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PARAMETRICES FOR THE BOUNDARY LAPLACIAN AND RELATED HYPOELLIPTIC DIFFERENTIAL OPERATORS

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1. Introduction. This is a discussion of some recent results on parametrices (approximate inverses) and estimates for a general class of second-order hypoelliptic partial differential operators. The operators to be considered are ones whose principal part is of the form $\sum a_{jk}X_jX_k$, where the X_j are real, smooth vector fields, and (a_{jk}) is a positive definite quadratic form. An example of such an operator which arises in several complex variables is the Laplacian $\Box_b$ of the boundary Cauchy-Riemann operator $\bar{\partial}_b$. Here the X_j are the real and imaginary parts of the tangential holomorphic vector fields. (See, e.g., Folland and Kohn [7] for an expository treatment of the Cauchy-Riemann complex and its boundary analogue.)

Kohn [15] proved that the operator $\Box_b$ is hypoelliptic under appropriate geometric conditions. The proof uses the observation that while the X_j themselves do not span the tangent space, the missing direction is obtained as a commutator $[X_j, X_k]$. A significant generalization of this regularity result was later obtained by Hörmander [12] (see also Kohn [16]). His theorem states that if $X_0, X_1, \cdots, X_n$ are real, smooth vector fields on a manifold M such that they and their commutators (up to some fixed length) span the tangent space, then the operator

$$\mathscr{L} = X_0 + \sum_{j=1}^{n} X_j^2$$

is hypoelliptic, i.e., $\mathscr{L}u = f, f \in C^\infty(U)$, implies $u \in C^\infty(U)$ for any open set U. The work to be discussed here involves the construction of a class of integral operators which contain parametrices for operators like $\Box_b$ or $\mathscr{L}$. The methods involve approximating the vector fields X_j by left-invariant vector fields on a nilpotent Lie

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group. This technique was employed by Folland and Stein **[8]** to find a parametrix for $\Box_b$ and was then extended to operators like $\mathscr{L}$ by Stein and the author **[18]**.

2. Theorems on parametrices and estimates. The results discussed here are proved in **[18]**.

In order to obtain the best possible smoothness properties for solutions of $\mathscr{L}u = f$, it is desirable to define new Sobolev spaces which take into account the differences in the directions determined by $X_0, X_1, \cdots, X_n$ and their higher commutators. Thus X_j, $1 \leqq j \leqq n$, will have weight one, while X_0 and each $[X_j, X_k]$, $1 \leqq j, k \leqq n$, has weight two. (For $\Box_b$, $X_0 \equiv 0$ and X_j, $1 \leqq j \leqq 2l = n$, is defined by $Z_j = \frac{1}{2}(X_j - i X_{j+l})$, where the Z_j range over a spanning set of tangential holomorphic vector fields.) For the monomial $\mathscr{X} = X_{i_1} X_{i_2} \cdots X_{i_s}$ let $\sigma(\mathscr{X}) = s + t$, where t is the number of i_j's which are zero. Write r for the smallest integer such that $X_0, X_1, \cdots, X_n$ and $\{[X_{i_1}, [X_{i_2}, \cdots, X_{i_s}]\cdots]: \sigma(X_{i_1} X_{i_2} \cdots X_{i_s}) \leqq r\}$ span the tangent spaces at each point.

The new Sobolev spaces S_k^p, $1 < p < \infty$, $k = 0, 1, 2, \cdots$, are defined by

$$S_k^p = \{f \in L^p: \mathscr{X} f \in L^p \text{ for all } \mathscr{X} \text{ with } \sigma(\mathscr{X}) \leqq k\},$$

with norm given by

$$\| f \|_{S_k^p} = \sum_{\sigma(\mathscr{X}) \leqq k} \| \mathscr{X} f \|_{L^p} + \| f \|_{L^p}, \qquad \mathscr{X} = X_{i_1} X_{i_2} \cdots X_{i_s}.$$

Since all results will be local, we replace M by a relatively compact subset of itself and assume $M \subset \boldsymbol{R}^m$. The classical Sobolev spaces $L_\alpha^p(M)$ may then be defined relative to $\boldsymbol{R}^m$. An operator T is bounded from $L_{\alpha_1}^p(M)$ to $L_{\alpha_2}^p(M)$ if aTb is bounded from $L_{\alpha_1}^p(\boldsymbol{R}^m)$ to $L_{\alpha_2}^p(\boldsymbol{R}^m)$ for all $a, b \in C_0^\infty(\boldsymbol{R}^m)$. Finally, we let Λ_α denote the classical Lipschitz spaces (see, e.g., **[20]**).

The main results are then as follows. An operator T (initially defined on $C_0^\infty(M)$) will be said to be *smoothing of order* λ, λ a nonnegative integer, if T extends to a bounded operator from

$$\left.\begin{array}{lll} S_k^p(M) \text{ to } S_{k+\lambda}^p(M), & & \\ L_\alpha^p(M) \text{ to } L_{\alpha+\lambda/r}^p(M), & \alpha \geqq 0, & \\ \Lambda_\alpha(M) \cap L^p \text{ to } \Lambda_{\alpha+\lambda/r}(M) \cap L^p, & \alpha > 0, & \\ L^\infty(M) \cap L^p \text{ to } \Lambda_{\lambda/r}(M) \cap L^p, & & \end{array}\right\} \quad 1 < p < \infty.$$

THEOREM 1. *Let* $\mathscr{L} = \sum X_j^2 + X_0$. *For any* $a \in C_0^\infty(M)$, *there exist operators* P, S, *and* S' *smoothing of orders two, one and one, respectively, such that* $\mathscr{L}P = aI + S$ *and* $P\mathscr{L} = aI + S'$.

The operator P is then called a *parametrix* for $\mathscr{L}$.

As a consequence of this result we obtain regularity for solutions of the corresponding differential equation.

THEOREM 2. *Suppose* $f \in L^p(M)$, $1 < p < \infty$, *and* $\mathscr{L}f = g$. *Then the following hold for* $1 < p < \infty$.

(a) *If* $g \in L_\alpha^p(M)$, *then* $f \in L_{\alpha+2/r}^p(M)$, $\alpha > 0$.
(b) *If* $g \in \Lambda_\alpha(M)$, *then* $f \in \Lambda_{\alpha+2/r}(M)$, $\alpha > 0$.
(c) *If* $g \in L^\infty(M)$, *then* $f \in \Lambda_{2/r}(M)$.
(d) *If* $g \in S_k^p(M)$, *then* $af \in S_{k+2}^p(M)$, *for each* $a \in C_0^\infty(M)$, $k = 0, 1, \cdots$.

Analogous results hold for the operator $\square_b$ acting on q-forms of a partially complex (or CR) manifold M. Recall that $\square_b$ is defined to be the Laplacian $\square_b = \bar{\partial}_b\vartheta_b + \vartheta_b\bar{\partial}_b$, where ϑ_b is the adjoint of the boundary Cauchy-Riemann operator $\bar{\partial}_b$ with respect to some Hermitian metric on M. In order to obtain regularity a condition must be imposed on the Levi form ρ of M: For any point $\xi \in M$

$$(1) \qquad p_1 \geqq \max(q + 1, l + 1 - q) \quad \text{or} \quad p_2 \geqq \min(q + 1, l + 1 - q),$$

where p_1 is the larger of the number of eigenvalues of $\rho(\xi)$ of the same sign, and p_2 is the number of pairs of eigenvalues of opposite signs. In order to state our result, we use the same notation $S_k^p(M)$, $L_\alpha^p(\Lambda)$, etc., to denote spaces of q-forms in which each scalar component satisfies the appropriate condition.

THEOREM 3. *Suppose* (1) *is satisfied and* $\square_b f = g$, *where* f *and* g *are* q-*forms with* f *in* $L^p(M)$. *Then* (a)—(d) *of Theorem* 2 *hold for* f.

3. The construction of Folland and Stein. In [8], a parametrix for $\square_b$ is constructed on a CR manifold with a definite Levi form using a nilpotent group as model. This is the basic approach to be used in the proofs of Theorems 1, 2 and 3.

We briefly outline some features of this construction. For simplicity, we shall assume that the Levi form is positive definite and that $\square_b$ is acting on 1-forms.

The "model" space will be $\boldsymbol{H}_l = \{(z, t): z \in \boldsymbol{C}^l, t \in \boldsymbol{R}\}$ with multiplication $(z, t) \cdot (z', t') = (z + z', t + t' + 2\,\mathrm{Im}\,(z \cdot z'))$, $z \cdot z' = \sum z_j \bar{z}_j$, a nilpotent Lie group. $\boldsymbol{H}_l$ may be given a CR structure by choosing the spanning holomorphic vector fields $Z_j, j = 1, 2, \cdots, l$, to be the left invariant vector fields agreeing with $\partial/\partial z_j$ at (0, 0). Then $Z_j = \partial/\partial z_j + i\bar{z}_j\,\partial/\partial t$. Writing $Z_j = \frac{1}{2}(Y_j - iY_{j+l})$ we obtain the important commutation relations

$$(2) \qquad [Y_j, Y_k] = -4\delta_{k,j+l}\,\partial/\partial t$$

if $j \leqq l$.

A calculation (see [8]) yields the following formula for $\square_b$ on 1-forms $\sum f_j \partial\bar{z}_j$ for the metric determined by the being an orthonormal basis $\{Z_j, \bar{Z}_j, \partial/\partial t\}$:

$$\square_b(\textstyle\sum f_j d\bar{z}_j) = -\sum(\mathscr{L}_l f_j) d\bar{z}_j,$$

where for any α, $\mathscr{L}_\alpha = \frac{1}{4}\sum Y_j^2 + i\alpha\,\partial/\partial t$.

To invert $\mathscr{L}_1$, and hence $\square_b$, it is necessary to consider homogeneity properties of $\boldsymbol{H}_l$. A crucial feature is the existence of a family of automorphisms $\{\delta_r\}$, given by

$$\delta_r(z, t) = (rz, r^2 t), \qquad r \geqq 0,$$

which act as dilations. Homogeneous functions may be defined with respect to these dilations: f is *homogeneous of degree* λ if $f(\delta_r(z, t)) = r^\lambda f(z, t)$ for all $r \geqq 0$. For any such homogeneous f, $Y_j f$, $1 \leqq j \leqq 2l$, is homogeneous of degree $\lambda - 1$, while $\partial f/\partial t$ is homogeneous of degree $\lambda - 2$. We shall therefore say that Y_j (resp. $\partial/\partial t$) is a *homogenous differential operator* of degree 1 (resp. 2). In this sense, each $\mathscr{L}_\alpha$ is homogeneous of degree 2. Exploiting this homogeneity as well as the invariance of $\mathscr{L}_\alpha$ under unitary transformations among the z_j, one finds

$$\phi_\alpha = (|z|^2 - it)^{-(l+\alpha)/2}(|z|^2 + it)^{-(l-\alpha)/2}$$

as a candidate for a fundamental solution of $\mathscr{L}_\alpha$. A calculation shows that $\mathscr{L}_\alpha\phi_\alpha =$

$c_\alpha \delta_\alpha$ with $c_\alpha \neq 0$ if and only if $\alpha \neq \pm(l + 2k)$ for any nonnegative integer k. Since $\mathscr{L}_\alpha$ is left-invariant, $\mathscr{L}_\alpha(\phi_\alpha * f) = c_\alpha f$, $f \in C_0^\infty$, where $*$ denotes the group convolution.

Note that $\phi_\alpha \in C^\infty(\boldsymbol{H}_l - \{0\})$ and is homogeneous of degree $-(2l + 2) + 2$. The number $2l + 2$, called the *homogeneous dimension* of $\boldsymbol{H}_l$, arises from the transformation of volume elements $d(\delta_r(z, t)) = r^{2l+2}\, d(z, t)$. Thus one is led to study operators of the form $f \to f * k$, where $k \in C^\infty(\boldsymbol{H}_l - \{0\})$, and homogeneous of degree $-(2l + 2) + \lambda$ for some λ, $0 \leqq \lambda < 2l + 2$. For the critical case $\lambda = 0$ (in which case the convolution must be taken in the principal value sense) it has been proved (see Knapp-Stein **[14]**) that such an operator extends to a bounded operator on $L^2(\boldsymbol{H}_l)$. From this and extensions to L^p one obtains the following regularity result.

THEOREM 4 (FOLLAND-STEIN **[8]**). *Suppose* $\mathscr{L}_\alpha f = g$, $\alpha \neq \pm(l + 2k)$, k *a nonnegative integer. If* $g \in L^p$, *locally*, $1 < p < \infty$, *then* $Y_j Y_k f \in L^p$, *locally*, $1 < j, k \leqq 2l$, *and* $\partial f/\partial t \in L_p$, *locally.*

This result says that the solution f is two degrees smoother in the "good" directions, i.e., those given by the Y_j, and one degree smoother in the "bad" direction.

Now consider an arbitrary Cauchy-Riemann manifold with positive definite Levi form. It is shown that there exists a Hermitian metric on M with respect to which one can choose an orthonormal basis $Z_1, \cdots, Z_l, \bar{Z}_1, \cdots, \bar{Z}_l, T$ of holomorphic and antiholomorphic vector fields, respectively, with T real, satisfying

$$[Z_j, \bar{Z}_k] = -\tfrac{1}{2}\delta_{jk} + \text{linear terms in } Z_s, \bar{Z}_t.$$

Writing $Z_j = \frac{1}{2}(X_j - iX_{j+l})$, X_j real, we have

$$\Box_b\left(\sum f_j\, d\bar{z}_j\right) = -\sum (L_1 f_j) d\bar{z}_j + O(\bar{Z}_j f, Z_j f, f)$$

where $L_\alpha = \sum X_j^2 + i\alpha T$ and $O(\bar{Z}_j f, Z_j f, f)$ is an error term involving only f and first derivatives of f with respect to Z_j or $\bar{Z}_k$.

In order to develop a class of operators on M which approximate group convolution it is necessary to define a map on $M \times M$ which replaces the group map $(x, y) \to x^{-1}y$. For $\xi, \eta \in M$, η sufficiently close to ξ, let $\Theta(\xi, \eta) = \operatorname{Exp}(\sum u_j y_j + u_0\, \partial/\partial t) \in \boldsymbol{H}_l$ where $\eta = \exp(\sum u_j X_j + u_0 T) \cdot \xi$. Here Exp and exp are the exponential transformations on $\boldsymbol{H}_l$ and M respectively. The corresponding integral operators on M are then given essentially by

$$f \to \int f(\eta) K(\xi, \eta)\, d\eta, \tag{3}$$

where $K(\xi, \eta) = k(\Theta(\eta, \xi))$ with $k \in C^\infty(\boldsymbol{H}_l \sim \{0\})$ homogeneous of degree $-(2l + 2) + \lambda$. In particular, a parametrix for L_α, $\alpha \neq$ the exceptional values, is given essentially by the kernel $K(\xi, \eta) = \phi_\alpha(\Theta(\eta, \xi))$.

4. An outline of the program. A careful examination of the operators (3) shows that the approximation of $\Box_b$ on a CR manifold by the corresponding operator on $\boldsymbol{H}_l$ is successful because of the similarity of the commutation relations,

$$[X_j, X_k] = -4\delta_{k,j+l}\, T + \text{linear terms in } X,$$

and (2), on M and $\boldsymbol{H}_l$ respectively. Therefore, in analyzing an operator of the form

$\mathscr{L} = X_0 + \sum X_j^2$, and $\Box_b$ for nondefinite Levi form we set forth the following steps.

Step 1. Find an appropriate nilpotent Lie group N with dilations to serve as a model for M. In particular, the commutation relations of the left-invariant vector fields on N should approximate the commutation relations of the X_j in some sense.

Step 2. Find a homogeneous fundamental solution for the corresponding operator on N.

Step 3. Define a map $\Theta: M \times M \to N$, locally and use this to define a class of operators on M which contains an approximate inverse for $\mathscr{L}$ (or $\Box_b$).

It will be clear from examples that this program has to be modified. Consider the operator $\mathscr{L} = \partial^2/\partial x^2 + x^2\, \partial^2/\partial y^2$ on $M = \boldsymbol{R}^2$. Here $X_1 = \partial/\partial x$, $X_2 = x\,\partial/\partial y$, $X_0 \equiv 0$. X_1, X_2 and $[X_1, X_2] = \partial/\partial y$ are needed to span the tangent space. However, since there are no nonabelian nilpotent Lie algebras of dimension two, Step 1 already presents a difficulty. However, let s be a new variable, and write $\tilde{X}_1 = \partial/\partial x$, $\tilde{X}_2 = \partial/\partial s + x\,\partial/\partial y$ on $\boldsymbol{R}^3$. Then the Lie algebra spanned by $\tilde{X}_1$, $\tilde{X}_2$, and $[\tilde{X}_1, \tilde{X}_2] = \partial/\partial y$ is isomorphic to the three-dimensional Heisenberg algebra.

With this example in mind we modify the above program as follows. Replace Step 1 by

Step 1′. Lift the vector fields X_j to $\tilde{X}_j$ on a larger manifold $\tilde{M} = M \times \boldsymbol{R}^q$, so that the $\tilde{X}_j$ may be approximated by generators $\{Y_j\}$ of a nilpotent Lie algebra with dilations.

Now Step 2 is as before, but Step 3 must be carried out on the extended manifold $\tilde{M}$. In order to obtain results on M a new step is needed.

Step 4. Define extension and restriction operators to pass from functions on M to functions on $\tilde{M}$ and conversely. This must be done in such a way that the class of operators thus obtained on M has appropriate smoothing properties.

For simplicity, in what follows we shall always assume $X_0 \equiv 0$.

5. Associating a nilpotent group as model. We briefly describe here some of the ideas involved in Step 1′ above.

(a) Suppose $X_1, X_2, \cdots, X_n$ already span the tangent space at each point. Then the commutators $[X_j, X_k]$ are not needed for hypoellipticity. Take $\tilde{X}_j = X_j$, $j = 1, \cdots, n$, and associate the abelian nilpotent Lie algebra with generators Y_j, $j = 1, 2, \cdots, n$ satisfying $[Y_j, Y_k] = 0$, all j, k.

(b) Suppose $X_1, X_2, \cdots, X_n$ and $\{[X_j, X_k],\ 1 \leqq j < k \leqq n\}$ span the tangent space. (This is the case when M is a CR manifold with nowhere vanishing Levi form.) Then assign the 2-step nilpotent Lie algebra $\mathfrak{N}_{n,2}$ with generators $Y_1, Y_2, \cdots, Y_n$ such that $Y_1, Y_2, \cdots, Y_n$, and $\{[Y_j, Y_k],\ j < k\}$ form a linearly independent set. To construct $\tilde{X}_j$, locally, first fix a point ξ. Then find vector fields of the form $F_j = \sum_k \alpha_{jk}(\eta, t)\,\partial/\partial t_k$, where $t_1, t_2, \cdots, t_q$ are new variables and α_{jk} are smooth functions defined for η close to ξ and $t \in \boldsymbol{R}^q$ such that the $\tilde{X}_j = X_j + F$ satisfy the following. First, $\{\tilde{X}_j, [\tilde{X}_k, \tilde{X}_l],\ 1 \leqq j \leqq n,\ 1 \leqq k < l \leqq n\}$ should be linearly independent at $\tilde{\xi} = (\xi, 0) \in M \times \boldsymbol{R}^q$. Second, the $\tilde{X}_j$ together with their first commutators $[\tilde{X}_j, \tilde{X}_k]$, $1 \leqq j, k \leqq n$, should span the tangent space at ξ.

In general one finds an r-step nilpotent Lie algebra $\mathfrak{N}$ with generators $Y_1, Y_2, \cdots, Y_n$ such that the dilations $\delta_t(Y_j) = tY_j$ extend to automorphisms of $\mathfrak{N}$. These then give automorphisms of the group N corresponding to $\mathfrak{N}$. Furthermore the construction yields a one-to-one correspondence between a set of vector fields $\{\tilde{X}_{jk}\}$

spanning the tangent space at ξ and a basis $\{Y_{jk}\}$ of $\mathfrak{N}$. Here $\tilde{X}_{1k} = \tilde{X}_k$ and $Y_{1k} = Y_k$, $1 \leqq k \leqq n$, and more generally, $\tilde{X}_{jk}$ and Y_{jk} correspond to the same commutators of length j of the $\{\tilde{X}_k\}$ and $\{Y_k\}$ respectively.

As in §3, we may define the map $\Theta: \tilde{M} \times \tilde{M} \to N$, where N is the group corresponding to $\mathfrak{N}$, by

$$\Theta(\xi, \tilde{\eta}) = \operatorname{Exp}\left(\sum u_{jk}\tilde{X}_{jk}\right) \in N, \tag{4}$$

if $\tilde{\eta} = \exp(\sum u_{jk}\tilde{X}_{jk}) \cdot \xi$. For each fixed $\xi \in \tilde{M}$, the map $\tilde{\eta} \to \Theta(\xi, \tilde{\eta})$ identifies a neighborhood of ξ in $\tilde{M}$ with a neighborhood of 0 in the graded Lie group N. One of the main features of this identification is the following. In analogy with the terminology of §3, we say that a differential operator D on N is homogeneous of degree λ if $D\phi$ is a homogeneous function of degree $\alpha - \lambda$, for any homogeneous function ϕ of degree α.

THEOREM 5. *In the coordinate system around $\xi \in \tilde{M}$ given by $\Theta(\xi, \tilde{\eta}) \leftrightarrow \tilde{\eta}$*

$$\tilde{X}_j = Y_j + R_j, \qquad 1 \leqq j \leqq n,$$

where each term of the Taylor expansion of R_j around 0 is a differential operator of degree $\leqq 0$.

The proof of this Theorem is quite long and complicated. One of the main techniques is the use of Baker-Campbell-Hausdorff formula (see, e.g., [19]) to express the product of formal power series $e^x e^y$ as a power series $e^{h(x,y)}$.

6. Operators and parametrices on an extended space $\tilde{M} = M \times R^q$. We shall assume that from the given vector fields $X_1, X_2, \cdots, X_n$, on the manifold M, we constructed the extended vector fields $\tilde{X}_1, \cdots, \tilde{X}_n$ on the manifold $\tilde{M} = M \times R^q$. Since all constructions and results are local, assume $\tilde{M}$ is shrunk so that the map $\theta: \tilde{M} \times \tilde{M} \to N$ is defined everywhere.

Our first task is to define an appropriate class of integral operators of $\tilde{M}$, using homogeneous distributions on N. Recall that $\mathfrak{N}$ is spanned by $\{Y_{jk}\}$, where each Y_{jk} is a differential operator of degree j. The *homogeneous dimension* of N is defined to be $Q = \sum_j \dim V^j$, where V^j is the linear span of the Y_{jk}. As in §3 an appropriate class of operators on N are those given by

$$f \to f * k, \qquad f \in C_0^\infty(N),$$

where $k \in C^\infty(N - \{0\})$ and is homogeneous of degree $-Q + \lambda$, for some λ, $0 \leqq \lambda < Q$, and $*$ denotes group convolution on N. (For $\lambda = 0$ the integral defining the convolution must be taken in the principal value sense and k must satisfy a mean value zero condition.) Then the operators of type λ on $\tilde{M}$ are given essentially by

$$f \to \int K(\xi, \tilde{\eta}) f(\tilde{\eta})\, d\eta, \qquad f \in C_0^\infty(M),$$

where $K(\xi, \tilde{\eta}) = k(\Theta(\tilde{\eta}, \xi))$, with k as above. In the formal definition, the kernel $k(\Theta(\tilde{\eta}, \xi))$ is multiplied by cut-off functions in $\tilde{\eta}$ and ξ. Furthermore, k is not required to be homogeneous, but merely to have an expansion in terms of homogeneous functions of degrees $\geqq -Q + \lambda$. We omit these technicalities and refer the reader to [8] or [18].

We now state the main properties of operators of type λ. First, using Theorem 5 one can prove that if T is an operator of type λ on $\tilde{M}$, then $\tilde{X}_k T$ and $T\tilde{X}_k$ are operators of type $\lambda - 1$. Next, we define the new Sobolev spaces $\tilde{S}^p_k(\tilde{M})$, $1 < p < \infty$, $k = 0, 1, 2, \cdots$, on $\tilde{M}$ as in §2, using $\tilde{X}_j$.

THEOREM 6. *An operator of type λ, $0 \leqq \lambda < Q$, is smoothing of order λ on $\tilde{M}$.*

In order to establish the analogue of Theorem 1 for the operator $\tilde{\mathscr{L}} = \sum \tilde{X}_j^2$, it will suffice to find an operator $\tilde{P}$ of type 2 such that $\tilde{\mathscr{L}}\tilde{P}$ and $\tilde{P}\tilde{\mathscr{L}}$ differ from the identity on compact sets by operators of type 1. For the construction of $\tilde{P}$, the following result on fundamental solutions of homogeneous differential operators on N is needed.

THEOREM 7. *Let D be a hypoelliptic differential operator on N which is left-invariant and homogeneous of degree λ, $0 < \lambda < Q$. Then there is a unique $k \in C^\infty(N - \{0\})$ which is homogeneous of degree $-Q + \lambda$ such that*

$$(Dk) * f(x) = D(k * f)(x) = f(x), \qquad f \in C_0^\infty(N).$$

For the proof of the above theorem see Folland [**6**]. This result was obtained independently by E. M. Stein and by R. Strichartz (unpublished).

We may now construct $\tilde{P}$. Suppose $a \in C_0^\infty(\tilde{M})$ is given. Let $k \in C^\infty(N—\{0\})$ define the fundamental solution of the homogeneous operator $D = \sum_{k=1}^n Y_{1k}^2$. Then let $\tilde{P}f(\tilde{\xi}) = \int \tilde{K}(\tilde{\xi}, \tilde{\eta}) f(\tilde{\eta})\, d\tilde{\eta}$, where $\tilde{K}(\tilde{\xi}, \tilde{\eta}) = a(\tilde{\xi}) k(\theta(\tilde{\eta}, \tilde{\xi})) b(\tilde{\eta})$, with $b \in C_0^\infty(\tilde{M})$ and $b \equiv 1$ on the support of a.

The analogous construction for the parametrix for $\tilde{\Box}_b$ on $\tilde{M}$ follows the same outline, but there are several significant differences. First, $\tilde{\Box}_b$ is defined on forms, not functions. Second, $\tilde{\Box}_b$ has an important lower order term which affects the conditions for hypoellipticity. Finally, the expression for $\tilde{\Box}_b$ involves varying coefficients, which means that the approximating differential operator on the group varies with $\tilde{\xi}$. Thus, the fundamental solutions involved vary with $\tilde{\xi} \in \tilde{M}$. The reader is referred to [**18**] for details.

7. Parametrix for $\sum_{j=1}^n X_j^2$. We have now lifted the original vector fields X_j to new vector fields $\tilde{X}_j$ on a higher dimensional space $\tilde{M} = M \times \boldsymbol{R}^q$, and constructed a parametrix for the lifted operator $\sum_{j=1}^n \tilde{X}_j^2$. The next step is to find an appropriate means of restricting operators on $\tilde{M}$ to M. For this purpose we define an extension mapping E taking functions on M to functions on $\tilde{M}$ by $(Ef)(\tilde{\xi}) = f(\xi)$, where $\tilde{\xi} = (\xi, t) \in \tilde{M}$, $\xi \in M$, $t \in \boldsymbol{R}^q$. Next, choose $\phi \in C_0^\infty(\boldsymbol{R}^q)$ with $\int_{\boldsymbol{R}^q} \phi(t)\, dt = 1$, and define the restriction map R by

$$(Rf)(\xi) = \int_{\boldsymbol{R}^q} f(\xi, t)\phi(t)\, dt$$

taking functions on $\tilde{M}$ to functions on M.

An operator T on M is said to be of *type* λ if $T = R\tilde{T}E$, for some operator $\tilde{T}$ of type λ, $0 \leqq \lambda < Q$, on $\tilde{M}$. Such operators are smoothing of order λ on M. To prove this, using Theorem 6 it suffices to show that E maps $S^p_k(M)$ to $S^p_k(\tilde{M})$, $L^p_\alpha(M)$ to $L^p_\alpha(\tilde{M})$ and $\Lambda_\alpha(M)$ to $\Lambda_\alpha(\tilde{M})$, and conversely for R. Finally, if $\tilde{P}$ is chosen to be a parametrix for $\tilde{\mathscr{L}} = \sum \tilde{X}_j^2$, then it can be shown that $P = R\tilde{P}E$ is a parametrix for

$\mathscr{L}$ in the sense of Theorem 1. This completes our brief outline of the proofs of the Theorems of §2.

8. Related results and problems. There is a considerable literature on hypoelliptic operators. We mention here a few results whose relationship to the present work is not yet understood, in the hope that some readers may be interested in pursuing this.

Oleinik and Radkevich **[17]** give sufficient conditions for hypoellipticity for second-order operators in a more general class. It would be interesting to construct parametrices for these operators. Grušin **[11]** gives necessary and sufficent conditions for hypoellipticity of certain homogeneous degenerate elliptic equations. By methods quite different from these, he constructs parametrices for the hypoelliptic operators. Beals **[1]** develops a general calculus of pseudodifferential operators which contains parametrices for certain hypoelliptic degenerate elliptic operators.

Boutet de Monvel **[2]** and Boutet de Monvel and Trèves **[3]**, **[4]** study pseudodifferential operators which are formally similar to $\Box_b$. Their method of constructing parametrices involves using Fourier integral operators (see **[13]** and **[5]**) to transform the original operator into a canonical form. Trèves **[21]** gives new methods for establishing hypoellipticity.

Greiner and Stein **[9]** have used the parametrix for $\Box_b$ of **[8]** to obtain a parametrix for the Cauchy-Riemann operator $\bar{\partial}$ in the interior. Very recently, Greiner, Kohn, and Stein **[10]** used the explicit computation of **[8]** to give necessary and sufficient conditions for local solvability of the Lewy equation $(\partial/\partial z + i\bar{z}\,\partial/\partial t)u = f$.

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