsilon = \{x \in \tilde{E} : \|x - e_0\|_E < \varepsilon\}. \tag{2.6}$$

Let $R: B_\varepsilon \to F$ be a smooth map with $R(e_0) = 0$. Assume that the linear map $R'(e_0): E \to F$ has a right continuous inverse $S: F \to E$, i.e.,

$R'(e_0)S = I_F$. Then we may write $E = E_0 \oplus E_1$, where $E_0 = \ker R'(e_0)$ and $E_1 = S(F)$. Note also that $R'(e_0)$ is an isomorphism from E_1 to F with inverse S.

Let $\mathcal{A} = \{x \in B_\varepsilon : R(x) = 0\}$. By the implicit function theorem, there exist $\eta > 0$ and a smooth map Φ: $\{x_0 \in E_0 : \|x_0\| < \eta\} \to E_1$, with $\Phi(0) = 0$, such that if ε is sufficiently small,

$$\mathcal{A} = \{x \in B_\varepsilon : x - e_0 = x_0 + \Phi(x_0),\ x_0 \in E_0\}. \tag{2.7}$$

This shows that $\mathcal{A}$ is a manifold. For $x \in \mathcal{A}$, we shall now describe $T_x\mathcal{A}$, the tangent space of $\mathcal{A}$ at x considered as a subspace of E. Note that for x close to e_0 in B_ε, $R'(x)S$ is an isomorphism of F into itself, since by assumption $R'(e_0)S$ is the identity map on F.

Lemma 2.8. *For ε sufficiently small and $x \in \mathcal{A}$ the tangent space $T_x\mathcal{A}$ is given by*

$$T_x\mathcal{A} = \{\hat{x} \in E : \hat{x} = \hat{x}_0 - S(R'(x)S)^{-1}[R'(x)\hat{x}_0],\ \hat{x}_0 \in E_0\}. \tag{2.9}$$

Proof. Let $\hat{x} \in E$. We may write uniquely $\hat{x} = \hat{x}_0 + S\hat{y}$ with $\hat{x}_0 \in E_0$ and $\hat{y} \in F$. Then $\hat{x} \in T_x\mathcal{A}$ if and only if $R'(x)\hat{x}_0 + R'(x)S\hat{y} = 0$. Using the invertibility of $R'(x)S$ and solving in $\hat{y}$ yield (2.9). q.e.d.

We shall apply the previous arguments, and in particular Lemma 2.8, to describe $T_A\mathcal{A}$ for $A \in \mathcal{A}$ given by (2.5). Here we take $E = \mathcal{D}_0^N$, $\widetilde{E} = \mathcal{D}_{p_0}^N$, $F = [C_*^{1,\alpha}(S^1)]^l$, and $e_0 = \langle p_0 \rangle$. Hence we have $B_\varepsilon = \mathcal{D}_{p_0,\varepsilon}^N$ given by (2.3). Recall that R is given by (2.4). To show that $R'(\langle p_0 \rangle): E \to F$ has a right continuous inverse S, it suffices to take S defined as follows. Since M is generic, the $l \times N$ complex matrix $\rho_Z(p_0)$ is of rank l. Choose an $N \times l$ matrix C with complex coefficients (as in (1.6)) such that

$$\rho_Z(p_0)C = I_{l \times l}. \tag{2.10}$$

For $f \in F$, we define $S(f)$ by

$$S(f)|_{S^1} = \tfrac{1}{2}C(f + iT_1 f), \tag{2.11}$$

where $T_1 f$ is the Hilbert transform of f vanishing at $\zeta = 1$. Note that T_1 maps F into itself.

For $\widehat{A} \in \mathcal{D}_0^N$ and $A \in \mathcal{D}_{p_0,\varepsilon}^N$ we have the following:

$$[R'(A)\widehat{A}](\zeta) = \rho_Z(A(\zeta))\widehat{A}(\zeta) + \rho_{\overline{Z}}(A(\zeta))\overline{\widehat{A}(\zeta)}. \tag{2.12}$$

It is easily checked from (2.11), (2.12) and the definition of E_0 and E_1, that we have $R'(\langle p_0\rangle)S = I_F$, and

$$E_0 = \{\widehat{A} \in \mathscr{D}_0^N : \rho_Z(p_0)\widehat{A} = 0\}, \qquad (2.13)$$
$$E_1 = \{\widehat{A} \in \mathscr{D}_0^N : \widehat{A} = C\widehat{D},\ \widehat{D} \in \mathscr{D}_0^l\},$$

where C is as in (2.10) and S is defined by (2.11).

In order to use (2.9) to describe $T_A\mathscr{A}$ for $A \in \mathscr{A}$, we need to compute the inverse of the map $R'(A)S$ of F into itself. We shall make use of the following lemma.

Lemma 2.14. *There is $\eta > 0$ such that if $m(\zeta)$ is an $l \times l$ matrix with complex-valued coefficients in $C^{1,\alpha}(S^1)$, satisfying $m(1)$ invertible and*

$$\|m(e^{i\theta}) - m(1)\|_{1,\alpha} \le \eta(\|m(1)^{-1}\|)^{-1}, \qquad (2.15)$$

then there exists a unique invertible $l \times l$ matrix $\nu(\zeta)$, whose coefficients are real-valued functions in $C^{1,\alpha}(S^1)$, with $\nu(1) = I_{l\times l}$, and such that the function $S^1 \ni \zeta \mapsto \nu(\zeta)m(\zeta)$ extends holomorphically to Δ as an invertible matrix in $\overline{\Delta}$. Moreover, if $f(\zeta)$ is an $l \times 1$ vector of real-valued functions in $C^{1,\alpha}(S^1)$ such that $S^1 \ni \zeta \mapsto {}^\tau m(\zeta)f(\zeta)$ extends holomorphically, then

$$f(\zeta) = {}^\tau\nu(\zeta)f(1). \qquad (2.16)$$

Proof. Let X be the Banach space of all $l \times l$ matrices whose coefficients are complex-valued functions in $C^{1,\alpha}(S^1)$. Denote by Y the closed subspace of X consisting of those matrices whose coefficients extend holomorphically to Δ. Also, denote by Z the subspace of X consisting of those matrices whose coefficients are real valued. Then the mapping $B\colon X \times Y \to Z$ defined by

$$B(n, h) = \operatorname{Im} hn \qquad (2.17)$$

is $\mathbf{R}$-bilinear. Note that if $I = I_{l,l}$ (the identity matrix), then $B(I, I) = 0$. By applying the implicit function theorem, it is easy to see that for any matrix $n \in X$ sufficiently close to I, there exists $h \in Y$, also close to I, such that $B(n, h) = 0$. In addition, h is unique up to left multiplication by an invertible real constant matrix. The first part of the lemma follows from the above statements by taking $n(\zeta) = m(1)m^{-1}(\zeta)$.

We now prove (2.16). Since ${}^\tau f(\zeta)m(\zeta)$ extends holomorphically, so does ${}^\tau f(\zeta)m(\zeta)(n\nu(z)m(\zeta))^{-1} = {}^\tau f(\zeta)\nu^{-1}(\zeta)$. Since the latter is real-valued, it must be constant, and hence (2.16) follows, since $\nu(1) = I$. This completes the proof of Lemma 2.14. q.e.d.

For $A \in \mathscr{D}^N_{p_0,\varepsilon}$, with ε sufficiently small, we shall apply Lemma 2.14 for the matrix

$$m(\zeta) = \rho_Z(A(\zeta))C, \tag{2.18}$$

and let $\nu(\zeta)$ be the matrix given by the lemma.

Lemma 2.19. *For ε sufficiently small and $A \in \mathscr{A}$, an analytic disc $\widehat{A} \in \mathscr{D}^N_0$ is in $T_A\mathscr{A}$ if and only if*

$$\widehat{A} = \widehat{A}_0 + C\widehat{D}, \tag{2.20}$$

where $\widehat{A}_0 \in E_0$ given by (2.13), *C is the matrix given by* (2.10) *and $\widehat{D} \in \mathscr{D}^l_0$ is given by*

(2.21)
$$-2\widehat{D}|_{S^1} = (\nu m)^{-1}[\nu\rho_Z(A)\widehat{A}_0 + \nu\rho_{\overline{Z}}(A)\overline{\widehat{A}}_0 + iT_1(\nu\rho_Z(A)\widehat{A}_0 + \nu\rho_{\overline{Z}}(A)\overline{\widehat{A}}_0)],$$

where $m(\zeta)$ is given by (2.18), *and $\nu(\zeta)$ is the associated matrix given by Lemma* 2.14.

Proof. To compute $(R'(A)S)^{-1}$, we let $f \in F$; by (2.11) and (2.12) we have

$$g = R'(A)Sf = \tfrac{1}{2}[\rho_z(A)C(f + iT_1f) + \rho_{\overline{Z}}(A)\overline{C}(f - iT_1f)]. \tag{2.22}$$

Applying Lemma (2.14) for m given by (2.18), and multiplying (2.22) on the left by ν yield

$$\nu m(f + iT_1f) = \nu g + iT_1(\nu g). \tag{2.23}$$

Hence

$$f + iT_1f = (\nu m)^{-1}[\nu g + iT_1(\nu g)]. \tag{2.24}$$

We then obtain (2.20) and (2.21) from(2.9), (2.12),(2.24), and the fact that $Sf = C\widehat{D}$, where $\widehat{D}|_{S^1} = \frac{1}{2}(f + iT_1f)$. q.e.d.

To conclude this section it should be noted that we have shown that if $\varepsilon > 0$ is sufficiently small then $\mathscr{A} = \mathscr{A}_{p_0,\varepsilon}(M)$ is an infinite-dimensional submanifold of $\mathscr{D}^N_{p_0,\varepsilon}$, and that $\mathscr{A}$ and its tangent space $T_A\mathscr{A}$ at any $A \in \mathscr{A}$ are parametrized by E_0 the closed subspace of $\mathscr{D}^N_0$ given by (2.13).

3. Defect of an analytic disc attached to a generic manifold

Let M be a generic manifold in $\mathbf{C}^N$ as in §1. Recall that $\Sigma(M)$, the conormal bundle of M defined by (2.4), is a real submanifold of dimension $2N$ contained in the complex manifold $\Lambda^{1,0}\mathbf{C}^N$. We consider analytic discs $B: \overline{\Delta} \to \Lambda^{1,0}\mathbf{C}^N$ attached to $\Sigma(M)$. The base projection of such a disc will then be an analytic disc in $\mathbf{C}^N$ attached to M. Taking $(Z_1, \cdots, Z_N, \lambda_1, \cdots, \lambda_N)$ for global holomorphic coordinates in $\Lambda^{1,0}\mathbf{C}^N$, we have

$$B(\zeta) = (A(\zeta), \lambda(\zeta)), \tag{3.1}$$

with $\lambda(\zeta) = (\lambda_1(\zeta), \cdots, \lambda_N(\zeta))$, where each λ_j is a scalar analytic disc. The disc B given by (3.1) is attached to $\Sigma(M)$ if and only if

$$A(S^1) \subset M \quad \text{and} \quad \sum_j \lambda_j(\zeta)\, dZ_j \in \Sigma_{A(\zeta)}(M), \qquad \zeta \in S^1. \tag{3.2}$$

For A attached to M, we denote by V_A the set of all discs of the form (3.1) attached to $\Sigma(M)$. Note that by (3.2), V_A can be equipped with a real vector space structure. For $\zeta \in S^1$, let $V_A(\zeta) \subset \Sigma_{A(\zeta)}M$ be defined by

$$V_A(\zeta) = \{\alpha \in \Sigma_{A(\zeta)}M : \alpha = \sum_j \lambda_j(\zeta)\, dZ_j,\ (A(\zeta), \lambda(\zeta)) \in V_A\}. \tag{3.3}$$

Hence $V_A(\zeta)$ is a subspace of $\Sigma_{A(\zeta)}M$, and by the identification (1.4), we have, for every $\zeta \in S^1$,

$$V_A(\zeta) \subset (T^c_{A(\zeta)}M)^{\perp}. \tag{3.4}$$

Definition 3.5. If $\zeta \in S^1$ and A is a disc attached to M, we define the *defect of A at ζ*, $\operatorname{def}_\zeta A$, as the dimension of the real vector space $V_A(\zeta)$.

The following shows that if a disc A attached to M is sufficiently close to a constant disc, then its defect is independent of $\zeta \in S^1$. Using the identification (1.5) we may regard $V_A(\zeta)$ as a subspace of $T^*_{A(\zeta)}M$. Making use of (1.7) we have:

Proposition 3.6. *If $\varepsilon > 0$ is sufficiently small and $A \in \mathscr{A}_{p_0,\varepsilon}(M)$, then $\operatorname{def}_\zeta A$ is independent of $\zeta \in S^1$. More precisely, for $\zeta_0 \in S^1$, a covector $\xi \in T^*_{A(\zeta_0)}M$ is in $V_A(\zeta_0)$ if and only if*

$$\xi = i\,{}^\tau b\nu(\zeta_0)\partial\rho(A(\zeta_0)), \tag{3.7}$$

with $b \in \mathbf{R}^l$, ν given by Lemma 2.14 *with $m(\zeta)$ defined by* (2.18), *and such that $\zeta \mapsto {}^\tau b\nu(\zeta)\rho_Z(A(\zeta))$ extends holomorphically to $\overline{\Delta}$.*

Proof. An analytic disc $B(\zeta)$ of the form (3.1) with $A \in \mathscr{A}$ is attached to $\Sigma(M)$ if and only if

$$\sum_j \lambda_j(\zeta)\, dZ_j = i\,{}^\tau t(\zeta)\partial\rho(A(\zeta)), \qquad \zeta \in S^1, \tag{3.8}$$

with $\lambda_j(\lambda)$ extending holomorphically to Δ and $S^1 \ni \zeta \mapsto t(\zeta) \in \mathbf{R}^l$ of class $C^{1,\alpha}(S^1)$. This implies that the map

$$S^1 \ni \zeta \mapsto {}^\tau t(\zeta)\rho_Z(A(\zeta)) \in \mathbf{C}^N$$

extends holomorphically. In particular, if C is the matrix given by (2.17), then the map given by

$$S^1 \ni \zeta \mapsto {}^\tau t(\zeta)\rho_Z(A(\zeta))C \in \mathbf{C}^l$$

also extends holomorphically. By applying the second statement in Lemma 2.14 to $m(\zeta)$ given by (2.18), we conclude that $t(\zeta) = {}^\tau\nu(\zeta)t(1)$, where $\nu(\zeta)$ is as given by the lemma. The characterization follows since $\nu(\zeta)$ is invertible for all $\zeta \in S^1$. This completes the proof of the proposition. q.e.d.

For $A \in \mathscr{A}$, we denote by $\operatorname{def} A$ the defect of A at any $\zeta \in S^1$.

4. Variations on a theorem of Tumanov

For every $\zeta_0 \in S^1$ fixed we define the evaluation mapping $\mathscr{F}_{\zeta_0} : \mathscr{D}^N \to \mathbf{C}^N$ given by

$$\mathscr{F}_{\zeta_0}(A) = A(\zeta_0). \tag{4.1}$$

Also, for $A \in \mathscr{D}^N$, we consider the push forward by A of the tangent vector of S^1, $\partial/\partial\theta|_{\theta=0}$, as a real tangent vector of $\mathbf{C}^N$ at $A(1)$. We introduce the derivative map $\mathscr{G}: \mathscr{D}^N \to T\mathbf{C}^N$ given by

$$\mathscr{D}^N \ni A \mapsto \mathscr{G}(A) = A_*\left(\frac{\partial}{\partial\theta}\Big|_{\theta=0}\right) \in T_{A(1)}\mathbf{C}^N. \tag{4.2}$$

We consider the restrictions of the mappings $\mathscr{F}_{\zeta_0}$ and $\mathscr{G}$, given by (4.1) and (4.2), to the submanifold $\mathscr{A} = \mathscr{A}_{p_0,\varepsilon}(M)$ of $\mathscr{D}^N$ defined by(2.5). Their differentials at $A \in \mathscr{A}$ are then real linear mappings

$$\mathscr{F}'_{\zeta_0}(A)\colon T_A\mathscr{A} \to T_{A(\zeta_0)}M, \tag{4.3}$$

$$\mathscr{G}'(A)\colon T_A\mathscr{A} \to T_{p_0}M. \tag{4.4}$$

Note that since $\mathscr{F}_1(A) = p_0$ for all $A \in \mathscr{A}$, we have $\mathscr{F}_1'(A) \equiv 0$. We can now state the following theorem.

Theorem 1. *Let $\zeta_0 \in S^1$ fixed, $\zeta_0 \neq 1$. If ε is sufficiently small, $A \in \mathscr{A}_{p_0,\varepsilon}(M)$, and $V_A(\zeta_0) \subset T^*_{A(\zeta_0)}M$ defined by* (3.3), *then*

(i) $\mathscr{F}'_{\zeta_0}(A)T_A\mathscr{A} = V_A(\zeta_0)^\perp$,

(ii) $\mathcal{F}'(A)T_A\mathcal{A} = V_A(1)^{\perp}$,
where orthogonality is taken in the sense of duality between TM *and* T^*M.

Statement (i) of Theorem 1 is essentially proved in Tumanov [20]; however, our formulation is expressed in a more geometric and invariant way. Statement (ii) is also inspired by a related result in [20].

Making use of (3.4), we obtain the following corollary of Theorem 1.

Corollary 4.5. *Under the assumptions of Theorem* 1, *the codimension of* $\mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A}$ *in* $T_{A(\zeta_0)}M$ *and the codimension of* $\mathcal{F}'(A)T_A\mathcal{A}$ *in* $T_{p_0}M$ *coincide and are equal to the defect of* A. *Furthermore the following inclusion holds*:

$$T^c_{A(\zeta_0)}M \subset \mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A}, \qquad T^c_{p_0}M \subset \mathcal{F}'(A)T_A\mathcal{A}. \tag{4.6}$$

Proof of Theorem 1. We shall first prove (i). For this, we shall begin by showing the inclusion

$$\mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A} \subset V_A(\zeta_0)^{\perp}. \tag{4.7}$$

Note that $\mathcal{F}'_{\zeta_0}(A)\widehat{A} = \widehat{A}(\zeta_0)$ for $\widehat{A} \in T_A\mathcal{A}$ given by (2.20). Then using Proposition 3.6, it suffices to prove that

$$^\tau b\nu(\zeta_0)\rho_Z(A(\zeta_0))[\widehat{A}_0(\zeta_0) + C\widehat{D}(\zeta_0)] = 0, \tag{4.8}$$

for $b \in \mathbf{R}^l$ such that $\zeta \mapsto {}^\tau b\nu(\zeta)\rho_Z(A(\zeta))$ extends holomorphically, and $\widehat{A}_0$, C, $\widehat{D}$ are as in Lemma 2.19. Replacing $\widehat{D}$ in (4.8) by using (2.21), and noting that $T_1 f = -if$ if f is a function on S^1 which extends holomorphically to $\overline{\Delta}$ and vanishes at 1, we obtain (4.8), which proves the inclusion (4.7).

To show that the opposite inclusion $\mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A} \supset V_A(\zeta_0)^{\perp}$ holds, we shall prove

$$[\mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A}]^{\perp} \subset V_A(\zeta_0). \tag{4.9}$$

For this, let $\xi \in T^*_{A(\zeta_0)}M$ such that

$$\langle \xi, \widehat{A}(\zeta_0)\rangle = 0 \tag{4.10}$$

for all $\widehat{A} \in T_A\mathcal{A}$. We must show that $\xi \in V_A(\zeta_0)$. By (1.7), we may take ξ in the form

$$\xi = {}^\tau\alpha\, dZ + {}^\tau\overline{\alpha}\, d\overline{Z} + i\,{}^\tau t\partial\rho(A(\zeta_0)), \tag{4.11}$$

with $\alpha \in \mathbf{C}^N$, ${}^\tau C\alpha = 0$, and $t \in \mathbf{R}^l$. We first choose $\widehat{A}_0$ in (2.20) so that $\widehat{A}_0(\zeta_0) = 0$. Then $\langle \mathrm{Re}({}^\tau\alpha\, dZ), \widehat{A}(\zeta_0)\rangle = 0$ and (4.10) becomes

$$i\,{}^\tau t\rho_Z(A(\zeta_0))C\widehat{D}(\zeta_0) - i\,{}^\tau t\rho_{\overline{Z}}(\zeta_0)\,\overline{C\widehat{D}(\zeta_0)} = 0. \tag{4.12}$$

Substituting the expression for $\widehat{D}$ from (2.21) in (4.12), we obtain

$$T_1[{}^\tau t\nu(\zeta_0)^{-1}\nu\rho_Z(A)\widehat{A}_0](\zeta_0) = 0. \tag{4.13}$$

Note that if $u(\zeta)$ is defined on S^1 we have

$$T_1 u(\zeta_0) = PV\frac{1}{\pi}\int_{S^1}\frac{u(\zeta)(1-\zeta_0)}{(\zeta-1)(\zeta-\zeta_0)}\,d\zeta. \tag{4.14}$$

For $d \in \mathbf{R}^N$ with $\rho_Z(p_0)d = 0$, we take $\widehat{A}_0(\zeta) = (\zeta-1)(\zeta-\zeta_0)\zeta^k d$, with k any nonnegative integer. Then (4.13) and (4.14) imply that

$$\zeta \mapsto {}^\tau t\nu(\zeta_0)^{-1}\nu(\zeta)\rho_Z(A(\zeta))\,d \tag{4.15}$$

extends holomorphically. This, together with the fact that $\nu(\zeta)\rho_Z(A(\zeta))C$ extends holomorphically, implies that $i\,{}^\tau t\partial\rho(A(\zeta_0)) \in V_A(\zeta_0)$.

To finish the proof of inclusion (4.9), it suffices to show that if ξ, given by (4.11), satisfies (4.10), then $\alpha = 0$. From inclusion (4.7) and the previous step, we have $\langle i\,{}^\tau t\partial\rho(A(\zeta_0)), \widehat{A}(\zeta_0)\rangle = 0$ for any $\widehat{A} \in T_A\mathscr{A}$. Hence by (4.10) and the fact that ${}^\tau C\alpha = 0$, we have $\mathrm{Re}({}^\tau\alpha\widehat{A}_0(\zeta_0)) = 0$. Since $\widehat{A}_0(\zeta_0)$ is an arbitrary vector in the kernel of $\rho_Z(p_0)$ in $\mathbf{C}^N$, it follows that ${}^\tau\alpha\widehat{A}_0(\zeta_0) = 0$, and hence $\alpha = 0$. This completes the proof of (i) of Theorem 1.

We shall now prove (ii) of Theorem 1. As in the proof of (i), we begin by showing the inclusion

$$\mathscr{G}'(A)T_A\mathscr{A} \subset V_A(1)^\perp. \tag{4.16}$$

Note that $\mathscr{G}'(A)\widehat{A} = d\widehat{A}(e^{i\theta})/d\theta|_{\theta=0}$ for $\widehat{A} \in T_A\mathscr{A}$ given by (2.20). Using Proposition 3.6, it suffices to prove

$${}^\tau b\rho_Z(p_0)\frac{d}{d\theta}[\widehat{A}_0(e^{i\theta}) + C\widehat{D}(e^{i\theta})]|_{\theta=0} = 0, \tag{4.17}$$

for $b \in \mathbf{R}^l$ such that $\zeta \mapsto {}^\tau b\nu(\zeta)\rho_Z(A(\zeta))$ extends holomorphically and $\widehat{A}_0$, C, and $\widehat{D}$ are as in Lemma (2.19). Since $\rho_Z(p_0)\widehat{A}_0(e^{i\theta}) \equiv 0$, we also have $\rho_Z(p_0)\,d\widehat{A}_0(e^{i\theta})/d\theta \equiv 0$. Hence the first term in (4.17) vanishes. Using the observations above, the rest of the proof of inclusion (4.16) is similar to that of (4.7).

To show that the opposite inclusion $\mathscr{G}'(A)T_A\mathscr{A} \supset V_A(1)^\perp$ holds, we shall prove

$$[\mathscr{G}'(A)T_A\mathscr{A}]^\perp \subset V_A(1). \tag{4.18}$$

For this, let $\xi \in T^*_{p_0}M$ such that

$$\langle \xi, \frac{d}{d\theta}\widehat{A}(e^{i\theta})|_{\theta=0}\rangle = 0 \tag{4.19}$$

for all $\widehat{A} \in T_A\mathscr{A}$. We must show that $\xi \in V_A(1)$. By (1.7), we may take ξ in the form

$$\xi = {}^\tau\alpha\, dZ + {}^\tau\overline{\alpha}\, d\overline{Z} + i^\tau t\partial\rho(p_0), \tag{4.20}$$

with $\alpha \in \mathbf{C}^N$, ${}^\tau C\alpha = 0$, and $t \in \mathbf{R}^l$. We first choose $\widehat{A}_0$ in (2.20) so that $d\widehat{A}_0(e^{i\theta})/d\theta|_{\theta=0} = 0$. Then $\langle \mathrm{Re}({}^\tau\alpha\, dZ),\ d\widehat{A}_0(e^{i\theta})/d\theta|_{\theta=0}\rangle = 0$ and since $\rho_Z(p_0)C = I$, (4.19) becomes

$$ {}^\tau t\left[\frac{d}{d\theta}\widehat{D}(e^{i\theta}) - \frac{d}{d\theta}\overline{\widehat{D}(e^{i\theta})}\right]|_{\theta=0} = 0. \tag{4.21}$$

Substituting the expression for $\widehat{D}$ from (2.21) and (4.21) yields

$$\frac{d}{d\theta}T_1[{}^\tau t\nu\rho_Z(A)\widehat{A}_0](e^{i\theta})|_{\theta=0} = 0. \tag{4.22}$$

Note that if $u(\zeta)$ is defined on S^1 with $u(1) = du(e^{i\theta})/d\theta|_{\theta=0} = 0$, we have

$$\frac{d}{d\theta}T_1u(e^{i\theta})|_{\theta=0} = \frac{1}{i\pi}\int_{S^1}\frac{u(\zeta)}{(\zeta-1)^2}\,d\zeta. \tag{4.23}$$

For $d \in \mathbf{R}^N$ with $\rho_Z(p_0)d = 0$, we take $\widehat{A}_0(\zeta) = (\zeta-1)^2\zeta^k d$, with k any nonnegative integer. Then (4.22) and (4.23) imply that

$$\zeta \mapsto {}^\tau t\nu(\zeta)\rho_Z(A(\zeta))\,d \tag{4.24}$$

extends holomorphically. This, together with the fact that $\nu(\zeta)\rho_Z(A(\zeta))C$ extends holomorphically, implies that $i^\tau t\partial\rho(p_0) \in V_A(1)$.

The rest of the proof of inclusion (4.18) is similar to that of (4.9). This completes the proof of Theorem 1.

Remark 4.25. Another consequence of Theorem 1 is that for $\varepsilon > 0$ sufficiently small, the mapping $\mathscr{A}_{p_0,\varepsilon}(M) \ni A \mapsto \operatorname{def} A \in \mathbf{Z}^+$ is upper semicontinuous. Indeed, the map $A \mapsto \operatorname{rank}\mathscr{F}_{\zeta_0}(A)$ is lower semicontinuous. Hence the claim follows from Theorem 1(i).

Following [21], we recall that M is *minimal* at p_0 if there is no germ N of a submanifold of M through p_0 with $T^c_pM \subset T_pN$ for all $p \in N$. The following corollaries are due to Tumanov [20].

Corollary 4.26. *If M is minimal at p_0, then there exist analytic discs A of defect 0 attached to M through p_0 with $\|A - \langle p_0\rangle\|_{1,\alpha}$ arbitrarily small.*

Proof. Choose a disc $A_* \in \mathscr{A}$ of minimal possible defect in a neighborhood of the constant disc $\langle p_0 \rangle$. After replacing $A_*(\zeta)$ by $A_*(\zeta^2)$, which does not change the defect, we may assume $A_*(-1) = p_0$. If A_* were not of defect 0, then it would follow from Theorem 1(i) that the image of a neighborhood of A_* in $\mathscr{A}$ by $\mathscr{F}_{-1}$ is a proper submanifold N of M containing p_0 for which $T_p^c M \subset T_p N$ for all $p \in N$, contradicting the minimality of M at p_0. q.e.d.

The following is also an immediate application of Theorem 1(ii).

Corollary 4.27. *If $A \in \mathscr{A}$ is of defect 0, then there is a smooth map, $r \mapsto A(r;\cdot)$, from an open neighborhood U of 0 in $\mathbf{R}^{2n+l}$ into $\mathscr{A}$, such that $A(0,\zeta) = A(\zeta)$ and $U \ni r \mapsto A_*(r; \partial/\partial\theta|_{\theta=0}) \in T_{p_0}M$ is of rank $2n+l$.*

Remark 4.28. If M is a generic manifold in $\mathbf{C}^N$, and $p_0 \in M$, one can prove by techniques similar to those used in §2 that the set $\mathscr{B}_{p_0,\varepsilon}(M)$ of all discs in $\mathscr{D}^N$ attached to M within ε distance from the constant disc $\langle p_0 \rangle$ forms an infinite-dimensional submanifold of the real Banach space $\mathscr{D}^N$. The manifold $\mathscr{A}_{p_0,\varepsilon}(M)$ of §2 is then a submanifold of $\mathscr{B}_{p_0,\varepsilon}(M)$. The tangent space to $\mathscr{B}_{p_0,\varepsilon}(M)$ at any point of this manifold can be parametrized by those analytic discs $\widehat{A} \in \mathscr{D}^N$ for which $\rho_Z(p_0)\widehat{A}(\zeta) \equiv ir$, with $r \in \mathbf{R}^l$. (See §2 for a similar calculation.) If $A_0 \in \mathscr{B}_{p_0,\varepsilon}(M)$ is of defect 0, by using the tangent space at A_0, one can show that the mapping $\mathscr{U} \times [0, 1] \ni (A, \xi) \mapsto A(\xi) \in \mathbf{C}^N$, where $\mathscr{U}$ is a neighborhood of A_0 in $\mathscr{B}_{p_0,\varepsilon}(M)$, covers an open wedge with edge M in $\mathbf{C}^N$. Tumanov [20] proves this result by solving the Bishop equation with parameters. His argument is replaced here by the use of the implicit function theorem for Banach spaces. If M is minimal at p_0, the filling of wedges by discs described here, together with the approximation theorem in [4], shows that every CR function in a neighborhood of p_0 in M extends holomorphically to a wedge in $\mathbf{C}^N$.

5. Discs of maximal defect. Minimal convexity

In this section we shall use the notation of §4. We begin with the following result.

Theorem 2. *If ε is sufficiently small, then for any $A \in \mathscr{A} = \mathscr{A}_{p_0,\varepsilon}(M)$ and any $\zeta_0 \in S^1$, $\zeta_0 = e^{i\theta_0}$, the following holds:*

$$(5.1) \qquad A_*\left(\frac{\partial}{\partial\theta}|_{\theta=\theta_0}\right) \in V_A(\zeta_0)^\perp.$$

Corollary 5.2. *Let A be as in Theorem 2. If A is of defect l, the codimension of M in $\mathbf{C}^N$, then for every $\zeta_0 \in S^1$, we have*

$$(5.3) \qquad A_*\left(\frac{\partial}{\partial\theta}|_{\theta=\theta_0}\right) \in T^c_{A(\zeta_0)}M,$$

where $\zeta_0 = e^{i\theta_0}$.

Proof of Corollary 5.2. Since A is of defect l, we have $\dim_{\mathbf{R}} V_A(\zeta_0) = l$ for all $\zeta_0 \in S^1$. On the other hand, by (3.4) we also have

$$(5.4) \qquad T^c_{A(\zeta_0)} \subset V_A(\zeta_0)^\perp.$$

Since $\dim_{\mathbf{R}} T^c_{A(\zeta_0)} = 2N - l$, we conclude by dimension that the inclusion in (5.4) is an equality when A is of defect l. Then (5.3) follows from (5.1).

Proof of Theorem 2. Since A is attached to M, we have $\rho(A(e^{i\theta})) \equiv 0$. Differentiating this identity in θ yields

$$(5.5) \qquad i\rho_Z(A(\zeta))\zeta A'(\zeta) - i\rho_{\overline{Z}}(A(\zeta))\overline{\zeta A'(\zeta)} = 0, \qquad \zeta \in S^1.$$

Let $\nu(\zeta)$ be as in Proposition (3.6). For $b \in \mathbf{R}^l$ such that

$$\zeta \mapsto {}^\tau b\nu(\zeta)\rho_Z(A(\zeta))$$

extends holomorphically to $\overline{\Delta}$, from (5.5) we obtain

$$(5.6) \qquad \operatorname{Re} i^\tau b\nu(\zeta)\rho_Z(A(\zeta))\zeta A'(\zeta) \equiv 0, \qquad \zeta \in S^1.$$

Since $\zeta \mapsto {}^\tau b\nu(\zeta)\rho_Z(A(\zeta))A'(\zeta)$ extends holomorphically, we conclude from (5.6) that ${}^\tau b\nu(\zeta)\rho_Z(A(\zeta))A'(\zeta)\zeta \equiv 0$, i.e., for every $\zeta_0 \in S^1$

$$(5.7) \qquad \left\langle i^\tau b\nu(\zeta_0)\rho_Z(A(\zeta_0)),\, A_*\left(\frac{\partial}{\partial\theta}|_{\theta=\theta_0}\right)\right\rangle = 0.$$

This proves the theorem, by using the description (3.7) of $V_A(\zeta_0)$ given by Proposition (3.6).

Another proof of Theorem 2. We give here a different proof of Theorem 2 which could be of independent interest. Note that for $c \in \Delta$ the mapping

$$(5.8) \qquad \Phi_c(\zeta) = (\zeta + c)(1 + \overline{c})/(1 + \overline{c}\zeta)(1 + c)$$

is an automorphism of $\overline{\Delta}$ fixing the point 1. Let $A \in \mathscr{A}$ and assume A to be of class $C^{2,\alpha}$. Fix $a \in \Delta$. Consider the one-parameter family of discs defined by

$$A(t, \zeta) = A(\Phi_{ta}(\zeta)). \tag{5.9}$$

If I is a small open interval containing the origin, then the map $I \ni t \mapsto A(t, \cdot) \in \mathscr{A}$ is a C^1 curve with $A(0, \cdot) = A(\cdot)$. Hence

$$\widehat{A}(\zeta) = \frac{\partial A(t, \zeta)}{\partial t}\Big|_{t=0} \in T_A\mathscr{A}. \tag{5.10}$$

Applying the chain rule, we obtain

$$\widehat{A}(\zeta) = A'(\zeta)(1-\zeta)(a+\overline{a}\zeta). \tag{5.11}$$

Hence for $\zeta_0 = e^{i\theta_0}$ we have

$$\widehat{A}(\zeta_0) = A_*\left(\frac{\partial}{\partial\theta}\Big|_{\theta=\theta_0}\right)\frac{(1-\zeta_0)(a+\zeta_0\overline{a})}{i\zeta_0}. \tag{5.12}$$

Assume $\zeta_0 \neq 1$ and take $a \in \Delta$ such that $a\overline{\zeta}_0 + \overline{a} \neq 0$. Since $\widehat{A}(\zeta_0) \in \mathscr{T}_{\zeta_0}(A)T_A\mathscr{A}$, (5.12) and Theorem 1(i) imply (5.1). If $\zeta_0 = 1$, differentiating (5.11) and taking $\zeta = 1$ yield $\widehat{A}(1) = -A'(a)(a+\overline{a})$, which implies

$$\mathscr{G}'(A)\widehat{A} = -A_*\left(\frac{\partial}{\partial\theta}\Big|_{\theta=0}\right)(a+\overline{a}). \tag{5.13}$$

With a real and nonzero, (5.13) and Theorem 1(ii) imply (5.1) for $\zeta_0 = 1$. We conclude the second proof of Theorem 2 by observing that the discs in $\mathscr{A}$ of class $C^{2,\alpha}(S^1)$ are dense in $\mathscr{A}$. q.e.d.

The following generalizes the notion of minimal convexity for hypersurfaces given in [3].

Definition 5.14. Let $\mathscr{T}: \mathscr{A}_{p_0,\varepsilon}(M) \to T_{p_0}M/T^c_{p_0}M$ be defined by $\mathscr{T}(A) = \pi(\mathscr{G}(A))$, where π is the canonical projection of $T_{p_0}M$ onto $T_{p_0}M/T^c_{p_0}$ and $\mathscr{G}$ is defined by (4.2). We say that M is *minimally convex* at p_0 if M is minimal at p_0 and there exists a closed, strictly convex cone $\Gamma \subset T_{p_0}M/T^c_{p_0}M$ such that $\mathscr{T}(A) \subset \Gamma$ for all $A \in \mathscr{A} = \mathscr{A}_{p_0,\varepsilon}(M)$ and ε sufficiently small.

Note that, as observed in [3], in the case where M is a hypersurface, i.e., $l = 1$, it follows from Tumanov [20] that if M is minimal at p_0, then either M is minimally convex at p_0 or every CR function on M near p_0 extends holomorphically to a full neighborhood of p_0 in $\mathbf{C}^N$.

Theorem 3. *If M is minimally convex at p_0, $A \in \mathscr{A}_{p_0,\varepsilon}(M)$, ε sufficiently small, and $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$, then A is of defect l and hence $A_*(\partial/\partial\theta) \in T^c_{A(e^{i\theta})}M$ for all $\theta \in \mathbf{R}$.*

Proof. By Corollary 5.2, it suffices to show that the assumption implies that A is of defect l. We note first that $\operatorname{rk}\mathscr{F}'(A) = 0$. Indeed, since $\mathscr{F}(A) = 0$ by assumption, if $\operatorname{rk}\mathscr{F}'(A) \geq 1$, then $\mathscr{F}(\mathscr{A})$ contains a nonsingular curve passing through the origin, contradicting the fact that $\mathscr{F}(\mathscr{A})$ lies in a strictly convex cone, since M is minimally convex. By Theorem 1(ii) and (and also (4.6)), the rank of $\mathscr{F}$ at A is $l - \operatorname{def} A$ for all $A \in \mathscr{A}$, which proves that $\operatorname{def} A = l$. q.e.d.

Using the remarks preceding Theorem 3 concerning minimally convex hypersurfaces and taking $l = 1$ in Theorem 3, we obtain the following.

Corollary 5.15. *Let M be a hypersurface in $\mathbf{C}^N$ minimal at p_0. Then one of the following occurs:*

(i) *M is not minimally convex at p_0.*

(ii) *There is $\varepsilon > 0$ which satisfies the following property. If $A \in \mathscr{A}_{p_0,\varepsilon}(M)$ and $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$, then we also have $A_*(\partial/\partial\theta) \in T^c_{A(e^{i\theta})}M$ for all θ.*

For a general generic manifold in $\mathbf{C}^N$ we have the following result.

Theorem 4. *Let M be a generic manifold and $p_0 \in M$. Then there exists $\varepsilon > 0$ such that the following holds. If $A \in \mathscr{A}_{p_0,\varepsilon}(M)$ and $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$, then one of the following must occur:*

(i) *Every CR function on M extends holomorphically to a full neighborhood of p_0 in $\mathbf{C}^N$.*

(ii) $\operatorname{def} A \geq 1$.

Proof. Let $\mathscr{F}$ be the map defined by Definition 5.14. Then we have $\mathscr{F}(A) = 0$ by assumption. If $\operatorname{def}(A) = 0$, then by Theorem 1(ii), the rank of $\mathscr{F}$ at A is l. An argument similar to the one used in Remark 4.28 (or inspection of the proof of Tumanov [20]) will then show that the set of discs attached to M through points near p_0 fill a full neighborhood of p_0 in $\mathbf{C}^N$; hence (i) holds. q.e.d.

If M is a generic manifold of codimension l in $\mathbf{C}^{n+l}$, we shall say that M is *homogeneous* if there exist holomorphic coordinates $Z = (z, w)$, with $z \in \mathbf{C}^n$, $w \in \mathbf{C}^l$, defining functions $\rho = (\rho_1, \cdots, \rho_l)$, and integers $m_1, \cdots, m_l$, $m_j \geq 2$, such that $\rho_w(0)$ is invertible, $\rho_z(0) = 0$, and $\rho_j(tz, \delta_t w) = t^{m_j}\rho_j(z, w)$, where $\delta_t w = (t^{m_1}w_1, \cdots, t^{m_l}w_l)$, $t > 0$.

Theorem 5. *Let M be a homogeneous generic manifold of codimension l in $\mathbf{C}^{l+1}$, minimal at 0. Then any nonconstant disc attached to M*

through 0 *is of defect less than* l. *In particular, if for some nonconstant disc* A *passing through* 0, $A_*(\partial/\partial\theta|_{\theta=0}) \in T_0^c M$, *then* M *is not minimally convex at* 0.

Proof. Note first that since M is smooth, homogeneous, and minimal at 0, it must be of finite type (in the sense of Bloom and Graham) and hence M is of finite type, and therefore minimal, at every point. Assume by contradiction that there is a nonconstant disc $A(\zeta) = (z(\zeta), w(\zeta))$ attached to M through 0 of defect l. Let

$$A(r, \zeta) = ((1+r)z(\zeta), \delta_{(1+r)}w(\zeta)).$$

For each $r \in \mathbf{R}$, $r > -1$, $A(r, \zeta)$ is attached to M through 0, by the homogeneity of M. Since A is not constant, $z(\zeta) \not\equiv 0$. We claim there is $\zeta_0 \in S^1$ such that $\operatorname{Re}\zeta_0 z'(\zeta_0)\overline{z(\zeta_0)} \neq 0$. Indeed, write $z(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ and $\zeta z'(\zeta)\overline{z(\zeta)} = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$; then $b_0 = \sum_{n=0}^{\infty} |a_n|^2 n \neq 0$. Hence the map $(r, \theta) \mapsto (1+r)z(e^{i\theta})$ is of rank 2 at $(0, \theta_0)$, where $e^{i\theta_0} = \zeta_0$. The map $(r, \theta) \mapsto \Psi(r, \theta) = A(r, e^{i\theta}) \in M$ is then also of rank 2 at $(0, \theta_0)$. We claim now that the image of Ψ contains a germ of a CR submanifold of M through $A(\zeta_0)$, which would contradict the minimality of M at 0 since $\dim T_p^c M = 2$ for all $p \in M$. Indeed, applying Theorem 1(i) we have $\Psi_*(\partial/\partial r|_{(r,\theta)}) \in (V_{A(r,\cdot)}(e^{i\theta}))^\perp$. Also, by Theorem 2, $\Psi_*(\partial/\partial\theta|_{(r,\theta)}) \in (V_{A(r,\cdot)}(e^{i\theta}))^\perp$. Since $A(r, \cdot)$ is of defect l, we have, by (5.4), $T^c_{A(r,\zeta)}M = (V_{A(r,\cdot)}(\zeta))^\perp$. Since $\operatorname{rk}\Psi'(r, \theta) = 2$ near $(0, \theta_0)$, this proves the claim and completes the proof of Theorem 5. q.e.d.

Taking $l = 1$ in Theorem 3, we obtain the following corollary.

Corollary 5.16. *Let* M *be a homogeneous hypersurface of finite type in* $\mathbf{C}^2$, *i.e., given by*

$$M = \{(z, w),\ z \in \mathbf{C},\ w \in \mathbf{C},\ \operatorname{Im} w = p(z, \bar{z})\},$$

where p *is a nonharmonic homogeneous polynomial. Then all nonconstant discs* A *attached to* M *through 0 are of defect 0. In particular, if for some nonconstant disc* A *passing through* 0, $A_*(\partial/\partial\theta|_{\theta=0}) \in T_0^c M$, *then* M *is not minimally convex at* 0.

We conclude this section with some remarks and examples.

Remark 5.17. If A is a disc attached to a generic manifold M such that $A(\Delta) \subset M$, we claim that $A_*(\partial/\partial\theta) \in T^c_{A(e^{i\theta})}M$ for all θ. Indeed, since $\rho(A(\zeta)) \equiv 0$, $\zeta \in \overline{\Delta}$, we have, by differentiating this identity, $\rho_z(A(\zeta))A'(\zeta) \equiv 0$, which proves the claim. Note, however, that $A(\Delta) \subset M$ does not imply that A is of positive defect, as shown by the following example.

Example 5.18. Let M be the hypersurface in $\mathbf{C}^3$ given by $\operatorname{Im} w = |z_1|^2 - |z_2|^2$ and $A(\zeta) = (z(\zeta), z(\zeta), 0)$, where $z(\zeta)$ is any nonconstant analytic disc valued in $\mathbf{C}$, with $z(1) = 0$. We have $A(\Delta) \subset M$, $A_*(\partial/\partial\theta) \in T^c_{A(e^{i\theta})}M$ for all θ by Remark 5.17, but A is of defect 0. Note also that all nonconstant discs through 0 are of defect 0 for this hypersurface.

Example 5.19. If M is a hypersurface in $\mathbf{C}^N$, $0 \in M$, and A is a small disc attached to M through 0, then the defect of A can be 1 although $A(\Delta) \not\subset M$. For this, let M be the hypersurface in $\mathbf{C}^3$ given by $\operatorname{Im} w = (\operatorname{Im} z_2 - |z_1|^2)^2$, and let $A(\zeta) = (z_1(\zeta), z_2(\zeta), 0)$, where the disc $B(\zeta) = (z_1(\zeta), z_2(\zeta))$ is any nonconstant disc attached to the Lewy hypersurface in $\mathbf{C}^2$ given by $\operatorname{Im} z_2 = |z_1|^2$. Then $A(\Delta) \cap M = \varnothing$ and A is of defect 1, since for $\zeta \in S^1$ $\rho_z(A(\zeta)) \equiv 0$ and $\rho_w(A(\zeta)) = 1/2i$.

Remark 5.20. If M is a tubular hypersurface in $\mathbf{C}^2$, i.e., M is given by $\operatorname{Im} w = \phi(\operatorname{Re} z)$, with ϕ a smooth, real-valued function satisfying $\phi(0) = \phi'(0) = 0$, then for A any analytic disc of defect 1 attached to M through 0, we have $A(\Delta) \subset M$. Indeed, in this case ρ_z is real valued and $\rho_w \equiv 1/2i$.

Example 5.21. We give here an example of a hypersurface in $\mathbf{C}^2$ given by $\operatorname{Im} w = \phi(z)$, with ϕ a smooth, real-valued function of two real variables, satisfying $\phi(0) = 0$, $\phi'(0) = 0$ with the following property. There is no nonconstant disc A attached to M through 0 with $A(\Delta) \subset M$; in addition for any $\varepsilon > 0$, there is a nonconstant disc A attached to M through 0 of defect 1 with norm $< \varepsilon$. For $n \geq 1$ let C_n be the circle in $\mathbf{C}$ given by $C_n = \{z \in \mathbf{C}: |z - 1/2^n| = 1/2^n\}$. Let $\phi \in C^\infty(\mathbf{C})$ with $\phi \geq 0$ and $\phi(z) = 0$ if and only if $z \in C = \bigcup_{n\geq 1} C_n$. The existence of such a function ϕ is standard by the Whitney Theorem. Note that since $\phi \geq 0$ we have necessarily $\phi'(z) = 0$ for $z \in C$. It is easy to check that the hypersurface M satisfies the above properties. In particular, for every $n \geq 1$, the disc $A_n(\zeta) = (2^{-n}(1-\zeta), 0)$ attached to M through 0 is of defect 1, and $A_n(\Delta) \not\subset M$. Note also that the only nonconstant discs attached to M through 0 of defect 1 are of the form $A(\zeta) = (z(\zeta), 0)$ where $z(\zeta)$ maps S^1 into C.

Remark 5.22. If M is a real analytic hypersurface in $\mathbf{C}^2$ minimal at $0 \in M$, we do not know whether there exist nonconstant analytic discs attached to M through 0 of defect 1 and of arbitrarily small norm. Note that Example 5.19 shows that such discs exist for minimal analytic hypersurfaces in $\mathbf{C}^3$.

Example 5.23. A real analytic hypersurface in $\mathbf{C}^2$ can have a nonconstant analytic disc of defect 1. Let M be the hypersurface in $\mathbf{C}^2$ given by $\operatorname{Im} w = 2|z|^2+2z\bar{z}^2+2\bar{z}z^2+|z|^4+z^2+\bar{z}^2$. The disc $A_0(\zeta) = (\zeta-1, 0)$ is attached to M and is of defect 1. However, since this hypersurface is strictly pseudoconvex in a neighborhood of 0, there cannot be a small disc of defect 1 through 0, since for any small disc $A(\zeta)$ we have $A_*(\partial/\partial\theta|_{\theta=0}) \notin T_0^c M$, which implies A is not of defect 1, by Corollary 5.2.

6. Analytic discs attached to CR manifolds

Let M be a smooth manifold in $\mathbf{C}^N$ of codimension l. For each $p \in M$, let $T_p^c M$ be the real subspace of $T_p M$ invariant under J, the complex involution of $\mathbf{C}^N$. Recall that M is a *CR manifold* if $\dim_{\mathbf{R}} T_p^c M$ is independent of p. Since $\dim_{\mathbf{R}} T_p^c M$ is necessarily even, if M is CR and $2n$ is the fiber dimension of the real bundle $T^c M$, then we shall say that M is of CR dimension n.

For the rest of this section we shall assume that M is a CR manifold in $\mathbf{C}^N$ of codimension l and CR dimension n. The following local characterization, whose proof is elementary, was given in [2].

Proposition 6.1 [2]. *Let M be a CR manifold as above, $p_0 \in M$. Then there exist holomorphic coordinates in $\mathbf{C}^N$ near p_0, vanishing at p_0, a generic submanifold M_1 in $\mathbf{C}^{N_1}$, where $N_1 = 2N-n-l$, and $N-n-l$ CR functions $\psi_1, \cdots, \psi_{N-n-l}$ on M_1 near 0, vanishing at 0, such that in a neighborhood of p_0 in $\mathbf{C}^N$, M is a graph of a CR mapping over M_1, i.e.,*

$$M = \{(Z, \psi(Z)) \in \mathbf{C}^N : Z \in M_1\}, \tag{6.2}$$

where $\psi = (\psi_1, \cdots, \psi_{N-n-l})$. The natural projection of M onto M_1 is a CR diffeomorphism, i.e., it carries $T^c M$ onto $T^c M_1$, the bundle for M_1, and is invariant under J. Finally, if M is real analytic, then the holomorphic coordinates can be chosen so that $\psi \equiv 0$.

Using the results of §2 and Proposition 6.1, one can check that if ε is sufficiently small, then again $\mathscr{A}_{p_0,\varepsilon}(M)$, the set of all analytic discs attached to M through p_0 and within ε distance of the constant disc $\langle p_0 \rangle$ forms an infinite-dimensional manifold. Let M_1 and Ψ be as in (6.2), and A_1 an analytic disc attached to M_1 and sufficiently close to the constant disc $\langle 0 \rangle$. By the approximation theorem [4], since each ψ_j is CR, the functions $\psi_j(A_1(\zeta))$ extends holomorphically from S^1 to $\bar{\Delta}$. Hence $A = (A_1(\zeta), \psi(A_1(\zeta))) \in \mathscr{A}_{p_0,\varepsilon}(M)$ if ε is sufficiently small. Conversely, any $A \in \mathscr{A}_{p_0,\varepsilon}(M)$ is obtained this way. Hence for $r > 0$ sufficiently small, there is a diffeomorphism

$$(6.3) \qquad \mathscr{A}_{0,r\varepsilon}(M_1) \ni A_1 \mapsto (A_1, \Psi(A_1)) \in \mathscr{A}_{p_0,\varepsilon}(M)$$

of $\mathscr{A}_{0,r\varepsilon}(M_1)$ onto a neighborhood of $\langle p_0 \rangle$ in $\mathscr{A}_{p_0,\varepsilon}(M)$.

As in §2, we let $\Sigma(M)$ denote the conormal bundle to M in $\mathbf{C}^N$. For $p \in M$, the restriction map (i.e., pullback) $r\colon \Sigma_p(M) \to T_p^*M$ given by $\langle r(\alpha), X\rangle = \langle \alpha, X\rangle$, $X \in T_pM$, is in general not injective. However it can be checked that M is CR if and only if the dimension of the image of r is constant. For $A \in \mathscr{A}_{p_0,\varepsilon}(M)$, we let V_A denote the set of all discs of the form $B(\zeta) = (A(\zeta), \lambda(\zeta))$, attached to $\Sigma(M)$, i.e., those analytic discs for which $A(S^1) \subset M$ and $\sum_j \lambda_j(\zeta)\, dZ_j \in \Sigma_{A(\zeta)}(M)$, $\zeta \in S^1$. For $\zeta \in S^1$, let $V_A(\zeta) \subset \Sigma_{A(\zeta)}M$ be defined by $V_A(\zeta) = \{\alpha \in \Sigma_{A(\zeta)}M\colon \alpha = \sum_j \lambda_j(\zeta)\, dZ_j,\ (A(\zeta), \lambda(\zeta)) \in V_A\}$. We define the *defect of A at ζ* by

$$(6.4) \qquad \operatorname{def}_\zeta A = \dim_{\mathbf{R}} r(V_A(\zeta)).$$

Proposition 6.5. *If $A = (A_1, \Psi(A_1))$ as in* (6.3), *then* $\operatorname{def}_\zeta A = \operatorname{def}_\zeta A_1$ *for every $\zeta \in S^1$. In particular, if ε is sufficiently small, then* $\operatorname{def}_\zeta A$ *is independent of ζ.*

Proof. Since the components of Ψ are CR functions on M_1, we may extend them to a neighborhood of 0 in $\mathbf{C}^{N_1}$ so that $\bar\partial \Psi_j|_{M_1} = 0$ for all j. We write $Z = (Z', Z'')$, with $Z' \in \mathbf{C}^{N_1}$ and $Z'' \in \mathbf{C}^{N-N_1}$. If M_1 is given near 0 by $\rho_j(Z') = 0$, $j = 1, \cdots, l_1$, then a disc $B(\cdot) = (A(\cdot), \lambda(\cdot)) \in V_A$ with $A = (A_1, \psi(A_1))$ satisfies for all $\zeta \in S^1$

$$(6.6) \qquad \sum_{j=1}^{N} \lambda_j(\zeta)\, dZ_j = \sum_{j=1}^{l_1} t_j(\zeta)\partial\rho_j(A_1(\zeta)) + \sum_{j=1}^{N-N_1} h_j(\zeta)(dZ_j'' - \partial\Psi_j(A_1(\zeta))),$$

where the t_j are real-valued functions, and the h_j and λ_j extend holomorphically to $\overline{\Delta}$. Since the pullback to M of the form defined by the second sum on the right-hand side of (6.6) vanishes identically, the proposition follows from the definition of the defect. q.e.d.

Note that the defect of A_1 is the one defined in §3, since M_1 is generic.

We may now state an analogue of Theorem 1 in the context of CR manifolds. Let $\mathscr{F}_{\zeta_0}$ and $\mathscr{F}$ be defined as in (4.1) and (4.2).

Theorem 1′. *Let M be a CR submanifold of $\mathbf{C}^N$, $p_0 \in M$, and $\zeta_0 \in S^1\setminus\{1\}$. If $\varepsilon > 0$ is sufficiently small, and $A \in \mathscr{A}_{p_0,\varepsilon}(M)$, then*

(i) $\mathscr{F}'_{\zeta_0}(A)T_A\mathscr{A} = r(V_A(\zeta_0))^{\perp}$,

(ii) $\mathscr{G}'(A)T_A\mathscr{A} = r(V_A(1))^{\perp}$,

where orthogonality is taken in the sense of duality between TM and T^*M.

The proof of Theorem 1′ follows from that of Theorem 1 and the results of this section. Details are left to the reader.

7. Applications to extension of holomorphic functions across a hypersurface and propagation of analyticity

We recall the following result, which is a consequence of the work of Diederich and Fornaess [9].

Proposition 7.1. *Let M be a smooth pseudoconvex hypersurface and $A \in \mathscr{A}_{p_0,\varepsilon}(M)$. If ε is sufficiently small, and $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$, then $A(\overline{\Delta}) \subset M$.*

Proof. Assume that M is a subset of $\partial\Omega$, where Ω is a pseudoconvex open set. The proposition follows by applying the Hopf Lemma to the function $\Delta \ni \zeta \mapsto \rho(A(\zeta))$, where ρ is a plurisubharmonic defining function for Ω as constructed in Diederich and Fornaess [9]. q.e.d.

We prove here some related results for nonpseudoconvex boundaries.

Theorem 6. *Let M be a hypersurface in $\mathbf{C}^N$ and $p_0 \in M$. Then there exists $\varepsilon > 0$ such that if there is $A \in \mathscr{A}_{p_0,\varepsilon}(M)$ with $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$ but $A_*(\partial/\partial\theta|_{\theta=\theta_0}) \notin T^c_{A(e^{i\theta_0})}M$ for some $\theta_0 \in \mathbf{R}$, then every holomorphic function defined on one side of M extends holomorphically to a full neighborhood of p_0 in $\mathbf{C}^N$.*

Proof. Let ε be given by Theorem 4. If there is a disc A as in the statement of the theorem, then $\operatorname{def} A = 0$ by Corollary 5.2. Hence Theorem 4(i) must hold, i.e., every function holomorphic on one side of M extends holomorphically across p_0. q.e.d.

Theorem 7. *Let M be a hypersurface in $\mathbf{C}^N$ and $p_0 \in M$. Then there exists $\varepsilon > 0$ such that if u is a CR function on M and $A_0 \in \mathscr{A}_{p_0,\varepsilon}(M)$ with $A_{0*}(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$, such that u extends holomorphically to one side of M near $p_1 = A_0(\zeta_1)$ for some $\zeta_1 \in S^1$, then u also extends holomorphically to the same side of M near p_0.*

Proof. We may assume that A_0 is of defect 1, since otherwise, by Theorem 4, every CR function extends to both sides of M near p_0. Let ρ be a local defining function for M near p_0, and $v_1 \in T_{p_1}\mathbf{C}^N$ nontangent to M and such that v_1 points to the side of M to which u extends holomorphically near p_1. Hence we may assume

$$\mathrm{Re}\langle \partial\rho(p_1), v_1\rangle > 0. \tag{7.2}$$

Let $\phi \in C_0^\infty(\mathbf{C}^N)$ with small support concentrated near p_1 and vanishing identically near p_0, $0 \le \phi(Z) \le 1$, and $\phi(p_1) = 1$. We let $\rho(Z, \eta)$ be the perturbation of ρ given by

$$\rho(Z, \eta) = \rho(Z - \eta\phi(Z)v_1), \tag{7.3}$$

and M_η the hypersurface defined by $\rho(Z, \eta) = 0$. Note that for $\eta > 0$ small, M_η coincides with M in any open set in M on which ϕ vanishes, and is contained on the side of M where u extends holomorphically near p_1. Hence, the restriction of u together with its holomorphic extension to M_η is a CR function on M_η. We shall show first that there is a disc A_η attached to M_η through p_0 such that $A_{\eta *}(\partial/\partial\theta|_{\theta=0}) \notin T^c_{p_0}M_\eta$, or more precisely,

$$\mathrm{Re}\left\langle \partial\rho(p_0), iA_{\eta *}\left(\frac{\partial}{\partial\theta}|_{\theta=0}\right)\right\rangle > 0, \tag{7.4}$$

i.e., $A'_\eta(1)$ points to the same side of M as v_1.

To construct a family of discs A_η attached to M_η for $\eta \in (-\eta_0, \eta_0)$, with A_0 the given disc, we shall use an argument similar to that in §2. That is we use the implicit function theorem for Banach spaces. For this, let $\mathscr{D}^N_{p_0, \varepsilon}$ be defined by (2.3). For $\eta_0 > 0$ sufficiently small, consider the map $R\colon \mathscr{D}^N_{p_0,\varepsilon} \times (-\eta_0, \eta_0) \to C^{1,\alpha}_*(S^1)$ defined by

$$\begin{aligned}\mathscr{D}^N_{p_0,\varepsilon} \times (-\eta_0, \eta_0) \ni (A, \eta) \mapsto R(A, \eta)(\cdot) \\ = \rho(A(\cdot), \eta) \in C^{1,\alpha}_*(S^1).\end{aligned} \tag{7.5}$$

To solve the equation $R(A_\eta, \eta) \equiv 0$ with $\eta \mapsto A_\eta$ smooth near $\eta = 0$ and A_0 the given disc, it suffices to show that the derivative $R'_A(A_0, 0)\colon \mathscr{D}^N_0 \to C^{1,\alpha}_*(S^1)$ has a right continuous inverse. For $\widehat{A} \in \mathscr{D}^N_0$, we have

$$R'_A(A_0, 0)\widehat{A} = \rho_Z(A_0)\widehat{A} + \rho_{\overline{Z}}(A_0)\overline{\widehat{A}}. \tag{7.6}$$

Invertibility then follows by an argument similar to that used in §2, since if ε is sufficiently small, A_0 is close to the constant disc $\langle p_0\rangle$. Note that the map $\eta \mapsto A_\eta$ is not unique due to its dependence on the choice of the matric C satisfying (2.10) as in §2.

We shall show that (7.4) holds for η small. We write

$$A_\eta = A_0 + \eta \dot{A} + O(\eta^2) \tag{7.7}$$

with $\dot{A} \in \mathscr{D}_0^N$. Differentiating the identity $R(A_\eta, \eta) \equiv 0$ and using (7.5) and (7.6) we obtain, after putting $\eta = 0$,

$$\mathrm{Re}\langle \partial\rho(A_0(\zeta)), \dot{A}(\zeta)\rangle = \mathrm{Re}\langle \partial\rho(A_0(\zeta)), \phi(A_0(\zeta))v_1\rangle. \tag{7.8}$$

Since A_0 is of defect 1, there exists $\nu(\cdot) \in C^{1,\alpha}(S^1)$ real valued and nowhere vanishing (say $\nu(\zeta) > 0$) such that the mapping

$$S^1 \ni \zeta \mapsto \nu(\zeta)\partial\rho(A(A_0(\zeta)) \in \Lambda^{1,0}\mathbf{C}^N$$

extends holomorphically to $\overline{\Delta}$. Hence the mapping

$$S^1 \ni \zeta \mapsto h(\zeta) = \langle \nu(\zeta)\partial\rho(A_0(\zeta)), \dot{A}(\zeta)\rangle \in \mathbf{C}$$

also extends holomorphically to $\overline{\Delta}$. We also denote by $h(\zeta)$ its holomorphic extension. We will apply the Hopf Lemma to the harmonic function $\mathrm{Re}\, h(\zeta)$. We claim first that $\mathrm{Re}\, h(\zeta) \geq 0$. For this, we observe that by (7.8) and the reality of $\nu(\zeta)$, we have

$$\mathrm{Re}\, h(\zeta) = \nu(\zeta)\phi(A_0(\zeta))\,\mathrm{Re}\langle \partial\rho(A_0(\zeta)), v_1\rangle, \tag{7.9}$$

which implies the claim if the support of ϕ is sufficient close to p_1. Since $h(1) = 0$, from the Hopf Lemma it follows that $\mathrm{Re}\,\partial h(1)/\partial\xi < 0$, so that $\partial\, \mathrm{Im}\, h(e^{i\theta})/\partial\theta|_{\theta=0} < 0$ by the Cauchy-Riemann equations. Since $\dot{A}(1) = 0$, we obtain

$$\mathrm{Im}\left\langle \partial\rho(p_0), \frac{\partial \dot{A}}{\partial\theta}(e^{i\theta})|_{\theta=0}\right\rangle < 0. \tag{7.10}$$

Using (7.7), (7.10), and the fact that A_0 is of defect 1 yields (7.4) for $\eta > 0$ small.

We claim that for η sufficiently small, the images of $\overline{\Delta}$ by all the discs attached to M_η near A_η cover, near p_0, the side of M_η defined by $A'_\eta(1)$, which is the same as the side defined by v_1. Indeed, by (7.4) and Theorem 1(ii), the disc A_η, for $\eta > 0$ and small, is of defect 0 for the hypersurface M_η. (Note that this does not imply that M_η is minimal at p_0. Indeed there need not exist *arbitrarily small* discs of defect 0 attached to M_η through p_0.) The claim follows by an argument similar to that outlined in Remark 4.28.

Remark 7.11. If M is a hypersurface which is not minimal at p_0, then there exists a germ at p_0 of a complex hypersurface $\mathscr{H}$ in $\mathbf{C}^N$ with $\mathscr{H} \subset M$. It is shown in [18] (see also Hanges and Treves [11] and [10]) that $\mathscr{H}$ propagates one-sided extendibility of CR functions on M. In this case, propagation of one-sided extendibility along the boundary of an analytic disc is equivalent to propagation along $\mathscr{H}$ since all $A \in \mathscr{A}_{p_0,\varepsilon}(M)$ are actually attached to $\mathscr{H}$, the boundaries of all such discs cover $\mathscr{H}$, and necessarily $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$ for all such discs. It should be noted, however, that Theorem 7 is a stronger result, since it implies that the side of extendibility is also preserved in the minimal case. Note also that if M is minimal at p_0, and the condition $A_{0*}(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$ is dropped in the statement of Theorem 7, then propagation of holomorphic extendibility to the same side need not hold.

We shall give here, as an application of Theorems 6 and 7, a completely different proof of one of the main results in [19].

Thereom 8 [19]. *Let M be a hypersurface in $\mathbf{C}^N$, and $p_0 \in M$. Assume that there is a CR submanifold V of $\mathbf{C}^N$ through p_0 with $V \subset M$, $T_{p_0}V \subset T^c_{p_0}M$, and V minimal at p_0. Then the following hold:*

(i) *If there is no open neighborhood U of p_0 in V such that $TU \subset T^cM$, then every CR function on M near p_0 extends holomorphically to a full neighborhood of p_0 in $\mathbf{C}^N$.*

(ii) *There exists a neighborhood U of p_0 in V such that any CR function u on M extending holomorphically to one side of M near $p_1 \in U$, also extends to the same side of M near p_0.*

Proof. To prove (i), we let $\mathscr{A}_{p_0,\varepsilon}(V)$ denote the manifold of all discs attached to V through p_0 of norm less than ε. Note that $\mathscr{A}_{p_0,\varepsilon}(V) \subset \mathscr{A}_{p_0,\varepsilon}(M)$, and by the assumption, for every $A \in \mathscr{A}_{p_0,\varepsilon}(V)$ we have $A_*(\partial/\partial\theta|_{\theta=0}) \in T^c_{p_0}M$. We shall prove the following lemma. Then Theorem 8(i) will follow from Theorem 4.

Lemma 7.12. *Under the assumption* (i) *of Theorem* 8, *for every $\varepsilon > 0$, sufficiently small, there is $A \in \mathscr{A}_{p_0,\varepsilon}(V)$ with $\operatorname{def}_M A = 0$, where def_M denotes the defect of A, regarded as an element of $\mathscr{A}_{p_0,\varepsilon}(M)$.*

Proof of Lemma. Let $\varepsilon > 0$ be sufficiently small and let p_1 be close to p_0 and satisfying $T_{p_1}V \not\subset T^c_{p_1}M$. Since V is minimal at p_0, if p_1 is sufficiently close to p_0, then by Theorem 1′ and its consequences in §6 there is $A \in \mathscr{A}_{p_0,\varepsilon}(V)$ with $\operatorname{def}_V A = 0$ and $A(-1) = p_1$. As in §2, we denote by $\mathscr{F}_{-1}$ the evaluation map at -1. By Theorem 1′ we obtain

$$T_{p_1}V = \mathcal{F}'_{-1}T_A\mathcal{A}_{p_0,\varepsilon}(V) \subset \mathcal{F}'_{-1}T_A\mathcal{A}_{p_0,\varepsilon}(M) \subset T_{p_1}M.$$

Note that we must have either $\mathcal{F}'_{-1}T_A\mathcal{A}_{p_0,\varepsilon}(M) = T^c_{p_1}M$ or $\mathcal{F}'_{-1}T_A\mathcal{A}_{p_0,\varepsilon}(M) = T_{p_1}M$, depending on whether $\operatorname{def}_M A = 1$ or $\operatorname{def}_M A = 0$, by Theorem 1(ii). Since by assumption $T_{p_1}V \not\subset T^c_{p_1}M$, we must have $\operatorname{def}_M A = 0$, which proves the lemma.

This completes the proof of part (i) of Theorem 8. To prove (ii), we note that if u extends to one side of M near $p_1 \in V$ sufficiently close to p_0, we may consider a disc $A \in \mathcal{A}_{p_0,\varepsilon}(V)$ with $A(-1) = p_1$ (by Theorem 1′ as in the proof of Lemma (7.12)). Then applying Theorem 7 yields the desired result, which completes the proof of Theorem 8. q.e.d.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
UNIVERSITÉ PARIS VI, PARIS