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Let \mathfrak{g} be a complex semisimple Lie algebra and \mathfrak{g}_0 a real form of \mathfrak{g} , i.e. $\mathfrak{g} = \mathfrak{g}_0 + \sqrt{-1}\mathfrak{g}_0$. The form \mathfrak{g}_0 is called *quasi-split* if there is a subalgebra $\mathfrak{b} \subset \mathfrak{g}_0$ such that $\mathfrak{b} + \sqrt{-1}\mathfrak{b}$ is a Borel subalgebra of \mathfrak{g} , and \mathfrak{g}_0 is said to be *full rank* if $\text{rank } \mathfrak{g} = \text{rank } \mathfrak{f}$, where \mathfrak{f} is a maximal compact subalgebra of \mathfrak{g}_0 . The purpose of this note is to show that these two properties characterize a unique real form of \mathfrak{g} .

THEOREM. *Every complex semisimple Lie algebra \mathfrak{g} contains a full rank, quasi-split real form, which is unique up to isomorphism.*

PROOF. We first show existence. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g}' and Δ a set of roots with respect to \mathfrak{h} . If \mathfrak{h}_0 is the real subspace of \mathfrak{h} spanned by Δ , then a real form \mathfrak{g}_0 is defined, up to isomorphism, by the choice of a complex-conjugate involution σ of \mathfrak{g} which leaves \mathfrak{h}_0 invariant. (The reader is referred to [1, Chapter IX] for proof of this and other facts about involutions and the classification of real forms, and to [2, Chapter V] for known facts about root systems and Weyl groups.) We write $\sigma|_{\mathfrak{h}_0}$ for the restriction of σ to \mathfrak{h}_0 . \mathfrak{g}_0 is quasi-split iff $\sigma|_{\mathfrak{h}_0}$ leaves invariant an open chamber of the Weyl group W acting on \mathfrak{h}_0 . Let Π be a set of simple positive roots for Δ , and let τ be the unique element of W which maps Π to $-\Pi$. Then $-\tau$ leaves the fundamental chamber invariant. Set $\sigma|_{\mathfrak{h}_0} = -\tau$ and extend to a complex-conjugate involution σ of \mathfrak{g} . Then the real form $\mathfrak{g}_0 = \{x \in \mathfrak{g} | \sigma x = x\}$ is quasi-split.

We claim \mathfrak{g}_0 is full rank. To see this, let $\mathfrak{h}_0^{+1}, \mathfrak{h}_0^{-1}$ be respectively the $+1$ and -1 eigenspaces of σ in \mathfrak{h}_0 . Put $i = \sqrt{-1}$. Then $\mathfrak{h}_0^{+1} + i\mathfrak{h}_0^{-1}$ is a Cartan subalgebra of \mathfrak{g}_0 , and $i\mathfrak{h}_0^{+1} + \mathfrak{h}_0^{-1}$ is a Cartan subalgebra of the associated compact form. One knows [1, p. 335, Theorem 3.2] that \mathfrak{g}_0 is full rank iff the involution of \mathfrak{h} defined by $\mu|_{i\mathfrak{h}_0^{-1}} = 1, \mu|_{\mathfrak{h}_0^{+1}} = -1$ is inner. But this follows immediately from [1, Proposition 2.5] since $\tau \in W$, and the linear extension of τ to all of \mathfrak{h} agrees with μ on $i\mathfrak{h}_0^{+1} + \mathfrak{h}_0^{-1}$.

To prove uniqueness, suppose \mathfrak{g}'_0 is another quasi-split, full rank real form defined by an involution σ' which leaves \mathfrak{h}_0 invariant. Then since \mathfrak{g}'_0 is quasi-split $\sigma'|_{\mathfrak{h}_0}$ leaves invariant a chamber of W in \mathfrak{h}_0 , and

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we may assume, up to conjugation of σ , that it is the fundamental chamber. By the above, since \mathfrak{g}_0 is full rank $\sigma|_{i\mathfrak{h}_0^{-1} + i\mathfrak{h}_0^{-1}}$ (extended linearly to \mathfrak{h}_0) is in W , and hence $-\sigma|_{\mathfrak{h}_0} = \tau$, since τ is the unique element of the Weyl group which maps Π to $-\Pi$. We now show that there exists η , an inner automorphism of \mathfrak{g} , such that $\sigma' = \eta^{-1}\sigma\eta$. Let $\{X_\alpha | \alpha \in \Pi\}$ be the simple root vectors of a Weyl basis of \mathfrak{g} . Then σ and σ' are completely determined by the complex numbers c_α and c'_α , where $\sigma X_\alpha = c_\alpha X_\alpha$ and $\sigma' X_\alpha = c'_\alpha X_\alpha$. Let η be the linear automorphism defined by $\eta|_{\mathfrak{h}}$ is the identity, and $\eta X_\alpha = \sqrt{c'_\alpha} / \sqrt{c_\alpha} X_\alpha$ for α simple. Then we claim $\eta^{-1}\sigma\eta X_\alpha = \sigma' X_\alpha$. Indeed,

$$\eta^{-1}\sigma\eta X_\alpha = \eta^{-1}\sigma\sqrt{c'_\alpha} / \sqrt{c_\alpha} X_\alpha = \eta^{-1}\sqrt{(c_\alpha c'_\alpha)} X_{\sigma\alpha}.$$

Since σ permutes the positive simple roots, $\sigma\alpha$ is again simple, and

$$\eta^{-1}\sqrt{(c_\alpha c'_\alpha)} X_{\sigma\alpha} = \sqrt{(c_\alpha c_{\sigma\alpha})} \sqrt{(c'_\alpha / c'_{\sigma\alpha})} X_{\sigma\alpha}.$$

But $\sigma^2 = 1$ and similarly for σ'^2 . Therefore

$$\sqrt{(c_\alpha c_{\sigma\alpha})} \sqrt{(c'_\alpha / c'_{\sigma\alpha})} X_{\sigma\alpha} = \sqrt{(c'_\alpha / c'_{\sigma\alpha})} X_{\sigma\alpha} = c'_\alpha X_{\sigma\alpha}$$

which proves that $\eta^{-1}\sigma\eta = \sigma'$. Hence \mathfrak{g}'_0 is isomorphic to \mathfrak{g}_0 .

The author wishes to thank Bertram Kostant for the idea of a proof not involving classification.

REMARK. One may construct a full rank quasi-split \mathfrak{g}_0 as follows. Any $\phi \in \Delta^+ = \{\phi \in \Delta | \phi > 0\}$ can be written $\phi = \sum_{\alpha \in \tau} n_\alpha \alpha$, where n_α is a nonnegative integer [1, p. 331, Theorem 2.2]. Write $\text{ht}(\phi) = \sum_{\alpha \in \tau} n_\alpha$, the *height of* ϕ . Let

$$\Delta_1 = \{\phi \in \Delta^+ | \text{ht}(\phi) \text{ is even}\}$$

and

$$\Delta_2 = \{\phi \in \Delta^+ | \text{ht}(\phi) \text{ is odd}\}.$$

Then $\mathfrak{g}_0 = i\mathfrak{h}_0 + \sum_{\phi \in \Delta_1} \mathcal{R}(X_\phi - X_{-\phi}) + \sum_{\phi \in \Delta_1} i\mathcal{R}(X_\phi + X_{-\phi}) + \sum_{\psi \in \Delta_2} \mathcal{R}(X_\psi - X_{-\psi}) + \sum_{\psi \in \Delta_2} i\mathcal{R}(X_\psi + X_{-\psi})$ is full rank, quasi-split, where $\mathcal{R}(X)$ denotes the real subspace spanned by $X \in \mathfrak{g}$. Note that

$$\mathfrak{f} = i\mathfrak{h}_0 + \sum_{\phi \in \Delta^+} \mathcal{R}(X_\phi - X_{-\phi}) + \sum_{\phi \in \Delta_1} i\mathcal{R}(X_\phi + X_{-\phi})$$

is a maximal compact subalgebra, and Δ_1 is a root system for \mathfrak{f} . Hence \mathfrak{g}_0 is completely determined by the condition that no simple root be in Δ_1 .

We list here the full rank, quasi-split real forms for the complex simple Lie algebras, using the notation of [1, p. 354].

	\mathfrak{g}	\mathfrak{g}_0	\mathfrak{f}
a_{2q-1}	$\mathfrak{sl}(2q, \mathbf{C})$	$\mathfrak{su}(q, q)$	$\mathfrak{su}(q) \oplus \mathfrak{su}(q) \oplus \mathbf{R}$
a_{2q}	$\mathfrak{sl}(2q+1, \mathbf{C})$	$\mathfrak{su}(q+1, q)$	$\mathfrak{su}(q+1) \oplus \mathfrak{su}(q) \oplus \mathbf{R}$
b_q	$\mathfrak{so}(2q+1, \mathbf{C})$	$\mathfrak{so}(q+1, q)$	$\mathfrak{so}(q+1) \oplus \mathfrak{so}(q)$
c_q	$\mathfrak{sp}(q, \mathbf{C})$	$\mathfrak{sp}(q, \mathbf{R})$	$\mathfrak{u}(q)$
d_{2q}	$\mathfrak{so}(4q, \mathbf{C})$	$\mathfrak{so}(2q, 2q)$	$\mathfrak{so}(2q) \oplus \mathfrak{so}(2q)$
d_{2q+1}	$\mathfrak{so}(4q+2, \mathbf{C})$	$\mathfrak{so}(2q+2, 2q)$	$\mathfrak{so}(2q+2) \oplus \mathfrak{so}(2q)$
e_6		E_{II}	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$
e_7		E_{V}	$\mathfrak{su}(8)$
e_8		E_{VIII}	$\mathfrak{so}(16)$
f_4		E_{I}	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
g		G	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

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