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PARAMETRICES WITH C^∞ ERROR FOR $\Box_b$ AND OPERATORS OF HORMANDER TYPE

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Introduction

We construct here parametrices for certain second order hypoelliptic differential operators including the boundary Laplacian $\Box_b$ of the Cauchy-Riemann operator. These classes of operators have been studied in Folland and Stein (1974), and Rothschild and Stein (1976), where parametrices were constructed to invert the operator modulo an error which is "smoothing" (see Section 2) of any preassigned finite order. Starting with the approximate inverses defined in Rothschild and Stein (1976) via singular integrals on nilpotent Lie groups, we obtain operators which are inverses modulo an error which is infinitely smoothing. In fact, the error will be given by an operator with a smooth kernel.

A differential operator D is called <u>hypoelliptic</u> if $Df = g$ with $g \in C^\infty(U)$ implies $f \in C^\infty(U)$ for any open set U. We shall consider two classes of hypoelliptic differential operators. First, let M be a partially complex (or C-R) manifold of dimension $2\ell + 1 = m$ with a fixed Riemannian metric. (See, for example, Folland and Kohn, 1972, for relevant definitions.) $\Box_b$ is defined as the Laplacian $\theta_b\bar\partial_b + \bar\partial_b\theta_b$, where θ_b is the formal adjoint of $\bar\partial_b$, the tangential Cauchy-Riemann operator. $\Box_b$ operates on (p,q) forms on M. Kohn (1964) has proved that the following conditions on the Levi form ρ on M imply that $\Box_b$ is hypoelliptic on (p,q)-forms.

Y(q) The Levi form ρ has at least $\min(q + 1, \ell - q + 1)$ pairs of eigenvalues of opposite sign or $\max(q + 1, \ell - q + 1)$ eigenvalues of the same sign.

We shall construct a two-sided inverse, modulo an infinitely smoothing error error, S, for $\Box_b$ acting on (p,q)-forms, provided the condition $Y(q)$

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is satisfied. Parametrices for $\Box_b$ have previously been given by Boutet de Monvel (1974) and by Sjöstrand (1974). However, our operators are proved to be bounded on the appropriate L^p Sobolev spaces for $1 < p < \infty$, while the methods used to construct the parametrices of Boutet de Monvel and Sjöstrand do not seem to lend themselves to such a proof.

A key step in the proof of hypoellipticity of $\Box_b$ is the observation that the highest order term is a negative definite quadratic expression in real vector fields which, together with their commutators, span the tangent space. Hörmander (1967) generalized this idea as follows. Let $X_1, X_2, \ldots, X_n$ be real vector fields on a manifold M, $m = \dim M$, such that $X_1, X_2, \ldots, X_n$, together with their commutators

$$[X_{i_1},[X_{i_2},\ldots[X_{i_1},X_{i_j}]\ldots],] \qquad j \leq r$$

up to length r span the tangent space at each point. Then

$$\mathcal{L} = \sum_{i=1}^{n} X_i^2$$

is hypoelliptic. We shall construct an inverse, modulo an infinitely smoothing operator, for $\mathcal{L}$. To our knowledge, there is no other method known for constructing such an operator. For the special case $r = 2$, however, the pseudodifferential operators defined by Beals (preprint), Boutet de Monvel (1974) and Sjöstrand (1974) contain parametrices for many operators of the form $\mathcal{L}$. Furthermore, Grusin (1970) constructs parametrices for certain operators of this type with arbitrary r. None of these classes of parametrices is shown to preserve L^p for all $1 < p < \infty$.

We remark here that our methods could also be applied to the more general operators

$$X_0 + \sum_{i=1}^{n} X_i^2$$

considered in Hörmander (1967). For simplicity, we restrict out attention to sums of squares.

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Main Results

Let $X_1, X_2, \ldots, X_n$ be real vector fields which, together with their commutators of length $\leq r$, span the tangent space, TM. Since we are interested in local results, we may replace M by a relatively compact subset and define the following function spaces in $\mathbb{R}^m$:

$$L^\infty = \{f : f \text{ essentially bounded}\}$$
$$L^p = \{f : \|f\|_{L^p} = (\int |f|^p dx)^{1/p} < \infty\}$$

The weighted Sobolev spaces S^p_k are defined for nonnegative integers k:

$$S^p_k = \{f \in L^p : X_{i_1} X_{i_2}, \ldots, X_{i_j} f \in L^p, \quad j \leq k$$
$$\text{with norm } \|f\|^p_{S^p_n} = \sum \|X_{i_1} \ldots X_{i_j} f\|^p_{L^p}$$

The sum is taken over all indices $(i_1, \ldots, i_j)$ $j \leq k$, $0 \leq i_\ell \leq n$, with

$$X_{i_0} = I$$

We say $f \in S^p_{k,loc}$ if $\phi f \in S^p_k$ for any $\phi \in C^\infty_0$. The classical L^p Sobolev spaces are defined as usual and denoted L^p_α, $L^p_{\alpha,loc}$ (cf. Stein, 1970). The Lipschitz spaces are also defined with reference to $\mathbb{R}^m$ ($\| \ \|$ denoting Euclidean length): For $0 < \beta < 1$

$$\Lambda_{\beta,loc} = \{f : \sup |\phi f(x+y) - \phi f(x)| / \|y\|^\beta < \infty, \quad \text{all } \phi \in C^\infty_0\}$$

with norm

$$\|\phi f\|_{\Lambda_\beta} = \|\phi f\|_\infty + \sup_{\|y\|>0} |\phi f(x+y) - \phi f(x)| / \|y\|^\beta$$

$$\Lambda_{1,loc} = \{f : \phi f \in L^\infty ; \sup_{\|y\|>0} |\phi f(x+y) + \phi f(x-y) - 2\phi f(y)| / \|y\| < \infty \quad \forall \phi \in C^\infty_0\}$$

Inductively, for $\alpha > 1$,

$$\Lambda_{\alpha,loc} = \{f : \phi f \in L^\infty \text{ and } \frac{\partial}{\partial x_j} (\phi f) \in \Lambda_{\alpha-1}, \quad \phi \in C^\infty_o\}$$

with norm

$$\|\phi f\|_{\Lambda_\alpha} = \|\phi f\|_{L^\infty} + \sum_{j=1}^m \left\| \frac{\partial}{\partial x_j} (\phi f) \right\|_{\Lambda_{\alpha-1}} .$$

Let L, L' be any two of the spaces defined above. A mapping $T : L \to L'$ will be said to be bounded from L_{loc} to L'_{loc} if $\phi_1 T \phi_2$ is bounded in the appropriate norms for any $\phi_1, \phi_2 \in C^\infty_0$. T will be said to be <u>smoothing of order</u> λ if T is bounded from $L^p_{\alpha,loc}$ to $L^p_{\alpha+\lambda/r,loc}$, for all real α, from $S^p_{k,loc}$ to $S^p_{k+\lambda,loc}$, all nonnegative k, from $\Lambda_{\alpha,loc}$ to $\Lambda_{\alpha+\lambda/r,loc}$ all $\alpha > 0$, and L^∞_{loc} to $\Lambda_{\lambda/r,loc}$. T is smoothing of infinite order if it is smoothing of order λ for all $\lambda > 0$.

THEOREM 1. Let

$$\mathcal{L} = \sum_{j=1}^{n} X_j^2$$

and $\phi \in C_0^\infty(M)$ be given. Then there exist operators P and P', smoothing of order 2, and S_∞, S'_∞, smoothing of infinite order, such that for $f \in L^p(M)$, $1 < p < \infty$,

$$P\mathcal{L}f = \phi f + S_\infty f$$

$$\mathcal{L}P'f = \phi f + S'_\infty f$$

Furthermore, $P\phi = \phi P'$. The errors S_∞ and S'_∞ are integral operators with infinitely smoothing kernels.

THEOREM 2. Let $\square_b^{(q)}$ be the boundary Laplacian acting on (p,q)-forms on a CR-manifold M. Suppose $Y(q)$ is satisfied. Then, for any $\phi \in C_0^\infty(M)$, there exist $P^{(')}$ and $S_\infty^{(')}$ as in Theorem 1 such that for $f \in C_0^\infty(M)$,

$$P\square_b f = \phi f + S_\infty f$$

$$\square_b P'f = \phi f + S'_\infty f$$

(All operators here are vector valued, and f is a (p,q)form.)

In Folland and Stein (1974), Theorem 2 is proved (in the case of a definite Levi form) with $P = P_k$ and the error, S_k, is smoothing of any preassigned order k. In Rothschild and Stein (1976), Theorem 1 and the general case of Theorem 2 are proved, also with an error smoothing of any finite order.

Approximation by Operators on a Nilpotent Lie Group

We shall construct P and P' for $\mathcal{L}$ only; the construction for $\square_b$ is similar. As in Rothschild and Stein (1976), we begin by extending each X_i to a smooth vector field $\tilde{X}_i$ on a product manifold $\tilde{M} = M \times \mathbb{R}^q$. We mention here some of the important features of this extension and refer to Rothschild and Stein (1976) for details

(i) $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ and their commutators of length at most r span $T\tilde{M}$.

(ii) $\tilde{X}_j = X_j$ on functions independent of the new variables.

(iii) The commutators of length j, $1 \leq j \leq r$, satsify as few linear relations as possible, i.e., the only linear relations are those generated by the Jacobi identity and antisymmetry.

As a consequence of (iii), it is possible to identify M with a nilpotent Lie group as follows. Let N be the "free" nilpotent group of step r on n generators. Then, there is a 1-1 correspondence

$$\tilde{X}_{jk} \leftrightarrow Y_{jk} \qquad 1 \leq j \leq r \tag{3.1}$$

between a basis $\{\tilde{X}_{jk}\}$ for $T\tilde{M}$, and a basis $\{Y_{jk}\}$ of the Lie algebra n of N. Here, each $\tilde{X}_{jk}$ is a commutator of the $\tilde{X}_\ell$ of length j, and each Y_{jk} is the commutator of length j of generators $Y_1, Y_2, \ldots, Y_n$ of n. We make the convention $Y_{1k} \equiv Y_k$, $\tilde{X}_{1k} = \tilde{X}_k$, $1 \leq k \leq n$.

For $\tilde{\xi} \in \tilde{M}$ fixed, the basis $\{\tilde{X}_{jk}\}$ provides a coordinate system around $\tilde{\xi}$ via the exponential map

$$\tilde{\eta} = \exp(\textstyle\sum u_{jk}\tilde{X}_{jk})\tilde{\xi} \leftrightarrow (u_{jk}) \tag{3.2}$$

The correspondences (3.1) and (3.2) give common coordinate systems for a neighborhood of $\tilde{\xi}$ in $\tilde{M}$ and a neighborhood of 0 in N:

$$\tilde{\eta} = \exp(\textstyle\sum u_{jk}\tilde{X}_{jk})\tilde{\xi} \leftrightarrow \mathrm{Exp}(\sum u_{jk}Y_{jk}) \leftrightarrow (u_{jk})$$

We define the important map $\Theta : \tilde{M} \times \tilde{M} \to N$ by

$$\Theta(\tilde{\xi}, \tilde{\eta}) = \mathrm{Exp}(\textstyle\sum u_{jk}Y_{jk}) \tag{3.3}$$

Integral Operators of Type λ

Following Rothschild and Stein (1976), we shall define the parametrix of $\sum \tilde{X}_j^2$ on $\tilde{M}$ as an integral operator obtained from "homogeneous" kernels on N. We first review homogeneity on N.

The dilations $\delta_s : Y_j \to sY_j$, $s > 0$ extend to automorphism of n, and via the exponential map to automorphisms of the group N. We denote the automorphisms again by $x \to \delta_s(x)$, $x \in N$. A function, f, is <u>homogeneous of degree</u> α if

$$f(\delta_s(x)) = s^\alpha f(x) \quad \text{for all } x$$

A differential operator D is called homogeneous of degree λ if $D(f\circ\delta_s) = s^\lambda(Df)\circ\delta_s$, $s > 0$, and of local degree $\leqslant \lambda$ if its Taylor expansion is a formal sum of homogeneous differential operators of degrees $\leqslant \lambda$. In this sense, the differential operator

$$\mathcal{D} = \sum_{j=1}^{n} Y_j^2$$

is homogeneous of degree 2, as is, for instance, $[Y_j, Y_k]$. By a careful choice of (3.1), one can prove (see Rothschild and Stein, 1976), that for each $\tilde{\xi}$, in the coordinates (u_{jk}), we have the important correspondence

$$\tilde{X}_j = Y_j + R_j \tag{4.1}$$

where R_j is of local degree $\leqslant 0$. It easily follows that

$$\sum_{j=1}^{n} \tilde{X}_j^2 = \sum_{j=1}^{n} Y_j^2 + D \tag{4.2}$$

where D is of local degree $\leqslant 1$.

A norm function on N is a mapping $x \to \langle x\rangle$, smooth away from $x = 0$ and homogeneous of degree 1 such that

(i) $\langle x\rangle = \langle x^{-1}\rangle$

(ii) $\langle x\rangle \geqslant 0$ for all x, and $\langle x\rangle = 0$ only if $x = 0$.

We define

$$\rho(\tilde{\xi},\tilde{\eta}) = \langle\Theta(\tilde{\xi},\tilde{\eta})\rangle \tag{4.3}$$

ρ is a pseudometric in the sense that we have the "triangle" inequality

$$\rho(\tilde{\xi},\tilde{\eta}) \leqslant c(\rho(\tilde{\xi},\tilde{\zeta}) + \rho(\tilde{\zeta},\tilde{\eta})) \tag{4.4}$$

The homogeneous dimension Q of N is

$$Q = \sum_{\alpha=1}^{r} \alpha \dim n^\alpha$$

where n^α is the eigenspace of δ_s with eigenvalue s^α. Any function g, smooth except at 0 and homogeneous of degree $-Q + \beta$, $0 < \beta$ defines a distribution by group convolution: $f \to f*g$. A homogeneous function g of degree $-Q$ smooth away from the origin whose mean value, $\int_{a\leqslant\langle u\rangle\leqslant b} g(u)\,du$, is zero for all a,b defines a distribution in principal value:

$$f \to \lim_{\varepsilon\to 0} \int_{<u>>\varepsilon} g(u^{-1}v)f(u)du$$

g will then be called a <u>kernel of type</u> β, for $0 \leq \beta < Q$. By the map Θ, these kernels define operators on $\tilde{M}$ as follows. A kernel of type on $\tilde{M}$ is a function $K(\tilde{\xi},\tilde{\eta})$ on $\tilde{M} \times \tilde{M}$ such that, for every integer $s > 0$, we can write

$$K(\tilde{\xi},\tilde{\eta}) = \sum_{i=1}^{\ell(s)} a_i(\tilde{\xi})k_{\tilde{\xi}}^{(i)}(\Theta(\tilde{\eta},\tilde{\xi}))b_i(\tilde{\eta})) + E_s(\tilde{\xi},\tilde{\eta})$$

such that

(a) $E_s \in C_0^s(\tilde{M}\times\tilde{M})$

(b) $a_i, b_i \in C_0^\infty(\tilde{M})$, $i = 1,2,\ldots,\ell$

(c) the functions $u \to k_{\tilde{\xi}}^{(i)}(u)$ are kernels of type $\geq s$, depending smoothly on $\tilde{\xi}$.

An <u>operator of type</u> β, $\beta > 0$ is one given by

$$T_K f(\tilde{\xi}) = \int K(\tilde{\xi},\tilde{\eta})f(\tilde{\eta})d\tilde{\eta}$$

where K is a kernel of type β; for $\beta = 0$, an operator of type β is given by a pair (K,a), where $a \in C_0^\infty$ and

$$T_K f(\tilde{\xi}) = \lim_{\varepsilon\to 0} \int_{<\Theta(\tilde{\xi},\tilde{\eta})>>\varepsilon} K(\tilde{\xi},\tilde{\eta})f(\tilde{\eta})d\tilde{\eta} + a(\tilde{\xi})f(\tilde{\xi})$$

We review here some important facts concerning kernels K_λ of type λ and the corresponding operators T_{K_λ}.

The important inequality

$$|K_\lambda(\tilde{\xi},\tilde{\eta})| \leq C\rho(\tilde{\xi},\tilde{\eta})^{-Q+\lambda} \tag{4.5}$$

is an almost immediate consequence of the definitions. Furthermore, since $\int_{a\leq|u|\leq b}|u|^{-Q+\lambda} \leq C(b^\lambda - a^\lambda)$ (see Folland and Stein, 1974), it can be shown that

$$\begin{aligned}\int|K_\lambda(\tilde{\xi},\tilde{\eta})|d\tilde{\eta} &\leq CC_{K,\lambda}\\ \int|K_\lambda(\tilde{\xi},\tilde{\eta})|d\tilde{\xi} &\leq CC_{K,\lambda}\end{aligned} \tag{4.6}$$

where

$$C_{K,\lambda} = \sup\{\rho(\tilde{\xi},\tilde{\eta}^{\lambda}) : K_{\lambda}(\tilde{\xi},\tilde{\eta}) \neq 0\}$$

PROPOSITION 4.1. For $\lambda > 0$, T_{K_λ} extends to a bounded operator on L^p, with norm $\leq CC_{K,\lambda}$.

The proposition follows from (4.6); see, for example, Rothschild and Stein (1976). The following are fundamental properties of kernels K_λ of type λ. (See Rothschild and Stein, 1976.)

If D is a differential operator of local degree $\leq d$, then $D^{\tilde{\eta}}K_\lambda$ and $D^{\tilde{\xi}}K_\lambda$ are kernels of type $\lambda - d$ for $d \leq \lambda$. (4.7)

T_{K_λ} is smoothing of order λ, $\lambda = 0,1,2,\ldots$ (4.8)

$X_i^{\xi}K = \sum X_j^{\eta}K_j + K_0$ where each K_j and K_0 is a kernel of type λ. (4.9)

Reduction to the Case $M = \tilde{M}$

Before proceeding to the construction we shall show that it suffices to assume $\tilde{M} = M$. For, suppose there exist $\tilde{P}$ smoothing of order 2 on $\tilde{M}$ and $\tilde{S}_\infty^{(1)}$ smoothing of infinite order as in Theorem 1. Let E be the extension operator (from functions on M to functions on $\tilde{M}$) and R the restriction operator (vice versa) defined in Rothschild and Stein (1976). Then, if T is an operator which is smoothing of order α on $\tilde{M}$, RTE is smoothing of order α on M. Let $P = R\tilde{P}E$ and $S_\infty^{(1)} = R\tilde{S}_\infty^{(1)}E$. Then $R\tilde{P}\tilde{\phi}E = R\tilde{\phi}IE + R\tilde{S}_\infty E$. However, since $\tilde{\mathcal{L}}E = E\mathcal{L}$, $P\mathcal{L} = \phi I + R\tilde{S}_\infty E$ where $P = R\tilde{P}E$ and $\tilde{\phi}$ is chosen so that $R\tilde{\phi}E = \phi$.

To construct the right parametrix, let $\mathcal{L}^t$ be the transpose of $\mathcal{L}$. Then

$$\mathcal{L}^t = \sum_{j=1}^{n} X_j^{t^2}$$

and so the above construction gives a left parametrix P' for $\mathcal{L}^t$ with error operator S'_∞. We shall show that P'^t and S'^t_∞ also have the desired smoothing properties, which will complete the proof.

Construction of P

As in Folland and Stein (1974) and Rothschild and Stein (1976), we begin with the existence of a kernel k of type 2 on N such that

$\mathcal{D}k = \delta$, the delta function (cf. Folland, 1975). Then define P_1 by

$$P_1 f = \int \phi_1(\xi)k(\Theta(\eta,\xi))\phi_2(\eta)f(\eta)d\eta$$

where $\phi_1, \phi_2 \in C_0^\infty(M)$ and $\phi_2 \equiv 1$ on the support of ϕ_1. To compute the error which results from using P_1 as a first approximation to the parametrix, note that $\mathcal{L}^\eta k(\Theta(\eta,\xi)) = \delta_{\eta=\xi} - S_1'$, where S_1' is a kernel of type 1. In fact, $S_1' = -D^\eta k(\Theta(\eta,\xi))$ by (4.2). Also, $\mathcal{L}$ is equal to its transpose modulo a differential operator R of local degree $\leqslant 1$. Thus, using (4.2)

$$\begin{aligned} P_1\mathcal{L}f &= \int \phi_1(\xi)k(\Theta(\eta,\xi))\phi_2(\eta)\mathcal{L}f(\eta)d\eta \\ &= \int \phi_1(\xi)(\mathcal{L} + R)^\eta(k(\Theta(\eta,\xi))\phi_2(\eta))f(\eta)d\eta \qquad (6.1) \\ &= \phi_1(\xi)f(\xi) + T_1 f(\xi) + T_2 f(\xi) - S_1\phi_2 f \end{aligned}$$

where

$$\begin{aligned} T_1 f &= \int \phi_1(\xi)k(\Theta(\eta,\xi))\phi_2''(\eta)f(\eta)d\eta \\ T_2 f &= \int \phi_1(\xi)K_1(\Theta(\eta,\xi))\phi_2'(\eta)f(\eta)d\eta \qquad (6.2) \\ S_1\phi_2 f &= \int \phi_1(\xi)K_1^{(1)}(\Theta(\eta,\xi))\phi_2(\eta)f(\eta)d\eta \end{aligned}$$

Here ϕ_2', ϕ_2'' denote first and second order differential operators, applied to ϕ_2, $K_1^{(1)} = (R + D)^\eta k$, which is, therefore, of type 1, and K_1 is a finite sum of kernels of the form $(X_j)k$, which are of type 1. Since $\phi_1(\xi)\phi_2'(\eta)$ and $\phi_1(\xi)\phi_2''(\eta)$ have support bounded away from the diagonal $\xi = \eta$, and $k(\Theta(\eta,\xi))$ is smooth away from the diagonal, T_1 and T_2 are infinitely smoothing operators.

Now, let $\xi_0 \in M$ be fixed and suppose a sequence $\{\phi_i\}$ of functions in $C_0^\infty(M)$ is chosen such that, for suitable ε,

$$\begin{aligned} &\text{(i)} \quad \sup\{\rho(\xi_0,\xi) : \xi \in \operatorname{supp} \phi_i, \ \text{some } i \leqslant \varepsilon\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.3) \\ &\text{(ii)} \quad \phi_{i+1} \equiv 1 \ \text{on} \ \operatorname{supp} \phi_i \end{aligned}$$

Define P_i, S_i, and R_i by

$$P_i f(\xi) = \int \phi_i(\xi)k(\Theta(\eta,\xi))\phi_{i+1}(\eta)f(\eta)d\eta$$

$$S_i\phi_{i+1}f(\xi) = -\int\phi_i(\xi)K_1^{(1)}(\Theta(\eta,\xi))\phi_{i+1}(\eta)f(\eta)d\eta$$

$$R_if(\xi) = -\int\phi_i(\xi)[k(\Theta(\eta,\xi))\phi''_{i+1}(\eta) + K_1(\Theta(\eta,\xi))\phi'_{i+1}(\eta)]f(\eta)d\eta$$

where the ϕ derivatives are as in (6.2). Then, for any N

$$[P_1 + S_1P_2 + S_1S_2P_3 + \dots + S_1S_2 \dots S_{N-1}P_N]\mathcal{L}f$$

$$= [\phi_1 f + (R_1+S_1R_2+S_1S_2R_3+\dots+S_1\dots S_{N-1}R_N)]f + (S_1S_2\dots S_N\phi_{N+1})f$$

In passing to the limit, we must choose the ϕ_j with care. All will have support in a fixed small ball. Since $\phi_j \equiv 1$ near $\operatorname{supp}\phi_{j-1}$,

$$\|\phi'_j\|_{L^\infty} \to \infty$$

as $j \to \infty$, and, therefore, the norm of $R_j \to \infty$ as $j \to \infty$. In fact, the norm of R_1 may be greater than one. This explains, also, why, in general, it is not possible to take the whole error ($S_1 + R_1$ at the first stage) and iterate it to obtain a fundamental solution with no error: the series will not converge. We choose the ϕ_j so that $\phi_i(\xi) = 1$ when

$$\rho(\xi,\xi_0) \leq \varepsilon \sum_{\ell=1}^{j} \frac{1}{\ell^2}$$

and

$$\operatorname{supp}\phi_j \subset \left\{\xi:\rho(\xi,\xi_0) \leq \varepsilon \sum_{\ell=1}^{j+1} \frac{1}{\ell^2}\right\}$$

with

$$|\phi'_j| \leq C(j+1)^{2r} \qquad |\phi''_j| \leq C(j+1)^{4r} \tag{6.4}$$

ε will be chosen later. (We could have chosen $\sum(1/\ell^s)$ for another $s > 1$ just as well.) To construct the ϕ_j, we begin by constructing auxilliary functions ψ_j on the group N. Choose $\psi_0 \in C_0^\infty(N)$ with support in $\{x:\langle x\rangle \leq \varepsilon\}$, the ball of radius ε around 0, such that $\psi_0 = 1$ for $\langle x\rangle \leq \varepsilon/2$. Define $\overline{\psi}_j$ as the dilation of ψ_0 by a factor of $2/(j+1)^2$, i.e., $\overline{\psi}_j(x) = \psi_0(\delta_{s_j}(x))$, where $s_j = (j+1)^2/2$. Then,

$$|\overline{\psi}'_j(x)| \leq ((j+1)^2/2)^r|\psi'_0(\delta_{s_j}(x))|$$

$$\overline{\psi}_j''(x) \leq ((j+1)^2/2)^{2r}|\psi_0''(\delta_{s_j}(x))|$$

so that $\overline{\psi}$ satisfies the inequalities (6.4) with ϕ_i replaced by $\overline{\psi}_i$. Then put

$$\psi_j = \begin{cases} 1 & \langle x\rangle < \varepsilon \sum_{\ell=1}^{j+1} \frac{1}{\ell^2} \\ \overline{\psi}_j(\delta_{c_j}(x)) & \varepsilon \sum_{\ell=1}^{j} \frac{1}{\ell^2} \leq \langle x\rangle \leq \varepsilon \sum_{\ell=1}^{j+1} \frac{1}{\ell^2} \\ 0 & \varepsilon \sum_{\ell=1}^{j+1} \frac{1}{\ell^2} < \langle x\rangle \end{cases}$$

where

$$c_j = 1 - \frac{\varepsilon}{\langle x\rangle}\left(\sum_1^j \frac{1}{\ell^2} - \frac{1}{(j+1)^2}\right)$$

Since the c_j have a common bound, (6.4) is again satisfied. We put

$$\phi_j(\xi) = \psi_j(\Theta(\xi_0,\xi)) \tag{6.5}$$

Now, by Proposition 4.1, the L^p norm of the operator S_i is $\leq d(\varepsilon)$, where $d(\varepsilon) \to 0$ as the diameter of the fixed ball containing the supports of all the ϕ_i goes to 0. Hence

$$\|S_1 \ldots S_N \phi_{N+1} f\|_{L^p} \leq d(\varepsilon)^N C^N \|\phi_{N+1} f\|_{L^p} \tag{6.6}$$

for $1 \leq p \leq \infty$, where C depends only on K. $d(\varepsilon)$ may be made arbitrarily small by shrinking supports. Thus, we may choose ε and the ϕ_i so that $d(\varepsilon)C < 1$ in (6.6). For such ε, the terms in (6.6) tend to zero as $N \to \infty$.

Next, we examine the limit of $E_N = R_1 + S_1R_1 + \ldots + S_1S_2 \ldots S_{N-1}R_N$ as $N \to \infty$.

Since the norm of R_{j+1} is $\leq C(j+1)^{4r}$, the norm of E_N on L^p is bounded by

$$\sum_{j=1}^{N} d(\varepsilon)^j C^j C(j+1)^{4r}$$

for $1 \leq p \leq \infty$, which converges as $N \to \infty$. Also, the L^p norm of the operator $S_1S_2\ldots S_NP_{N+1}$ is bounded by $cd(\varepsilon)^N C^N$.

We have, therefore, proved

LEMMA 6.1. Let $S \;= R$

$$S_\infty = R_1 + \sum_{N=1}^{\infty} S_1S_2\ldots S_NR_{N+1}$$

and

$$P = P_1 + \sum_{N=1}^{\infty} S_1S_2\ldots S_NP_{N+1}$$

with ϕ_i chosen as above. Then, both sums converge as operators on L^p for $1 \leqslant p \leqslant \infty$, and

$$P\mathcal{L} = \phi_1 + S_\infty$$

S_∞ is Infinitely Smoothing

We shall first show that S_∞ is bounded from L^p to S_k^p for any k. This is true for any single term $S_1S_2\ldots S_jR_{j+1}$ since R_{j+1} is bounded from L^p to S_k^p and each S_i is bounded on S_k^p. By (4.8) we may thus exclude the terms $S_1S_2\ldots S_jR_{j+1}$ for $j < k$. For any k-tuple $(i_1,\ldots,i_k)$, the operator

$$X_{i_1}\ldots X_{i_k}S_1\ldots S_k$$

is bounded from L^p to L^p by (4.9), since each X_j is of local degree $\leqslant 1$ and therefore any X_jS_ℓ is bounded on L^p by (4.7) and (4.9). Thus, we need only show that

$$\sum_{N=k+2}^{\infty} S_{k+1}\ldots S_{N-1}R_N$$

is bounded on L^p, which follows as in the proof of Lemma 6.1, since each term has norm $\leqslant d(\varepsilon)^{N-k}C^{N-k}C(N+1)^{4r}$, with $Cd(\varepsilon) < 1$.

The proof that S_∞ is bounded from L^p to L_k^p for all k follows immediately from the above and that easy fact that $S_{rk}^p \subset L_k^p$. Finally, to prove that S_∞ is smoothing with respect to the Lipschitz spaces, it suffices to apply Sobolev's Lemma to show that for any α, k can be chosen so large that the inclusion map $L_k^2 \to \Lambda_\alpha$ is bounded. Since the inclusion $L_{loc}^\infty \to L_{loc}^2$ is bounded, the proof is complete. We have, thus, proved

LEMMA 7.1. S_∞ is smoothing of infinite order.

The Limit $P = P_1 + S_1P_2 + S_1S_2P_3 + \ldots$

As above, we begin with the S_k^p spaces.

LEMMA 8.1. For any k,

$$P^{(k)} = \sum_{N=k}^{\infty} S_1S_2\ldots S_{N+3}P_{N+4}$$

is bounded from L^p to S^p_{k+2}.

Proof. Since

$$X_{i_1}X_{i_2}\ldots X_{i_{k+2}}S_1S_2\ldots S_{k+2}$$

is bounded from L^p to L^p, the lemma follows from the convergence, in L^p norm, of the operator

$$\sum_{N=k}^{\infty} S_{k+3}S_{k+4}\ldots S_{N+3}P_{N+4}$$

From the lemma, and the fact that each term is bounded from S_k^p to S^p_{k+2}, it follows that P is bounded from S_k^p to S^p_{k+2}. For the usual Sobolev spaces L^p_α, choose $k > r\alpha + 2$. Then, $S_k^p \subset L^p_{\alpha+2/r}$. Lemma 8.1 then shows the sum $S_1S_2\ldots S_NP_{N+1} + S_1S_2\ldots S_{N+1}P_{N+2} + \ldots$ to be bounded from $L^p \subset L^p_\alpha$ to $S_k^p \subset L^p_{\alpha+2/r}$. Since each term is bounded from L^p_α to $L^p_{\alpha+2/r}$, so is P. The argument for the Lipschitz spaces is similar: each term is bounded from Λ_α to $\Lambda_{\alpha+2/r}$ and the infinite sum is bounded from $L^\infty \subset L^2$ to $L^2_k \supset \Lambda_{\alpha+2/r}$.

The Transposes P^t and S^t_∞

Since the right parametrix is obtained as the transpose of the left parametrix for $\mathcal{L}^t$, we must prove that the operators P and S_∞ constructed above have appropriately smoothing transposes. These involve

$$S_i^t f(\eta) = -\int K_i(\Theta(\eta,\xi))\phi_i(\xi)f(\xi)d\xi$$

$$P_i^t f(\eta) = \int \phi_{i+1}(\eta)K_2(\Theta(\eta,\xi))\phi_i(\xi)f(\xi)d\xi$$

$$R_i^t f(\eta) = -\int \{\phi'_{i+1}(\eta)k(\Theta(\eta,\xi)) + K_1(\Theta(\eta,\xi)\phi''_{i+1}(\eta)\}\phi_i(\xi)f(\xi)d\xi$$

and

$$P^t = P_1^t + P_2^t S_1^t + P_3^t S_2^t S_1^t + \dots + P_N^t S_{N-1}^t \dots S_1^t + \dots$$

$$S_\infty^t = R_1^t + R_2^t S_1^t + R_3^t S_2^t S_1^t + \dots + R_N^t S_{N-1}^t \dots S_1^t + \dots$$

The convergence in L^p norms of these infinite sums of operators is proved as before. However, the smoothing properties are proved differently; the reason is that the terms

$$X_{i_1} \dots X_{i_{k+2}} R_N^t S_{N-1}^t \dots S_1$$

and

$$X_{i_1} \dots X_{i_k} P_N^t S_{N-1}^t \dots S_1^t$$

may entail $k + 2$ derivatives on ϕ_N.

To handle this, we choose the ϕ_j with slightly more care. We shall still have $\phi_j(\xi) = 1$ for

$$\rho(\xi_0, \xi) \leq \varepsilon \sum_{\ell=1}^{j} \frac{1}{\ell^2}$$

and $\phi_j(\xi) = 0$ for

$$\rho(\xi_0, \xi) > \varepsilon \sum_{\ell=1}^{j+1} \frac{1}{\ell^2}$$

However, we shall require that ϕ_j also satisfy

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \phi_j \right| \leq CC^{|\alpha|} (|\alpha|!)^2 (j + 1)^{2|\alpha| r} \tag{9.1}$$

This may be accomplished by defining ϕ_j by (6.5) as before, but requiring the function ψ_0 used in the construction to be (for example) of Gevrey class two, i.e., satisfying

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha \psi_0 \right| \leq CC^{|\alpha|} (|\alpha|!)^2$$

Assuming the cutoff functions have been defined as above, we now prove the smoothing properties.

LEMMA 9.1. S_∞^t is smoothing of infinite order.

Proof. As before, it suffices to prove that S_∞^t is bounded from L^p to L_k^p for all k, $1 < p < \infty$. Now,

$$X_{i_1} X_{i_2} \ldots X_{i_k} R_N^t S_{N-1}^t \ldots S_1^t$$

is a finite sum of operators with j derivatives falling on the cutoff functions ϕ'_{N+1} and ϕ''_{N+1} of R_N in the η variable, and $k - j$ derivatives (in η) falling on the kernels $k(\Theta(\eta,\xi))$ and $K_2(\Theta(\eta,\xi))$ of R_N^t. By (4.5)

$$\left| X_{i_1}^\eta X_{i_2}^\eta \ldots X_{i_{(k-j)}}^\eta k(\Theta(\eta,\xi)) \right| \leq C(k,j) \left|\rho(\xi,\eta)\right|^{-Q-k+j-1} \tag{9.2}$$

where $C(k,j)$ is a constant depending on k and j.

If $\rho(\xi,\eta) \geq \varepsilon/2(N+1)^2$ on the support of $\phi'_{N+1}(\eta)\phi_N(\xi)$,

$$\rho(\xi,\eta)^{-Q-k+j+1} \leq \left(\frac{2C(N+1)^2}{\varepsilon}\right)^{k+Q-j-1} \leq \left(\frac{2C(N+1)^2}{\varepsilon}\right)^{k+Q} \tag{9.3}$$

Using (9.2), (9.3) and (9.1),

$$\left\| X_{i_1}^\eta X_{i_2}^\eta \ldots X_{i_k}^\eta R_N^t \right\|_{L^p} \leq C'(k) \frac{(N+1)^{2(k+Q)}}{\varepsilon^{k+Q}} (k!)^2 (N+1)^{2kr}$$

where $\| \ \|_{L^p}$ denotes operator norm, and $C'(k)$ is independent of N. Therefore,

$$\left\| X_{i_1}^\eta X_{i_2}^\eta \ldots X_{i_k}^\eta R_N^t S_{N-1}^t \ldots S_1^t \right\|_{L^p} \leq C''(k) \frac{(N+1)^{(2k+2Q+2kr)}}{\varepsilon^{k+Q}} CC^N d(\varepsilon)^N$$

so, as in Section 7, it suffices to choose $d(\varepsilon) < C^{-1}$.

Finally, we must show

LEMMA 9.2. P^t is smoothing of order two.

Proof. As before, it suffices to show that, for any $(i_1,\ldots,i_{k+2})$,

$$\| X_{i_1} \ldots X_{i_{k+2}} P_{N+1}^t S_N^t S_{N-1}^t \ldots S_{N-k}^t \|_{L^p}$$
$$= \| X_{i_1} \ldots X_{i_{k+2}} \phi_{N+2} P_0^t \phi_{N+1} S_0^t \phi_N S_0^t \ldots S_0^t \phi_{N-k} \|_{L^p}$$

is bounded by a polynomial in N, where $P_0g(\xi) = \phi_0(\xi)\int k(\Theta(\eta,\xi))\phi_0(\eta)g(\eta)d\eta$ and $S_0g(\xi) = \phi_0(\xi)\int k_1(\Theta(\eta,\xi))\phi_0(\eta)g(\eta)d\eta$, $\phi_0 \in C_0^\infty(\{\xi:\rho(\xi_0,\xi) < \varepsilon\})$, and is identically one near the support of ϕ_i for all i. Using (4.9) to pass X's across P_0^t and the S_0^t's, we obtain $c(k)$ terms, the worst of which, namely the one with all derivatives on ϕ_{N+2}, has L^p norm bounded by $c_k(k!)^2(N + 2)^{2r}$.

The Kernels of S_∞ and S_∞'

We now prove that the kernels of S_∞ and S_∞' are functions in $C_0^\infty(M)$. In Lemmas 7.1 and 9.1, we have shown that S_∞ and its transpose $S_\infty'^t$ are both infinitely smoothing operators. The same is true of S_∞' and $S_\infty'^t$. Hence, the result follows from a more general result. Lacking an explicit reference, we give a simple proof.

PROPOSITION 10.1. Let $s(x,y)$ be compactly supported and in L^1, separately in x and y and set $Tf(x) = \int_{R^m} s(x,y)f(y)dy$. Suppose that for $f \in L^2$, and all α,

$$|D^\alpha Tf| \leq C_\alpha \|f\|_{L^2}$$

$$|D^\alpha T^*f| \leq C_\alpha \|f\|_{L^2}$$

Then, $s(x,y) \in C_0^\infty(R^m \times R^m)$.

Proof. Let $(1 - \Delta_z)^t$ denote the pseudodifferential operator with symbol $(1 + |\zeta|^2)^{t/2}$, t real, with $|\ |$ the Euclidean norm.

It suffices to show that, for some ℓ, $(1 - \Delta_x)^{-\ell}(1 - \Delta_y)^{-\ell}s(x,y) \in C^\infty$. Choose ℓ even, so that $(1 - \Delta_x)^{-\ell}\delta_{x=x_0} \in L^2_{loc}(R^m)$. Then,

$$|D_x^\alpha \int s(x,y)(1 - \Delta_y)^{-\ell}\delta_{y=y_0}dy| = |(D_x(1 - \Delta_y)^{-\ell}s(x,y))(x,y_0)|$$

$$\leq C_\alpha \quad \text{uniformly in } y_0$$

and, similarly,

$$|(D_y^\beta(1 - \Delta_x)^{-\ell}s(x,y))(x_0,y)| \leq C_\beta \quad \text{uniformly in } x_0$$

But, then, it follows that, for any α', β'

$$|D_x^{\alpha'}D_y^{\beta'}(1 - \Delta_x)^{-\ell}(1 - \Delta_y)^{-\ell}s(x,y)| \leq C_{\alpha'\beta'}$$

which completes the proof.

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